

TAYLOR MATRIX SOLUTION OF THE MATHEMATICAL MODEL OF THE RLC CIRCUITS

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Abstract- The RLC circuit is a basic building block of the more complicated electrical circuits and networks. The present study introduces a novel and simple numerical method for the solution this problem in terms of Taylor polynomials in the matrix form. Particular and general solutions of the related differential equation can be determined by this method. The method is illustrated by a numerical application and a quite good agreement is observed between the results of the present method and those of the exact method.

Key Words- Taylor Matrix Method, Electrical Circuits, Differential Equation, Mathematical Model

1. INTRODUCTION

The RLC circuit is a basic building block of the more complicated electrical circuits and networks. As shown in Fig. 1, it consists of a **resistor** with a resistance of R ohms, an **inductor** with an inductance of L henries, and a **capacitor** with a capacitance of C farads, in series with a source of electromotive force (such as a battery or a generator) that supplies a voltage of $E(t)$ volts at time t . If the switch of the circuit shown in Fig. 1 is closed, this results in a current of $I(t)$ amperes in the circuit and a charge of $Q(t)$ coulombs on the capacitor at time t . The relation between the functions $Q(t)$ and $I(t)$ is

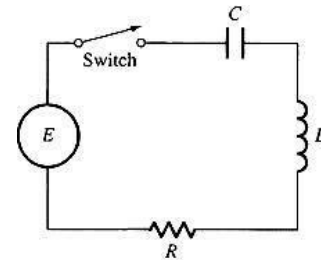


Figure 1. The series RLC circuit

$$\frac{dQ(t)}{dt} = Q'(t) = I(t)$$

The second order linear differential equation of this simple RLC circuit is [1]

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t) \quad (1)$$

for the charge $Q(t)$, under the assumption that the voltage $E(t)$ is known.

In this study we introduce a novel and simple method in terms of Taylor polynomials in matrix form. These polynomials have been used for the solution of

differential and integral equations by many researchers. Sezer [2] used this method to find the approximate solution of the second-order linear differential equation with specified associated conditions in terms of Taylor polynomials about any point. Sezer, Karamete and Gülsu [3] gave Taylor polynomial solutions of the systems of linear differential equations with variable coefficients. Gülsu and Sezer [4] expanded this method for solving differential-difference equations. Yalçınbaş and Sezer [5] developed a Taylor method to find the approximate solution of high order linear Volterra-Fredholm integro differential equations under the mixed conditions in terms of Taylor polynomials about any point. Sezer and Akyüz-Daşcıoğlu [6] developed a similar Taylor polynomial method to find an approximate solution of high order linear Volterra-Fredholm integro differential equations with variable coefficients under the mixed conditions. Kurt and Çevik [7] gave an example of a mechanical vibration problem for solving single degree of freedom system by this method. Çevik [8] expanded the method for the longitudinal vibration analysis of rods. Wang and Li [9] established a reliable algorithm for solving ordinary differential equations by using the theories and method of mathematics analysis and computer algebra. They also established a Maple procedure based on Taylor polynomial method.

The following steps are used in this work. First the governing differential equation of the RLC circuit is represented in matrix form. The initial conditions are also written in matrix form. Then the steady periodic and general solutions of the problem are obtained. Next, the method is illustrated by a numerical example. Finally, the results are discussed.

2. MATRIX REPRESENTATION OF THE PROBLEM

In most practical problems, it is the current $I(t)$ rather than the charge $Q(t)$ that is of primary interest, so we differentiate both sides of Eq. (1) and substitute $I(t)$ for $Q'(t)$ to obtain the system differential equation.

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t) \quad (2)$$

with initial values

$$I(0) = I_0 \quad (3a)$$

$$Q(0) = q_0 \quad (3b)$$

The solution of Eq. (2) is expressed in the Taylor polynomial form as

$$I(t) = \sum_{n=0}^N x_n (t-c)^n, \quad x_n = \frac{x^{(n)}(c)}{n!}, \quad (4)$$

and obtained by determining the Taylor coefficients $x_n, n = 1, 2, \dots, N$. We may put Eq. (4) into the following matrix form

$$[I(t)] = \mathbf{T}(t)\mathbf{X} \quad (5)$$

where

$$\mathbf{T}(t) = \begin{bmatrix} 1 & (t-c) & (t-c)^2 & \cdots & (t-c)^N \end{bmatrix} \quad (6)$$

$$\mathbf{X} = [x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_N]^T \quad (7)$$

