Mathematical and Computational Applications, Vol. 17, No. 1, pp.1-8, 2012.

A METHOD FOR DISCRIMINATING EFFICIENT CANDIDATES WITH RANKED VOTING DATA BY COMMON WEIGHTS

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Abstract- Ranked voting data arise when voters select and rank more than one candidate with an order of preference. Cook et al.[1] introduced data envelopment analysis (DEA) to analyze ranked voting data. Obata et al.[2] proposed a new method that did not use information obtained from inefficient candidates to discriminate efficient candidates.
Liu et al.[3] ranked efficient DMUs on the DEA frontier with common weights. They proposed a methodology to determine one common set of weights for the performance indices of all DMUs. Then, these DMUs were ranked according to the efficiency score weighted by the common set of weights. In this paper, we use one common set of weights for ranked voting data.
Key Words- Data envelopment analysis (DEA), Ranked voting data, Ranking of candidates, Common weight

1. INTRODUCTION

In the recent papers, we have considered ranked voting data which are obtained when voters select and rank more than one candidate. Here, it is assumed that a voter selects k(k>0) candidates from a set of $m(m \geq k)$ candidates and ranks them from top to the kth place. Let v_{ij} be the number of the jth place votes of candidate i(i=1,...,m, j=1,...,k). A preference score Z_i, of candidate i should be calculated as a weighted sum of the votes with certain weights w_j, i.e.,

$$Z_i = \sum_{j=1}^{k} w_j v_{ij} \qquad (1)$$

By using data envelopment analysis (DEA) [4], Cook et al.[1] have proposed a method for estimating preference scores without imposing any fixed weights on outputs. Each candidate's score is calculated with the most favorable weight for the outputs.
Their formulation is as follows:

$$\begin{array}{lll} Z_0^* = \text{Max} & \sum_{j=1}^{k} w_j v_{0j} & \\ \text{s.t.} & \sum_{j=1}^{k} w_j v_{ij} \leq 1, & i=1,...,m. \qquad (2) \\ & w_j - w_{j+1} \geq d(j,\varepsilon), & j=1,...,k-1. \\ & w_k \geq d(k,\varepsilon), & \end{array}$$

Where d(.,ε), called the discrimination intensity function, is nonnegative and non decreasing in ε, and satisfied d(.,0)=0, parameter ε is nonnegative.

This is solved for each candidate $o,(o \in \{1,...,m\})$. The resulting score Z_0^* is the preference score of the candidate. This model obtains favorable weights that are different for each DMU.

Liu et al. [3] ranked efficient DMUs using common weight but the object of this paper is ranked voting in the method of Obata et al. [2] with common weight; therefore, we use Liu et al. [3] method. In previous methods, for example method of Cress et al. [1], weights changed from one DMU to another DMU. But in the proposed model we have a common weight for all of DMUs that makes the model more valuable. Also in the proposed model, we solve one linear programming, but in the previous models n linear programming had to be solved.

In this paper we want to rank units by common weights. In section 2, we will introduce a method for finding the common weights for this method. We will use CWA-efficiency (Common Weights Analysis) method. In section 3, we will propose a ranking rule for ranking the candidates. We will give an example for our method in section 4, and Section 5 will be comparison with other models. section 6 will provide our conclusion.

2. OUR PROPOSED METHOD

We defined v_{ij} to be the number of the jth place votes of candidate i, (i=1,...,m, j=1,...,k). We define DMU_i with coordinate $(1, v_{i1}, v_{i2}, ..., v_{ik})$ i=1,...,m. Therefore, we have m DMUs with k outputs, and a single input with one value.

In Fig.1, both the vertical and the horizontal axes are outputs. Similar to Liu et al. [3], we define the benchmark level in the two-dimensional outputs space as the benchmark level (Ox), which is one straight line that passes through the origin with slope 1.0 in outputs space. w_j (j=1,...,k) in the weighted sum denotes the decision variable of the common weights for the outputs. We want to compare the DMUs with the benchmark level and obtain the common weights and rank them with these common weights.

The notation of a decision variable with superscript symbol" $'$ " represents an arbitrary assigned value. For any two DMUs, DMU_M and DMU_N , given one set of weights $w_j^i(j = 1,...,k)$, the coordinates of points M' and N' in Fig.1 are $(\sum_{j=1}^{k} w_j{}' v_{Mj})$ and $(\sum_{j=1}^{k} w_j{}' v_{Nj})$, respectively. The virtual gaps between points M' and M'^P the horizontal and vertical axes are denoted by Δ_M^1, Δ_M^2 , respectively. Similarly, for points N' and N'^P the gaps are denoted Δ_N^1, Δ_N^2, respectively. Therefore, in view of points M' and N', we observe that there exists a total virtual gap $\Delta_M^1 + \Delta_M^2 + \Delta_N^1 + \Delta_N^2$, to the benchmark line. Let the notation of a decision variable with superscript " * " represent the optimal value of the variable.

