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# ON THE DARBOUX VECTOR OF RULED SURFACES IN PSEUDO-GALILEAN SPACE

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**Abstract-** In the Euclidean space the Darboux vector may be interpreted kinematically as the direction of the instantaneous axis of rotation in the moving trihedron. In this paper we mainly study the Darboux vector of ruled surfaces in pseudo-Galilean space. We obtain relationships between Darboux and Frenet vectors of each type of ruled surfaces in pseudo-Galilean space. Moreover we observe that in the pseudo-Galilean space the Darboux vector can be interpreted kinematically as a shear along the absolute line.
**Key Words-** Ruled Surface, Darboux Vector, Pseudo-Galilean Space

## 1. INTRODUCTION

The motion in $n$-dimensional Galilean space $\mathbf{G}^n$ and pseudo-Galilean space $\mathbf{G}_1^n$ has the form

$$\dot{x}^1 = x^1 + a^1, \qquad \dot{x}^i = A_1^i x^1 + A_j^i x^j + a^i, \quad (i, j = 2,3,...,n)$$

where $A_j^i$ is an $(n-1)\times(n-1)$-orthogonal or pseudo-orthogonal matrix. These formulas coincide with the transformation formulas of orthogonal coordinates in the $n$-dimensional isotropic or pseudo-isotropic space $I^n$ or $I_1^n$, respectively. Thus, the isotropic space $I^4$ is the space-time of classical mechanics of Galilei-Newton. The name of Galilean space is explained by the coincidence of the formulas of motion in $\mathbf{G}^4$ with the transformation formulas of orthogonal coordinates in $I^4$. Therefore, Kotelnikov, who defined the space $\mathbf{G}^4$, believed that it is the space-time of classical mechanics of Galilei-Newton, and this opinion was supported by Rosenfeld and Maryukova in [5].

I. M. Yaglom explained basics of Galilean geometry in [3]. Differential geometry of the Galilean space $\mathbf{G}^3$ and especially the geometry of ruled surfaces in this space has been largely developed in O. Röschel's paper [6]. Some more results about ruled surfaces in $\mathbf{G}^3$ have been given in paper by I. Kamenarovic [4]. Pseudo-Galilean space $\mathbf{G}_1^3$ has been explained in details in [1, 8, 9].

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries equipped with the projective metric of signature (0, 0, +, -), as in [1]. The absolute figure of the pseudo-Galilean geometry consists of an ordered triple $(\omega, f, I)$, where $\omega$ is the ideal (absolute) plane, $f$ the real line (absolute line) in $\omega$ and $I$ the fixed hyperbolic involution of points of $f$.

A plane is called pseudo-Euclidean plane if it contains $f$, otherwise it is called isotropic. The planes $x = \text{constant}$ are pseudo-Euclidean plane and so is the planes $\omega$. Other planes are isotropic. A vector $\boldsymbol{u} = (u_1, u_2, u_3)$ is said to be non-isotropic if $u_1 \neq 0$. All unit non-isotropic vectors are of the form $\boldsymbol{u} = (1, u_2, u_3)$. For isotropic vectors $u_1 = 0$ holds. There are four types of isotropic vectors; spacelike if $u_2^2 - u_3^2 > 0$, timelike if $u_2^2 - u_3^2 < 0$ and two types of lightlike vectors if $u_2 = \pm u_3$. A non-lightlike isotropic vector is a unit vector if $u_2^2 - u_3^2 = \pm 1$ [1].

Let $\boldsymbol{a} = (x, y, z)$ and $\boldsymbol{b} = (x_1, y_1, z_1)$ be vectors in the pseudo-Galilean space. The scalar product is defined by

$$\boldsymbol{ab} = xx_1 \tag{1}$$

If $\boldsymbol{ap} = 0$ then $\boldsymbol{a} \perp \boldsymbol{p}$ (in the sense of the pseudo-Galilean Geometry) implies, $\boldsymbol{a}^2 \neq 0$ that $\boldsymbol{p} = (0, y, z)$ is an isotropic vector.

For special vectors (isotropic) in pseudo-Galilean space $\boldsymbol{p} = (0, y, z)$ and $\boldsymbol{q} = (0, y_1, z_1)$ the scalar product is defined by

$$\boldsymbol{pq} = yy_1 - zz_1$$

The orthogonality of these vectors, $\boldsymbol{p} \perp \boldsymbol{q}$ means that $\boldsymbol{pq} = 0$ [1].