The relation between the matrix $\mathbf{T}(t)$ and its first derivative $\mathbf{T}'(t)$ can be expressed as [4]

$$\mathbf{T}'(t) = \mathbf{T}(t)\mathbf{B} \quad (8)$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (9)$$

The second derivative can be written similarly,

$$\mathbf{T}''(t) = \mathbf{T}'(t)\mathbf{B} = \mathbf{T}(t)\mathbf{B}^2 \quad (10)$$

and the matrix representation of the right-hand side term of Eq. (2) can be written in the form

$$[E'(t)] = \sum_{n=0}^N e_n (t-c)^n = \mathbf{T}(t)\mathbf{E}, \quad e_n = \frac{e^{(n+1)}(c)}{n!} \quad (11)$$

where

$$\mathbf{E} = [e_0 \quad e_1 \quad e_2 \quad \cdots \quad e_N]^T \quad (12)$$

The matrix form for the initial conditions (3a, 3b) can be obtained using (5) and (8)

$$[I_0] = \mathbf{T}(0)\mathbf{X} \quad (13)$$

$$\left[\frac{E(0) - RI_0 - q_0 / C}{L} \right] = \mathbf{T}(0)\mathbf{B}\mathbf{X} \quad (14)$$

where we can write equation (14) due from RLC model which satisfy the basic circuit equation

$$LI' + RI + \frac{1}{C}Q = E(t) \quad (15)$$

Finally, we can obtain the matrix representation of the problem using Eqs. (5), (8), (10) and (11) as

$$\left\{ \mathbf{LB}^2 + \mathbf{RB} + \frac{1}{C} \mathbf{I} \right\} \mathbf{X} = \mathbf{E} \quad (16)$$

where \mathbf{I} is the identity matrix.

3. MATRIX SOLUTION OF THE PROBLEM

The general solution of Eq. (2) is the sum of transient current I_{tr} that approaches zero as $t \rightarrow \infty$ (under the assumption that the coefficients in Eq. (2) are all positive, so roots of characteristic equation have negative real part), and a steady periodic current I_{sp} ; thus

$$I = I_{sp} + I_{tr} \quad (17)$$

Therefore we can easily obtain transient current solution by taking the difference of general and steady periodic solutions.

3.1. The steady periodic solution

In order to determine the steady periodic solution I_{sp} of the problem, (16) is written briefly in the form

$$\mathbf{WX} = \mathbf{E} \text{ or } [\mathbf{W}; \mathbf{E}] \quad (18)$$

where

$$\mathbf{W} = [w_{ij}] = \mathbf{LB}^2 + \mathbf{RB} + \frac{1}{C} \mathbf{I}, \quad i, j = 0, 1, \dots, N \quad (19)$$

By consequence,

$$\mathbf{X} = \mathbf{W}^{-1} \mathbf{E} \quad (20)$$

which yields the desired Taylor coefficients $x_n, n = 0, 1, \dots, N$ of the steady periodic current I_{sp} .

3.2. General Solution

To determine the general solution, the matrix form (16) of the boundary conditions [2] is written as

$$[\mathbf{U}; \boldsymbol{\lambda}] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & ; & \lambda_0 \\ 0 & 1 & 0 & \cdots & 0 & ; & \lambda_1 \end{bmatrix} \quad (21)$$

The first row of matrix (21) is derived from equation (13) and the second row from equation (14).

Now, to solve the problem, the following augmented matrix [7] is constructed by replacing the last 2 rows of $[\mathbf{W}; \mathbf{E}]$ of (18) by the 2-row matrix $[\mathbf{U}; \boldsymbol{\lambda}]$

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{E}}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & ; & e_0 \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & ; & e_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{(N-2)0} & w_{(N-2)1} & w_{(N-2)2} & \cdots & w_{(n-2)N} & ; & e_{N-2} \\ 1 & 0 & 0 & \cdots & 0 & ; & \lambda_0 \\ 0 & 1 & 0 & \cdots & 0 & ; & \lambda_1 \end{bmatrix} \quad (22)$$

If $\det(\tilde{\mathbf{W}}) \neq 0$, then one can write

$$\tilde{\mathbf{X}} = (\tilde{\mathbf{W}})^{-1} \tilde{\mathbf{E}} \quad (23)$$

which yields the Taylor coefficients of the general solution; that is, the fundamental Eq. (2) with initial conditions (3a) and (3b) has a unique solution.

