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CORRELATION PROPERTIES OF CHAOS: CUMULANT APPROACH*

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Abstract- The current trend in the statistical analysis of chaos shows certain gaps particularly regarding the engineering applications. This paper, which is a sequel of previous publications from the authors [1-5], develops an application of the cumulant approach to the analysis of the covariance properties of chaotic signals. A general approach for the analysis of two-moment cumulants is considered, particular emphasis is made in the covariance function and the third order cumulant behavior. The cumulant functions of the Lorenz and Chua strange attractors are considered as examples.

Keywords- Chaos, strange attractors, cumulants, degenerated equations

1. INTRODUCTION

The chaotic modeling (chaos models) is an effective tool for solution of many theoretical and practical problems. In the electrical engineering field, chaotic models are used for solving problems of electromagnetic compatibility (interference problems), wide-band communications, channel modeling, etc. [6]. Recently the authors proposed an effective approximate approach for solving statistical problems of chaos based on the cumulant calculus and named it “degenerated cumulant equations” [1-4], which can be regarded as a generalization of the approach founded and developed by A. N. Malakhov [7, 8] for the case of chaotic signals.

It was shown (see [1, 3] and the references therein) that the application of the “cumulant concept” to chaotic signals provides a rather simple approximate method for the calculus of many statistical parameters that are relevant for applications (see [1-4] for example).

The cumulants, the cumulant brackets, the cumulant equations, etc. are not well known tools in the engineering statistical analysis. Therefore, to fill this gap, a thorough explanation of all those issues was introduced in [3, 7, 8], which can help the interested reader in the process of getting familiar with the topic.

In recent publications [5] the authors showed that the non-linear filtering of chaotic signals offers significant and promising advantages and is rather opportunistic for

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applications. Although the relevance of the chaotic models in practical problems is clear, it is also clear that there are very few statistical methods developed for such models.

Regarding the cumulants and cumulant analysis of chaos it is worth to note that up to now we have an acceptable level of analysis for "one-moment" cumulants, one-dimensional probability density function (PDF), etc. To the best of the authors' knowledge the covariance function of the chaos generated by certain attractors was studied mainly in an experimental way (see [9, 10]). At the same time, the application of "two-moment" statistical characteristics can significantly improve the accuracy of the non-linear filtering algorithms[†], for example. Therefore the need of the two-moment statistical properties of chaos is doubtless.

This paper is dedicated to the development of the degenerated cumulant equations method for two-moment cumulants of chaos. Section II describes the general form of the degenerated cumulant equations with emphasis on the equations for the covariance functions. Section III is dedicated to solutions for the covariance functions of the first components of Lorenz and Chua attractors. In section IV the two-moment cumulants of the third order are presented. The conclusions are stated in section V.

2. DEGENERATED CUMULANT EQUATIONS FOR TWO-MOMENT CUMULANTS

Let us consider the dissipative continuous time system (strange attractor) in the way:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} \in \boldsymbol{R}^n, \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{1}$$

where $\boldsymbol{f}(\cdot)$ is a differentiable vector function, all notations are the same as in [1].

Equation (1) can be represented in the form of a stochastic differential equation (SDE) if one considers an extension of (1) where external noises with their intensities tending to zero (see details at [3, 7 , 8]) are included:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{\varepsilon\xi}(t), \tag{2}$$

where $\boldsymbol{\xi}(t)$ is a vector of a weak external white noises with a positive defined matrix of intensities $\boldsymbol{\varepsilon} = [\varepsilon_{ij}]^{n\times n}$.