Let $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$ be vectors in the pseudo-Galilean space. The cross product of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\boldsymbol{u} \wedge \boldsymbol{v} = \begin{vmatrix} 0 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (0, u_3 v_1 - u_1 v_3, u_2 v_1 - u_1 v_2). \tag{2}$$

A trihedron $(T_0; \boldsymbol{e}_1; \boldsymbol{e}_2; \boldsymbol{e}_3)$, with a proper origin

$$(T_0;\, \boldsymbol{e}_1;\, \boldsymbol{e}_2;\, \boldsymbol{e}_3) \approx (1 : x_0 : y_0 : z_0)$$

is orthonormal in pseudo-Galilean sense if the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are of following forms

$$\boldsymbol{e}_1 = (1,\, y_1,\, z_1); \quad \boldsymbol{e}_2 = (0,\, y_2,\, z_2); \quad \boldsymbol{e}_3 = (0,\, \varepsilon z_2,\, \varepsilon y_2) \tag{3}$$

with $y_2^2 - z_2^2 = \delta$, where $\delta, \varepsilon$ is +1 or -1. Such trihedron $(T_0;\, \boldsymbol{e}_1;\, \boldsymbol{e}_2;\, \boldsymbol{e}_3)$ is called positively oriented if $\det(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = 1$ holds for the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ i.e. if $y_2^2 - z_2^2 = \varepsilon$.

If a curve $C$ of the class $C^r (r \geq 3)$ is given by the parametrization

$$r(x) = (x, y(x), z(x))$$

then $x$ is a pseudo-Galilean invariant of the arc length on $C$. The associated invariant moving trihedron is given by

$$\begin{aligned} \boldsymbol{t} &= (1, y'(x), z'(x)), \\ \boldsymbol{n} &= \frac{1}{\kappa}(0, y''(x), z''(x)), \\ \boldsymbol{b} &= \frac{1}{\kappa}(0, \varepsilon z''(x), \varepsilon y''(x)) \end{aligned} \tag{4}$$

where $\kappa = \sqrt{|y''(x)^2 - z''(x)^2|}$ is the curvature and $\tau = \frac{1}{\kappa^2}\det[r', r'', r''']$ is the torsion. Also $\varepsilon = \pm 1$, chosen by criterion $\det[\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}] = 1$.

In contrary to the geometrical interpretation of curvature in Euclidean and Galilean space, in pseudo-Galilean case $\kappa$ measures the absolute value of the change of the angle between tangents in neighbour points.

The curve $r(x)$ given by (4) is timelike (resp. spacelike) if $\boldsymbol{n}(x)$ is a spacelike (resp. timelike) vector. The principal normal vector or simply normal is spacelike if $\varepsilon = +1$ and timelike if $\varepsilon = -1$ [1].

Frenet formulas may be written as

$$\frac{d}{dx}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}.$$

A general equation of a ruled surface in $\mathbf{G}_1^3$ is

$$\varphi(x, v) = r(x) + v\boldsymbol{a}(x), \quad v \in R, \quad r(x), \boldsymbol{a}(x) \in C^3 \tag{5}$$

where the curve $r(x)$ is parametrized by the pseudo-Galilean arc length.

We say that the ruled surface given by (5) is regular if

$$\varphi_x \neq 0, \quad \varphi_v \neq 0, \quad \varphi_x \wedge \varphi_v \neq 0.$$

According to the absolute figure of $\mathbf{G}_1^3$, we distinguish the following three types of ruled surfaces in $\mathbf{G}_1^3$:

*Type I.* The ruled surfaces of type I are nonconoidal or conoidal ruled surfaces whose directional straight line at infinity is not the absolute line. The striction curve of these surfaces does not lie in a pseudo-Euclidean plane;

*Type II.* Ruled surfaces with the striction curve in a pseudo-Euclidean plane;

*Type III.* Conoidal ruled surfaces whose directional straight line at infinity is the absolute line.