In case $\det(\tilde{\mathbf{W}}) = 0$, any other two rows of $[\mathbf{W}; \mathbf{E}]$ of (18) are replaced by the 2-row matrix $[\mathbf{U}; \boldsymbol{\lambda}]$ of (21) until the Taylor coefficients matrix $\tilde{\mathbf{X}}$ is yielded.

4. NUMERICAL APPLICATION

Consider an RLC circuit with $R=30\Omega$, $L=10\text{ H}$, and $C=0.02\text{ F}$. At time $t=0$, $I(0)=Q(0)=0$ and the circuit is connected to input voltage $E(t)=50\sin 2t\text{ V}$ [1]. The matrix operations in this section are performed by using MAPLE 13 software package [10].

Substituting the numerical values yields

$$10I''(t) + 30I'(t) + 50I(t) = 100\cos 2t$$

with initial conditions

$$I(0) = 0$$

$$I'(0) = \frac{E(0) - RI_0 - q_0/C}{L} = 0$$

First, we find a polynomial solution around the origin ($t=0$)

According to (12), taking $N=6$

$$\mathbf{E} = [100 \quad 0 \quad -200 \quad 0 \quad 66.6667 \quad 0 \quad -8.8889]^T$$

and according to (19)

$$\mathbf{W} = 10\mathbf{B}^2 + 30\mathbf{B} + 50\mathbf{I} = \begin{bmatrix} 50 & 30 & 20 & 0 & 0 & 0 & 0 \\ 0 & 50 & 60 & 60 & 0 & 0 & 0 \\ 0 & 0 & 50 & 90 & 120 & 0 & 0 \\ 0 & 0 & 0 & 50 & 120 & 200 & 0 \\ 0 & 0 & 0 & 0 & 50 & 150 & 300 \\ 0 & 0 & 0 & 0 & 0 & 50 & 180 \\ 0 & 0 & 0 & 0 & 0 & 0 & 50 \end{bmatrix}$$

One may also write (21) for the given boundary conditions

$$[\mathbf{U}; \boldsymbol{\lambda}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \end{bmatrix}$$

Therefore, the augmented matrix (22) becomes

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{E}}] = \begin{bmatrix} 50 & 30 & 20 & 0 & 0 & 0 & 0 & ; & 100 \\ 0 & 50 & 60 & 60 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 50 & 90 & 120 & 0 & 0 & ; & -200 \\ 0 & 0 & 0 & 50 & 120 & 200 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 50 & 150 & 300 & ; & 66.6667 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \end{bmatrix}$$

Performing the necessary matrix operations, the general solution is determined as

$$I(t) = 5t^2 - 5t^3 + 1.25t^5 - 0.42778t^6$$

in polynomial form.

The steady periodic solution is determined without inserting the initial conditions, as

$$I_{sp}(t) = -0.18317 + 2.61888t + 1.5926t^2 - 3.712t^3 + 0.47999t^4 + 0.64t^5 - 0.17778t^6$$

In order to obtain a solution in a interval sufficiently large to observe the solutions, $N=80$ is taken. The matrix operations are performed by Maple13 [10].

The exact solution of the problem is given as [1]

$$I(t) = \frac{-10}{37\sqrt{11}} e^{-3t/2} \left(\sqrt{11} \cos \frac{\sqrt{11}}{2} t + 27 \sin \frac{\sqrt{11}}{2} t \right) + \frac{10}{37} (\cos 2t + 6 \sin 2t).$$

Fig. 2 illustrated both the Taylor matrix solution and exact solution of the problem in the interval $0 \leq t \leq 9$, comparatively.