Then assuming as an invariant measure the physical measure in the form of transient PDF $W(\boldsymbol{x},t|\boldsymbol{x}_0,t_0)$ for solution of SDE (2), one can operate with the linear Kolmogorov-Fokker-Plank (FPK) operator and its adjoin inverse Kolmogorov operator $L^{+}(\boldsymbol{x})$ [3]:

$$L^{+}(\boldsymbol{x}) = \boldsymbol{K}_{1i}(\boldsymbol{x})\frac{\partial}{\partial x_i} + \boldsymbol{K}_{2_{ij}}(\boldsymbol{x})\frac{\partial^2}{\partial x_i \partial x_j}, \tag{3}$$

here $\boldsymbol{K}_{1i}(\boldsymbol{x}) = f(\boldsymbol{x})$ and $\boldsymbol{K}_{2_{ij}} = [\varepsilon_{ij}]^{n\times n}$ are kinetic coefficients, ε_{ij} - white noise intensities defined at (2). Taking into account that the two-moment PDF:

$$W(\boldsymbol{x}_0,t_0,\boldsymbol{x},t) = W(\boldsymbol{x}_0,t_0)W(\boldsymbol{x},t|\boldsymbol{x}_0,t_0),$$

satisfies the same FPK equation as $W(\boldsymbol{x},t|\boldsymbol{x}_0,t_0)$, one can apply the same methodology to obtain degenerated cumulant equations as it was presented in [3, part 3], (see also [1, 7, 8]).

[†] An attempt to achieve this was presented at [5].

For stationary conditions, which are assumed in the following, t - t_0 is defined as τ and all two-moment cumulants will be two-moment cumulant functions that depend on τ.

Finally one can get:

$$\begin{aligned}
&\frac{d}{d\tau} < x, x_{\alpha_\tau} > = < x, K_{\alpha_\tau}(\boldsymbol{x}_\tau) > \\
&\frac{d}{d\tau} < x, x_{\alpha_\tau}, x_{\beta_\tau} > = < x, x_{\alpha_\tau}, K_{\beta_\tau}(\boldsymbol{x}_\tau) > + < x, x_{\beta_\tau}, K_{\alpha_\tau}(\boldsymbol{x}_\tau) >, \\
&\frac{d}{d\tau} < x, x_{\alpha_\tau}, x_{\beta_\tau}, x_{\gamma_\tau} > = 3\{< x, x_{\alpha_\tau}, x_{\beta_\tau}, K_{\gamma_\tau}(\boldsymbol{x}_\tau) >\}_S
\end{aligned} \tag{4}$$

where $K_{\alpha_\tau} \stackrel{\Delta}{=} K_\alpha(\boldsymbol{x}_\tau)$; $\alpha, \beta, \gamma = \overline{1,n}$ are kinetic coefficients; $3\{< x, x_{\alpha_\tau}, x_{\beta_\tau}, K_{\gamma_\tau}(\boldsymbol{x}_\tau) >\}_S$ are Stratonovich symmetrization brackets (see [3 part 2] as well as [8]), where symmetrization must be done across all indexes α, β, γ ; $<x, y, z>$ is a cumulant bracket (for details see [3, 8]).

From (4) it follows (indirectly), that all two-moment cumulant functions depend on the whole infinite set of all mutual cumulant functions of the solution of (2).

To make it clear, let us introduce the joint cumulant function of the s-order (see details in [3, 8]) as:

$$\left\langle x, x_{\alpha_\tau}^{[k_1]}, x_{\beta_\tau}^{[k_2]}, ... x_{\gamma_\tau}^{[k_\gamma]} \right\rangle = \kappa_{1,k_1...k_\gamma}^{x, x_{\alpha_\tau}, ... x_{\gamma_\tau}}, \tag{5}$$

$s = k_1 + k_2 + \dots k_\gamma$ is the order of the cumulant function.

Then, after some cumbersome algebra (see details at [7, 8]) one can get the following differential equation for the second order cumulant function:

$$\frac{d}{d\tau} < x, x_{\alpha_\tau} > = \sum_{k_1}^{\infty} \sum_{+\ k_2}^{\infty} \dots \sum_{+ \dots + k_\gamma > 0}^{\infty} \frac{1}{k_1! \cdots k_\gamma!} \left\langle \frac{\partial^{k_1 + \dots + k_\gamma}}{\partial x_1^{k_1} \dots \partial x_\gamma^{k_\gamma}} k_\gamma(\boldsymbol{x}) \right\rangle \kappa_{1,k_1,\dots,k_\gamma}^{x, x_{\alpha_\tau}, \dots x_{\gamma_\tau}}. \tag{6}$$

For two-moment cumulant functions of any order it is possible to obtain, in the same way, equations similar to (6). All those cumulant equations, both degenerated and non-degenerated, have the same important property: the operator of the system of this equations is linear [7], so any finite number of those equations has an analytical solution. In the following we will apply for each strange attractor under study, the system (4) because the analysis there can be made in a more attractive and friendly way, as it requires only the simple utilization of cumulant and Stratonovich symmetrization brackets already introduced at [3, 7, 8].