## 2. DARBOUX VECTOR OF A RULED SURFACE OF TYPE I IN PSEUDO-GALILEAN SPACE

The equation of a ruled surface of type I in $\mathbf{G}_1^3$ is given by the parametrization

$$\varphi(x, v) = r(x) + v\boldsymbol{a}(x)$$

where $r(x) = (x, y(x), z(x))$ is the directrix curve and $\boldsymbol{a}(x) = (1, a_2(x), a_3(x))$ is a unit vector field. The associated trihedron of the ruled surface of type I in pseudo-Galilean space is defined by

$$\begin{aligned} \boldsymbol{t} &= (1, a_2, a_3), \\ \boldsymbol{n} &= \frac{1}{\kappa}(0, a_2', a_3'), \\ \boldsymbol{b} &= \frac{1}{\kappa}(0, a_3', a_2') \end{aligned} \tag{6}$$

where $\kappa=\sqrt{\left|(a_2')^2-(a_3')^2\right|}$ is curvature and $\boldsymbol{n}$ is the central isotropic timelike normal vector field. In this study, $\boldsymbol{n}$ is taken as timelike. If one takes it spacelike, similar procedures will be applied.

The Frenet formulas are as follows:

$$\frac{d}{dx}\begin{bmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ 0 & 0 & \tau\\ 0 & \tau & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{bmatrix} \tag{7}$$

where $\tau=-\frac{1}{\kappa^2}\det[\boldsymbol{a},\boldsymbol{a}',\boldsymbol{a}'']$ is the torsion of the ruled surface.

The shear motion is determined by an angular velocity vector given by Frenet vector

$$f=-\tau\,\boldsymbol{t}+\kappa\,\boldsymbol{b} \tag{8}$$

which satisfies $\frac{d\boldsymbol{t}}{dx}=f\wedge\boldsymbol{t}$, $\frac{d\boldsymbol{n}}{dx}=f\wedge\boldsymbol{n}$ and $\frac{d\boldsymbol{b}}{dx}=f\wedge\boldsymbol{b}$.

The Frenet vector of the associated trihedron can be separated into two shear motion: $\boldsymbol{b}$ binormal vector shear with $\kappa$ angular speed along the absolute line, that is

$$\boldsymbol{t}'=(\kappa\boldsymbol{b})\wedge\boldsymbol{t}$$

and $\boldsymbol{n}$ normal vector shear with $-\tau$ angular speed along the absolute line, that is

$$\boldsymbol{n}'=(-\tau\,\boldsymbol{t})\wedge\boldsymbol{n}$$

The surface frame $\{\boldsymbol{O},\boldsymbol{S}_n,\boldsymbol{S}_b\}$ is defined as

$$\boldsymbol{O}=\boldsymbol{a}(x),\quad \boldsymbol{S}_n=\frac{\varphi_x\wedge\varphi_v}{\left|\varphi_x\wedge\varphi_v\right|},\quad \boldsymbol{S}_b=\boldsymbol{S}_n\wedge\boldsymbol{O} \tag{9}$$

where $\boldsymbol{S}_n$ is the isotropic timelike normal vector of ruled surface of type I. Let $\phi$ be the hyperbolic angle between the isotropic timelike vectors $\boldsymbol{S}_n$ and $\boldsymbol{n}$. Then we may express results in matrix form as

$$\begin{bmatrix}\boldsymbol{O}\\ \boldsymbol{S}_n\\ \boldsymbol{S}_b\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cosh\phi & \sinh\phi\\ 0 & \sinh\phi & \cosh\phi\end{bmatrix}\begin{bmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{bmatrix}. \tag{10}$$

By a straightforward computation, we have

$$\begin{bmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cosh\phi & -\sinh\phi\\ 0 & -\sinh\phi & \cosh\phi\end{bmatrix}\begin{bmatrix}\boldsymbol{O}\\ \boldsymbol{S}_n\\ \boldsymbol{S}_b\end{bmatrix}. \tag{11}$$

Differentiating (10) with respect to $x$ then substituting (11) and (7) into the result, one obtains

$$\frac{d}{dx}\begin{bmatrix}\boldsymbol{O}\\ \boldsymbol{S}_n\\ \boldsymbol{S}_b\end{bmatrix}=\begin{bmatrix}0 & \kappa\cosh\phi & -\kappa\sinh\phi\\ 0 & 0 & d\phi+\tau\\ 0 & d\phi+\tau & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{O}\\ \boldsymbol{S}_n\\ \boldsymbol{S}_b\end{bmatrix}. \tag{12}$$