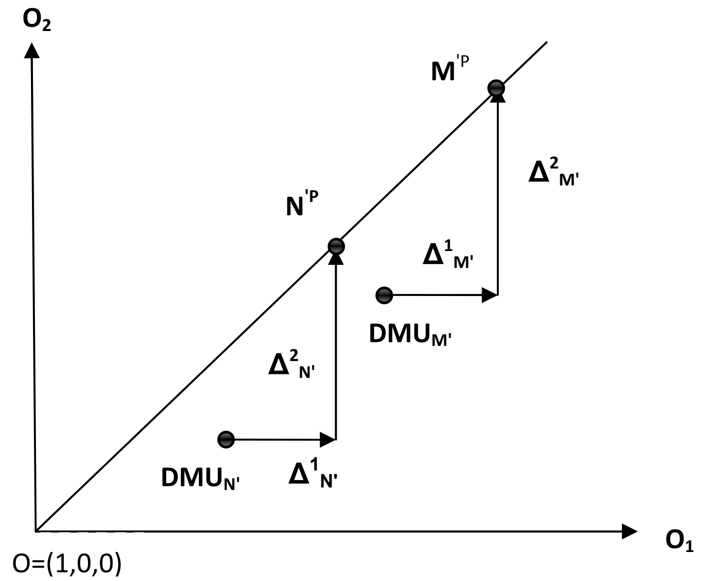

Figure 1. Gap analysis showing the DMUs below the virtual benchmark line.

We want to determine an optimal set of weights w_j^*(j=1,...,k) so that both points M^* and N^* below the benchmark line could be as close to their projection points, M^{*P} and N^{*P} on the benchmark line, as possible. In other words, by adopting the optimal weights, the total virtual gap $\Delta_M^{*1}+\Delta_M^{*2}+\Delta_N^{*1}+\Delta_N^{*2}$, to the benchmark line is shortest to both DMUs.

As for the constraint, the weighted sum of outputs plus the vertical virtual gap $\Delta_i\,(i=1,...,m)$ equals 1. This constraint implies that the direction closest to the benchmark line is upwards and rightwards at the same time. This equality to 1 in the constraint means that the projection point on the benchmark line is reached. ε is a positive non-Archimedean infinitesimal constant. We also avoid a case of zero value of indices obtained by choosing the set of zero weights.

$$
\begin{aligned}
\Delta^{*}=\text{Min} \quad & \sum_{i=1}^{m}\Delta_i \\
\text{s.t.} \quad & \sum_{j=1}^{k} w_j v_{ij} + \Delta_i = 1, && i=1,...,m, \\
& w_j - w_{j+1} \geq d(j,\varepsilon), && j=1,...,k-1, \\
& w_k \geq d(k,\varepsilon), \\
& \Delta_i \geq 0, && i=1,...,m.
\end{aligned} \tag{3}
$$

(3) could be rewritten as the equivalent linear programming problem (4) by the following equation:

$$\Delta_i = 1 - \sum_{j=1}^{k} w_j v_{ij},$$

$$\sum_{i=1}^{m}\Delta_i = \sum_{i=1}^{m} 1 - \sum_{i=1}^{m}\sum_{j=1}^{k} w_j v_{ij} = m - \sum_{i=1}^{m}\sum_{j=1}^{k} w_j v_{ij}$$

$$\text{Min } \sum_{i=1}^{m}\Delta_i = \text{Min } m - \sum_{i=1}^{m}\sum_{j=1}^{k} w_j v_{ij} = m - Max \quad \sum_{i=1}^{m}\sum_{j=1}^{k} w_j v_{ij}$$

$$
\begin{aligned}
m-\Delta^{*}=\text{Max} \quad & \sum_{i=1}^{m}\sum_{j=1}^{k} w_j v_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{k} w_j v_{ij} \leq 1, && i=1,...,m, \\
& w_j - w_{j+1} \geq d(j,\varepsilon), && j=1,...,k-1, \\
& w_k \geq d(k,\varepsilon),
\end{aligned} \tag{4}
$$