In order to achieve tractable approximate results we will use systematically the ergodic hypothesis assuming that while τ grows, the statistical dependence between two stochastic variables, divided by τ, have to diminish and tend to certain product of the one-moment cumulants presented at [1-4] .

3. COVARIANCE FUNCTION EQUATIONS OF STRANGE ATTRACTORS

In the following we will consider only the first component of the attractors under analysis. The other components can be analyzed in a similar way.

Lorenz attractor

Equations of the Lorenz attractor are [3]:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_{2_\tau} - x_{1_\tau}) \\ \dot{x}_2 &= Rx_{1_\tau} - x_{2_\tau} - x_{3_\tau} \ , \\ \dot{x}_3 &= x_{1_\tau} x_{2_\tau} - Bx_{3_\tau} \end{aligned} \tag{7}$$

where σ , R and B are attractor parameters [3].

As in [1-4] let us make the following notations: $< x_1, x_{1_\tau} >= \kappa_2^1(0,\tau)$ - is a covariance function of the first component. Then using (4) one can get:

$$\begin{aligned} &\frac{d}{d\tau}\kappa_2^1(0,\tau) = -\sigma\kappa_2^1(0,\tau) + \sigma\kappa_{1,1}^{1,2}(0,\tau) \\ &\text{or} \qquad\qquad\qquad\qquad\qquad\qquad , \\ &\frac{d}{d\tau}\ln\kappa_2^1(0,\tau) = \sigma\left(1 - \frac{\kappa_{1,1}^{1,2}(0,\tau)}{\kappa_2^1(0,\tau)}\right) \end{aligned} \tag{8}$$

Then

$$\kappa_2^1(0,\tau) = \kappa_2^1 \exp\left[-\sigma\left(1 - \frac{\kappa_{1,1}^{1,2}(0,\tau)}{\kappa_2^1(0,\tau)}\right)|\tau|\right] \ . \tag{9}$$

Let us consider two asymptotic cases from (9): $\tau \ll \tau_{\text{corr}}$ and $\tau \gg \tau_{\text{corr}}$.

When $\tau \ll \tau_{\text{corr}}$ $\kappa_{1,1}^{1,2} \approx \kappa_2^1$ and $\kappa_2^1(0,\tau) \approx \kappa_2^1$, so it is easy to show, that:

$$\kappa_2^1(0,\tau) = \kappa_{2_0}^1\left(1 - \frac{(\Delta f_{eff}\tau)^2}{2}\right), \tag{9a}$$

where Δf_{eff} - is the effective bandwidth for the first component of the statistically linearized attractor; $\tau_{\text{corr}} \sim \frac{1}{\Delta f_{eff}}$ (see [3] for details).

If $\tau \gg \tau_{\text{corr}}$ and supposing that $\lim\limits_{\tau\to\infty}\kappa_{1,1}^{1,2}(0,\tau) \sim 0$, it is reasonable to assume that the tendency to zero, when $\tau \to\infty$, is "faster" for $\kappa_{1,1}^{1,2}(0,\tau)$ than for $\kappa_2^1(0,\tau)$.

Then for $\tau \to\infty$:

$$\kappa_2^1(0,\tau) \to 0 \, , \tag{9b}$$

Chua attractor

Equations of the Chua attractor are:

$$\begin{aligned} \dot{x}_1 &= \beta_1(x_{2_\tau} - x_{1_\tau}) - \alpha h(x_{1_\tau}) \\ \dot{x}_2 &= \beta_2(x_{1_\tau} - x_{2_\tau}) + \beta_4 x_{3_\tau} \quad , \\ \dot{x}_3 &= \beta_3 x_{2_\tau} \end{aligned} \tag{10}$$

where α , β_1, β_2 and β_3 are parameters of the attractor and $h(x) = \begin{cases} -L & x < -L \\ x & |x| < L \\ L & x > L \end{cases}$. For details see [3] and the references therein.