Thus, from (12) the normal curvature, the geodesic curvature and the relative torsion are given by

$$\begin{aligned} k_n &= \kappa \cosh\phi, \\ k_g &= -\kappa \sinh\phi, \\ \tau_g &= d\phi + \tau. \end{aligned} \tag{13}$$

respectively. Substituting (13) into (12), it can be written as

$$\frac{d}{dx}\begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{S}_n \\ \boldsymbol{S}_b \end{bmatrix} = \begin{bmatrix} 0 & k_n & k_g \\ 0 & 0 & \tau_g \\ 0 & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{S}_n \\ \boldsymbol{S}_b \end{bmatrix}. \tag{14}$$

Moreover the Darboux vector of the ruled surface of type I in pseudo-Galilean space is

$$U = -\tau_g\, \boldsymbol{O} + k_g \boldsymbol{S}_n + k_n\, \boldsymbol{S}_b \tag{15}$$

which satisfies $\dfrac{d\boldsymbol{O}}{dx} = U \wedge \boldsymbol{O}$, $\dfrac{d\boldsymbol{S}_n}{dx} = U \wedge \boldsymbol{S}_n$ and $\dfrac{d\boldsymbol{S}_b}{dx} = U \wedge \boldsymbol{S}_b$.

Notice that relationship between Darboux and Frenet vectors may also be found. Since the vectors $\boldsymbol{O}$ and $\boldsymbol{t}$ are coincident, one may obtain $\boldsymbol{t}'$ as

$$\boldsymbol{t}' = U \wedge \boldsymbol{O} = f \wedge \boldsymbol{t}.$$

Then simple calculation implies that

$$U = f + \lambda\, \boldsymbol{t}. \tag{16}$$

Then from (15) and (8), we have

$$-\tau_g\, \boldsymbol{O} + k_g \boldsymbol{S}_n + k_n\, \boldsymbol{S}_b = -\tau\, \boldsymbol{t} + \kappa\, \boldsymbol{b} + \lambda\, \boldsymbol{t}.$$

Finally using (13) and (16), it is obvious that we have the following relationship between Darboux and Frenet vectors of the ruled surface of type I in pseudo-Galilean space:

$$U = f - d\phi\, \boldsymbol{t}. \tag{17}$$

## 3. DARBOUX VECTOR OF A RULED SURFACE OF TYPE II IN PSEUDO-GALILEAN SPACE

In this section, we investigate the ruled surface of type II which has striction line in pseudo-Euclidean plane. Since the striction line is a base curve, it may also be written as

$$r(x) = (0,\ y(x),\ z(x)). \tag{18}$$

The ruled surface of type II in pseudo-Galilean space is parametrized as

$$\varphi(x, v) = r(x) + v\, \boldsymbol{a}(x) \tag{19}$$

where $r(x)$ is the directrix curve and $\boldsymbol{a}(x) = (1,\ a_2(x),\ a_3(x))$ is a unit vector field.

The associated trihedron of the ruled surface of type II in the 3-dimensional pseudo-Galilean space is defined by

$$\begin{aligned} \boldsymbol{t} &= (1, a_2, a_3), \\ \boldsymbol{n} &= (0, z', y'), \\ \boldsymbol{b} &= (0, y', z'). \end{aligned} \tag{20}$$

Then the Frenet formulas are

$$\frac{d}{dx}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{21}$$

where the curvature and the torsion of the ruled surface in pseudo-Galilean space are, respectively

$$\kappa = -\frac{a_2'}{z'}, \quad \tau = \frac{y''}{z'}. \tag{22}$$

The Frenet vector of ruled surface of type II is

$$f = -\tau\, \boldsymbol{t} + \kappa\, \boldsymbol{b} \tag{23}$$

which satisfies $\frac{d\boldsymbol{t}}{dx} = f \wedge \boldsymbol{t}$, $\frac{d\boldsymbol{n}}{dx} = f \wedge \boldsymbol{n}$ and $\frac{d\boldsymbol{b}}{dx} = f \wedge \boldsymbol{b}$. For ruled surfaces of type I and type II in pseudo-Galilean space we have the same Frenet vector, associated trihedron and also surface frame. Therefore (17) holds for ruled surfaces of type II in pseudo-Galilean space too.