So (3) translates to the following model:
We need dual of model (4) for a better analysis. Here, we use $\{\lambda_i \mid i=1,...,m\}$ as the standard dual variable associated with the m first constraints, and the variables $\{\mu_j, \mu_k \mid i=1,...,k-1\}$ are the dual variables associated with the k-1 and first, second and third constraints. In order to obtain more information, we transform (4) to its dual form (5).

$$
\begin{aligned}
\text{Min} \quad & \sum_{i=1}^{m}\lambda_i - \sum_{j=1}^{k}\mu_j d(j,\varepsilon) \\
\text{s.t.} \quad & \sum_{i=1}^{m}\lambda_i v_{i1} - \mu_1 = \sum_{i=1}^{m} v_{i1}, \\
& \sum_{i=1}^{m}\lambda_i v_{ij} - \mu_{j-1} - \mu_j = \sum_{i=1}^{m} v_{ij}, && j=2,...,k, \\
& \lambda_i \geq 0, && i=1,...,m, \\
& \mu_j \geq 0, && j=1,...,k,
\end{aligned} \tag{5}
$$

We can rank voting for each DMU using models (4), (5). In these model λ_j is value of reference share of DMU_j and value of λ_j can be used for ranking candidates.

3. PROPOSED RANKING RULE

First, we solve models (4) and (5) and obtain $w^*=(w_1^*,...,w_k^*)$, Δ^* and $\lambda^*=(\lambda_1^*,...,\lambda_m^*)$, which are the optimal solutions. Then, we calculate the preference score of the DMU under evaluation (DMU_0). That is, we obtain $Z_0^*=\sum_{j=1}^{k} w_j^* v_{0j}$

By the value of Z_0^*, we can rank the voting data.

Definition 1: Candidate i has the first rank if $Z_0^*=1$.

Definition 2: The preference score of candidate i is better than that of candidate j if $Z_i^* > Z_j^*$.

Definition 3: If $Z_i^* = Z_j^* < 1$, then the preference score of candidate i is better than that of candidate j, if $\Delta_i^* < \Delta_j^*$.

Definition 4: If $Z_i^* = Z_j^* = 1$, then the preference score of candidate i is better than that of candidate j, if $\lambda_i^* > \lambda_j^*$.

We can be sure that there exists at least one candidate that has a preference score of 1.0.

Theorem 1: There is at least one candidate such as $DMU_i(i=1,...,m)$, with $Z_i^* = \sum_{j=1}^{k} w_j^* v_{ij} = 1$.

Proof: *We will use contradiction to prove the existence of the above theorem. Assume that there is not a candidate that has a preference score of 1.0. So, there is* $\Delta_i^* > 0$ *so that* $\sum_{j=1}^{k} w_j^* v_{ij} + \Delta_i^* = 1$, i=1,...,m.

We can obtain $a_i>1$ *so that* $\sum_{j=1}^{k} a_i w_j v_{ij} = 1$, *i=1,...,m. Let a be the minimum of set* $\{a_i \mid i=1,...,m\}$ *Then we can obtain another feasible common set of weights* w_j *leads to a smaller objective function, and this contradicts the assumption. Hence, there is at least one candidate with* $Z_0^*=1$. ■

4. NUMERICAL EXAMPLE

We assume that 10 people take part in a voting. Each person can vote for three candidates. We can see the data obtained from this voting in Table 1.

Table 1. Voting data

candidates	Firs t vote	Second vote	Third vote
1	5	3	2
2	10	9	7
3	1	5	10
4	6	7	2
5	3	2	1
6	5	2	7
7	1	4	3
8	9	10	5
9	2	6	7
10	1	2	3

In this paper, v_{ij} denotes the number of the jth place votes of candidate (i=1,...m, j=1,...,K). Also we define $DMU_i=(1,v_{i1}, v_{i2},...,v_{ik})$. Therefore, we have 10 DMUs with 3 outputs and a single input with one value.

Table 2. DMUs

DMU	v_{i1}	v_{i2}	v_{i3}
1	3	0	1
2	1	3	2
3	1	1	2
4	0	1	0
5	2	1	1
6	1	1	0
7	0	1	2
8	0	0	0
9	1	1	0
10	1	1	1

In Table 2, the coordinate of DMU_8 is (0,0,0), which means that nobody has voted for it so, it has the last rank and we exclude it from the evaluation. After solving models (4), (5), the following results are obtained (see Table 3).