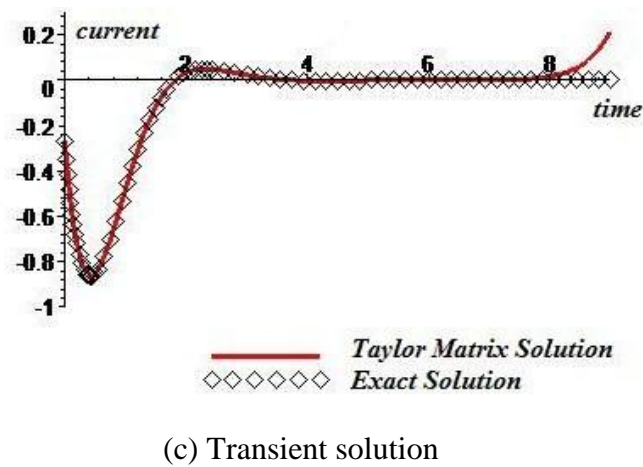
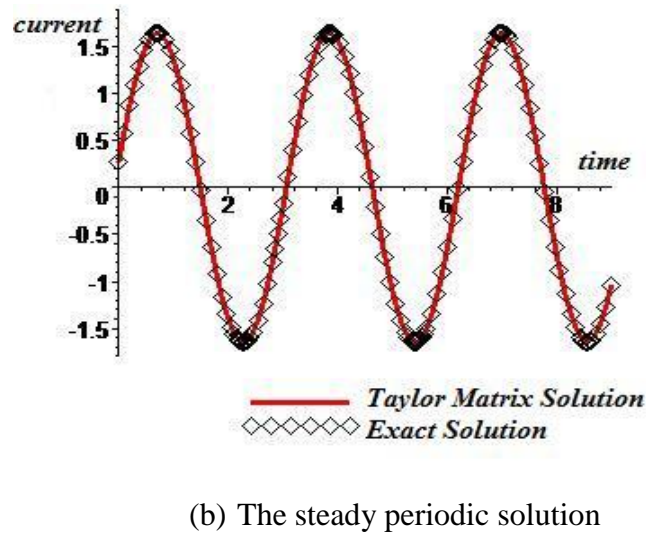
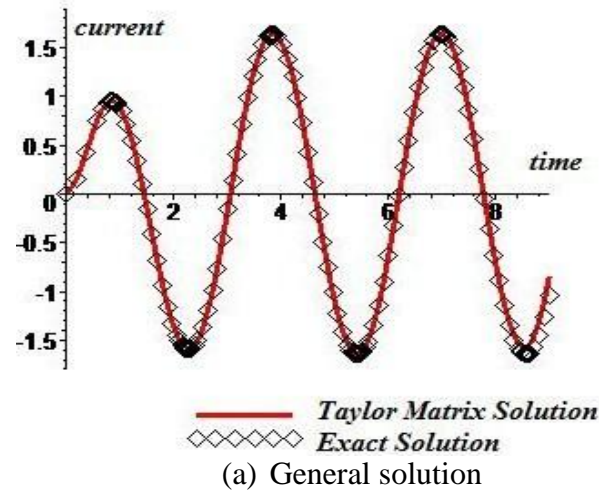


Figure 2. Time response of the system obtained by the Taylor matrix method and by the method of undetermined coefficients.

Polynomial solutions by Taylor matrix method diverges for values of t (time) greater than 9. The truncation limit should be increased to expand the solution interval and to have a better approximation.

In order to determine the value of the solution function at any arbitrary point other than zero, a very low truncation limit would be sufficient; that is, the result would be obtained with great ease. Table 1 shows the convergence of the results of Taylor solution to those of the exact solution, as N increases.

Table.1 Convergence of the Taylor results ($N=20, 40, 50$ and 100) to those of exact solution around $t=10$ (chosen arbitrarily)

Time (s)	Exact Solution	Taylor N=20	Taylor N=40	Taylor N=50	Taylor N=100
9.6	0.8105706	0.9495388	0.8127466	0.8105704	0.8105709
10.0	1.5907448	1.6687802	1.5896402	1.5907451	1.5907446
10.4	1.4059944	1.4316286	1.4043838	1.4059948	1.4059943
10.8	0.3683869	0.3670531	0.3673268	0.3683876	0.3683868
11.2	-0.8926788	-0.9015528	-0.8931118	-0.8926788	-0.8926790
11.6	-1.6122576	-1.6195311	-1.6123122	-1.6122578	-1.6122578
12.0	-1.3538624	-1.3574805	-1.3537801	-1.3538640	-1.3538628
12.4	-0.2742324	-0.2751778	-0.2741208	-0.2742562	-0.2742502
12.8	0.9717432	0.9720105	0.9718382	0.9716978	0.9717544

5. CONCLUSIONS

This paper presented a Taylor matrix method for solving the mathematical equation of the RLC circuits. This method uses orthogonal Taylor polynomials as basis functions and employs matrices to increase its competency by expanding up to any number of desired terms. Both steady periodic and general solutions of the system differential equation can be determined by this method. The results show a very good agreement with those of the exact solution. The main advantage of this method is that the solution can be obtained easily with symbolic computation software after writing an algorithm.

6. REFERENCES

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