## 4. DARBOUX VECTOR OF A RULED SURFACE OF TYPE III IN PSEUDO-GALILEAN SPACE

The surfaces of type III are conoidal surfaces with isotropic generator field. They can be parametrized by

$$\varphi(x,v) = r(x) + v\,\boldsymbol{a}(x)$$

where $r(x) = (x, y(x), 0)$ is the directrix curve and $\boldsymbol{a}(x) = (0, a_2(x), a_3(x))$ is a unit vector field. The associated trihedron of ruled surfaces type III in pseudo-Galilean space is defined as

$$\begin{aligned} \boldsymbol{t} &= (1, y', 0), \\ \boldsymbol{n} &= (0, a_2, a_3), \\ \boldsymbol{b} &= (0, a_3, a_2). \end{aligned} \tag{24}$$

Let $\theta$ be the hyperbolic angle between the plane $z = 0$ and $\boldsymbol{n}$, then Frenet formulas are

$$\frac{d}{dx}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa\cosh\theta & -\kappa\sinh\theta \\ 0 & 0 & 1/\delta \\ 0 & 1/\delta & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{25}$$

where the curvature and the torsion of the ruled surface of type III are, respectively,

$$\kappa = y'', \quad \delta = \frac{a_3}{a_2'}.$$

The Frenet vector of the ruled surface of type III is given by

$$f = -\frac{1}{\delta}\, \boldsymbol{t} - \kappa\sinh\theta\, \boldsymbol{n} + \kappa\cosh\theta\, \boldsymbol{b} \tag{26}$$

which satisfies $\frac{d\boldsymbol{t}}{dx} = f \wedge \boldsymbol{t}$, $\frac{d\boldsymbol{n}}{dx} = f \wedge \boldsymbol{n}$ and $\frac{d\boldsymbol{b}}{dx} = f \wedge \boldsymbol{b}$.

The surface frame $\{\boldsymbol{O}, \boldsymbol{S}_n, \boldsymbol{S}_b\}$ is defined as

$$\boldsymbol{O} = \boldsymbol{a}(x), \quad \boldsymbol{S}_n = \frac{\varphi_x \wedge \varphi_v}{|\varphi_x \wedge \varphi_v|}, \quad \boldsymbol{S}_b = \boldsymbol{S}_n \wedge \boldsymbol{O}$$

where $\boldsymbol{S}_n$ is the isotropic timelike normal vector of ruled surface of type III. Let $\phi$ be hyperbolic angle between the isotropic timelike vectors $\boldsymbol{S}_n$ and $\boldsymbol{n}$.

We may express the relationship between the frames $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and $\{\boldsymbol{O}, \boldsymbol{S}_n, \boldsymbol{S}_b\}$ in matrix form as

$$\begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{S}_n \\ \boldsymbol{S}_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\phi & \sinh\phi \\ 0 & \sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}. \tag{27}$$

By a straightforward computation, we have

$$\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\phi & -\sinh\phi \\ 0 & -\sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{S}_n \\ \boldsymbol{S}_b \end{bmatrix}. \tag{28}$$

Differentiating (27) with respect to $x$ then substituting (28) and (25) into the result, one obtains

$$\frac{d}{dx}\begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{S}_n \\ \boldsymbol{S}_b \end{bmatrix} = \begin{bmatrix} 0 & \kappa\cosh\psi & -\kappa\sinh\psi \\ 0 & 0 & d\phi + 1/\delta \\ 0 & d\phi + 1/\delta & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{S}_n \\ \boldsymbol{S}_b \end{bmatrix} \tag{29}$$

where $\psi = \phi + \theta$. The normal curvature $k_n$, the geodesic curvature $k_g$ and the relative torsion $\tau_g$ are given by

$$\begin{aligned} k_n &= \kappa\cosh\psi, \\ k_g &= -\kappa\sinh\psi, \\ \tau_g &= d\phi + \frac{1}{\delta} \end{aligned} \tag{30}$$

respectively. Substituting (30) into (29), we have

$$\frac{d}{dx}\begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{S}_n \\ \boldsymbol{S}_b \end{bmatrix} = \begin{bmatrix} 0 & k_n & k_g \\ 0 & 0 & \tau_g \\ 0 & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{S}_n \\ \boldsymbol{S}_b \end{bmatrix}. \tag{31}$$

Moreover the Darboux vector of ruled surface of type III in pseudo-Galilean space is

$$U = -\tau_g \boldsymbol{O} + k_g \boldsymbol{S}_n + k_n \boldsymbol{S}_b \tag{32}$$

which satisfies $\dfrac{d\boldsymbol{O}}{dx} = U \wedge \boldsymbol{O}$, $\dfrac{d\boldsymbol{S}_n}{dx} = U \wedge \boldsymbol{S}_n$ and $\dfrac{d\boldsymbol{S}_b}{dx} = U \wedge \boldsymbol{S}_b$.