$w^* = (w_1^*, w_2^*, w_3^*) = (0.285716, 0.142861, 0.142851)$

Table 3. Results

DMU_j	Z_j^*	Δ_j^*	λ_j^*	Rank
1	1.00000	0.000000	2.21	2
2	1.00000	0.000000	3.35	1
3	0.714279	0.285721	0.00	4
4	0.142861	0.857139	0.00	8
5	0.857144	0.142856	0.00	3
6	0.428577	0.571423	0.00	6
7	0.428562	0.571438	0.00	7
8	-	-	-	8
9	0.428577	0.571423	0.00	6
10	0.571428	0.428572	0.00	5

Note that DMU_6 and DMU_9 have the same coordinate; therefore, both of them are in rank 6.

5. COMPARISON WITH OTHER MODELS

In this section, we intend to compare proposed model to the mentioned ones. Therefore we apply given data in Obata et al.[2] in Table 4. We consider $d(.,\varepsilon) = 0$ as Cook and Kress did.

Table 4. Sample data(m=6,k=2)

Candidate	First rank	Second rank
A	32	10
B	28	20
C	13	36
D	20	27
E	27	19
F	30	8
G	0	30

The results of model (1) from Cook et al. [1] are showed in table 5.

Table 5. Results of Cook and Kress's model

DMU_j	A	B	C	D	E	F	G
Z_j^*	1.0000	1.0000	1.0000	0.9693	0.9611	0.9375	0.6122

The feasible solutions of Cook and Kress's model on the weight space are w_1, w_2, w_3, w_4 so that $w_1 = (1/32,0), w_2 = (1/36,1/90), w_3 = (4/187,15/748), w_4 = (1/49,1/49)$. The longest sets of favorable weight vectors for the candidates A, B and C are $P_A = [w_1, w_2], P_B = [w_2, w_3], P_C = [w_3, w_4]$, respectively, where $[w_i, w_j]$ means a line segment from w_i to w_j.

Obata et al. [2] scaled these weights to:

$\hat{w}_1 = (1,0), \hat{w}_2 = (5/7,2/7), \hat{w}_3 = (16/31,15/31), \hat{w}_4 = (1/2,1/2)$

When the L_1-norm was used, the preference scores of A,B and C on their own territory were estimated $[\hat{w}_1, \hat{w}_2],[\hat{w}_2, \hat{w}_3]$ and $[\hat{w}_3, \hat{w}_4]$, respectively. The results showed that the score of A is maximum at $\hat{w}_1$ and the normalized preference score was $\hat{z}_A^* = 32$. Similarly, $\hat{z}_B^* = 25.714$ (at $\hat{w}_2$) and $\hat{z}_C^* = 24.5$ (at $\hat{w}_4$). So Obata et al. [2] could judge that the winner was candidate A, and was followed by B then C. Obata et al. [2] could rank the efficient candidates.

Now, by solving model (4) with data of table 4, one could obtain $(w_1, w_2) = (0.021390,0.020053)$. Then, z_j^* is calculated from (1). Their results are shown in the second column in table 6. In this method, there are just two efficient candidates. For ranking efficient candidates, model (5) is solved and λ_j^* (j=B,C), $\lambda_B^* = 4.61 > \lambda_C^* = 1.60$ are obtained; therefore, the winner is candidate B. Ranking of all candidates is shown in the last column of table 6.

Table 6. Results of the proposed model

DMU_j	Z_j^*	Δ_j^*	λ_j^*	Rank
A	0.885027	0.114973	0.00	5
B	1.000000	0.000000	4.61	1
C	1.000000	0.000000	1.60	2
D	0.969200	0.030749	0.00	3
E	0.958556	0.041440	0.00	4
F	0.802139	0.197861	0.00	6
G	0.601604	0.398396	0.00	7

Previous models solved one LP for each candidate and obtained one weight corresponding with each LP for the evaluated candidate. Although, proposed method solved just one LP for all candidates with a common set of weights.

The results of Obata's model are almost the same as those of proposed model; however, they are different in ranking. It's worth mentioning that our model has less series of calculations and applied just one common set of weights for all candidates.

6. CONCLUSION

In this paper, we have briefly surveyed ranked voting data and its analysis with DEA. Our model is based on the ranking of units by DEA with common weights. This paper obtains one common set of weights that is the most favorable for determining the absolute efficiency for DMUs at the same time. As for its practical application, this methodology is aimed at the ranking of voting data. New ranking rules, obtained from absolute efficiency, could help decision makers understand the performance of candidates. The CWA (Common Weights Analysis) methodology helps us in ranking voting data.

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