Almost in the same way as it was done above for the Lorenz case, one can get:
- For τ << τ_{corr} :

$$\kappa_2^1(0,\tau) \approx \kappa_{2_0}^1 \left(1 - \Delta f_{eff} \mid \tau \mid\right), \tag{10a}$$

- For τ >> τ_{corr} :

$$\kappa_2^1(0,\tau) \xrightarrow{\tau\to\infty} 0 \ . \tag{10b}$$

One can see that when τ << τ_{corr} the behavior of $\kappa_2^1(0,\tau)$ for Lorenz and Chua attractors is completely different. Experimentally it was obtained in [9, 10], but here it is shown analytically. The approximate method for calculus of $\kappa_{2_0}^1$ was presented at [3].

So, the asymptotic solutions presented above as well as the cumulant calculus in [1-4] provide not only a qualitative but a quantitative description (at least with broken line approximations) of the covariance function.

4. TWO-MOMENT CUMULANTS OF THE THIRD ORDER.

Lorenz attractor

In the same way as before, one can find that the equations for the cumulant $\kappa_{1,1,1}^{x_1,x_{1\tau},x_{2\tau}}(0,\tau,\tau)$ of the third order is, (see (4)):

$$\frac{d}{dt}\kappa_{1,1,1}^{x_1,x_{1\tau},x_{2\tau}}(0,\tau,\tau) = -(\sigma-1) + \frac{R\kappa_{1,2}^{x_1,x_{1\tau}}(0,\tau) + \sigma\kappa_{1,2}^{x_1,x_{2\tau}}(0,\tau) - \kappa_{1,1,1}^{x_1,x_{1\tau},x_{3\tau}}(0,\tau,\tau)}{\kappa_{1,1,1}^{x_1,x_{1\tau},x_{2\tau}}(0,\tau,\tau)} \ . \tag{11}$$

Assuming that τ << τ_{corr} and applying the material from [3] one can get:

$$\kappa_{1,1,1}^{x_1,x_{1\tau},x_{2\tau}}(0,\tau,\tau) \overset{\tau\to 0}{\approx} \exp\left[-\kappa_{2_0}^1\left(\frac{R-1}{B}\right)\right]\cdot\left(1-(\sigma-1)\mid\tau\mid\right). \tag{12}$$

When $\tau \to\infty$, $\kappa_{1,1,1}^{x_1,x_{1\tau},x_{2\tau}}(0,\tau,\tau) \to 0$.

Chua attractor

In a similar way for Chua attractor, it yields:

$$\kappa_{1,1,1}^{x_1,x_{1\tau},x_{2\tau}}(0,\tau,\tau) \overset{\tau\to 0}{\approx} \exp\left[-\frac{\beta_1\beta_4}{\beta_4}\kappa_{2,3}^{2,1}\right]\cdot\left(1-\beta_4\mid\tau\mid\right) \tag{13}$$

and

$$\kappa_{1,1,1}^{x_1,x_{1\tau},x_{2\tau}}(0,\tau,\tau) \xrightarrow{\tau\to\infty} \to 0$$

From (12) and (13) it follows that the two-moment cumulants of the third order **might be non-monotonic** functions of τ in the region τ << τ_{corr}. Please note that in [1, 3, etc.] it is thoroughly discussed how to find the cumulants $\kappa_{2,3}^{2,1}$. To the best of our knowledge, such features of the cumulants of the third order have not been reported in the literature before.

5. CONCLUSIONS

- The cumulant calculus allows predicting the behavior of the covariance properties of the strange attractors.

- The degenerated cumulant equations are also an effective tool to study the two-moment cumulants of the highest order for qualitative and quantitative analysis as well: for example it is possible to find some cases where the behavior of those cumulants is non-monotonic, etc. Such kind of analysis is important for the development of some quasi-optimum algorithms for nonlinear filtering of chaos for practical implementations.

- It is worth to emphasize that all the two-moment cumulant properties follow directly from the degenerated cumulant equations of chaos, i.e. from the analysis of concrete strange attractors.

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