Consequently, it is obvious that we have the following relationship between Darboux and Frenet vectors of ruled surface of type III in pseudo-Galilean space:

$$U = f - d\phi \boldsymbol{t}. \tag{33}$$

**Example 4.1** Let us now consider the ruled surface of type I in pseudo-Galilean space, shown in Figure 1. It can be parametrized by

$$\varphi(x,v) = (x, A\sinh\frac{x}{p}, A\cosh\frac{x}{p}) + v(1, \frac{1}{B}\cosh\frac{x}{p}, \frac{1}{B}\sinh\frac{x}{p})$$

where $A, B, p \in$ **Rç** $A \neq 0, B \neq 0, p \neq 0$ and $|AB - p| > |v|$.

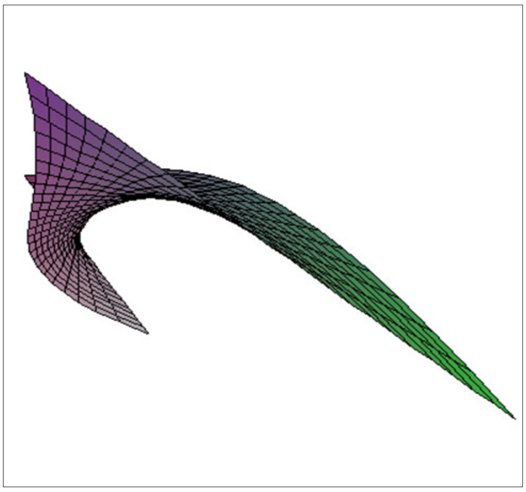

Figure 1

The associated trihedron of the ruled surface of type I in pseudo-Galilean space is defined by

$$\begin{aligned} \boldsymbol{t} &= (1, \frac{1}{B}\cosh\frac{x}{p}, \frac{1}{B}\sinh\frac{x}{p}), \\ \boldsymbol{n} &= (0, \sinh\frac{x}{p}, \cosh\frac{x}{p}), \\ \boldsymbol{b} &= (0, \cosh\frac{x}{p}, \sinh\frac{x}{p}) \end{aligned}$$

where $\kappa = \frac{1}{Bp}$. Frenet formulas are as follows:

$$\frac{d}{dx}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & 1/Bp & 0 \\ 0 & 0 & 1/p \\ 0 & 1/p & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}$$

where $\tau = \frac{1}{p}$ is called the torsion of the ruled surface and Frenet vector is

$$f = (-\frac{1}{p}, 0, 0).$$

Let $\phi$ be the hyperbolic angle between isotropic timelike vectors $\boldsymbol{S}_n$ and $\boldsymbol{n}$.

Using (13), we obtain

$$k_n = \frac{1}{Bp}\cosh\phi, \quad k_g = -\frac{1}{Bp}\sinh\phi, \quad \tau_g = d\phi + \frac{1}{p}.$$

Then we have

$$U = -(d\phi + \frac{1}{p})\boldsymbol{O} - \frac{1}{Bp}\sinh\phi\boldsymbol{S}_n + \frac{1}{Bp}\cosh\phi\boldsymbol{S}_b.$$

Consequently, $U$ takes the form as $U = f - d\phi\boldsymbol{t}$.

## 5. CONCLUDING REMARKS

G. Darboux was the first to point out the geometric significance of the Darboux vector for the natural trihedral of a space curve. The vector, since its discovery, has come into great prominence and proved of considerable interest especially in the theory of curves and surfaces. A Galilean space is a pseudo-Euclidean space of nullity one and it may be considered as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics. Consequently we shall bear in mind that our geometry arises naturally out of mechanical considerations connected with Galileo's principle of relativity. This implies that, in our case, properties of geometric significance are really properties of mechanical significance.

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