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ON CHARACTERIZATION OF INEXTENSIBLE FLOWS OF CURVES ACCORDING TO TYPE-2 BISHOP FRAME E^3

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Abstract-In this paper, we study inextensible flows of curves according to type-2 Bishop frame in Euclidean 3-space. Necessary and sufficient conditions for an inextensible curve flow are expressed as a partial differential equation involving the curvature.

Key Words: Inextensible flows, Type-2 Bishop frame.

1. INTRODUCTION

The flow of a curve is called to be inextensible if the arc-length of a curve is preserved. Inextensible curve flows have growing importance in many applications such as engineering, computer vision, structural mechanics and computer animation [1-5]. The terms "inextensible" and "extensible" mostly come up in physics. There are inextensible and extensible collisions in physics. In extensible collision, both the kinetic energy and momentum are conserved. In inextensible collision, the kinetic energy is not conserved in the collision; however, the momentum is conserved. One of the oldest topics in the calculus of variations is the study of the elastic rod which, according to Daniel Bernoulli's idealization, minimizes total squared curvature among curves of the same length and first order boundary data. The classical term extensible refers to a curve in the plane or E^3 which represents such a rod in equilibrium.

Physically, inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example can be described by inextensible curve flows. Such motions arise quite naturally in a wide range of physical applications. For example, both Chirikjian and Burdick [6] and Mochiyama *et al*. [7] study the shape control of hyper-redundant, or snake-like, robots. Gage and Hamilton [8] and Grayson [9] investigated shrinking of closed plane curves to a circle via the heat equation. Kwon and Park [10] derived the evolution equation for an inextensible plane and space curve . Besides, Latifi, Razavi [11] studied inextensible flows of curves in Minkowskian space. In this paper, we study inextensible flows of curves according to type-2 Bishop frame in Euclidean 3-space E^3. We hope that these results will be helpful to mathematicians who are specialized on this area.

2. PRELIMINARIES

Let $\alpha : I \to E^3$ be an arbitrary curve in E^3. Recall that the curve α is said to be of unit speed if $\langle \alpha', \alpha' \rangle = 1$, where $\langle , \rangle$ is the standard scalar (inner) product of E^3 given by

$\langle x,y\rangle = x_1y_1 + x_2y_2 + x_3y_3$, for each $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)\in E^3$. Particularly, the norm of a vector $x\in E^3$ is given by $\|x\| = \sqrt{\langle x,x\rangle}$. Denote by $\{T(s),N(s),B(s)\}$ the moving Frenet frame along the unit speed curve α. Then the Frenet formulas are given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1}$$

$$\begin{aligned} \langle T,T\rangle &= \langle N,N\rangle = \langle B,B\rangle = 1, \\ \langle T,N\rangle &= \langle N,B\rangle = \langle B,T\rangle = 0. \end{aligned}$$

Here, T,N and B are the tangent, the principal normal and the binormal vector fields of the curves, respectively. $\kappa(s)$ and $\tau(s)$ are called, curvature and torsion of the curve α, respectively.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame.

Let $\alpha=\alpha(s)$ be a unit speed regular curve in E^3. The type-2 Bishop frame of the $\alpha(s)$ is defined by [12,13]

$$\begin{bmatrix} N_1' \\ N_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_1 \\ 0 & 0 & -k_2 \\ k_1 & k_2 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}. \tag{2}$$

The relation matrix between Frenet-Serret and type-2 Bishop frames can be expressed

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}. \tag{3}$$

Here, the type-2 Bishop curvatures are defined by

$$k_1(s) = -\tau\cos\theta(s), k_2(s) = -\tau\sin\theta(s). \tag{4}$$

It can be also deduced as

$$\theta' = \kappa = \frac{\left(\dfrac{k_2}{k_1}\right)'}{1+\left(\dfrac{k_2}{k_1}\right)^2}. \tag{5}$$

The frame $\{N_1, N_2, B\}$ is properly oriented, and τ and $\theta(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve $\alpha = \alpha(s)$. We shall call the set $\{N_1, N_2, B, k_1, k_2\}$ as type-2 Bishop invariants of the curve $\alpha = \alpha(s)$.

3. INEXTENSIBLE FLOWS OF CURVES ACCORDING TO TYPE-2 BISHOP FRAME E^3

We assume that $F:[0,l]\times[0,w] \to E^3$ is a one parameter family of smooth curve in Euclidean space E^3, where l is the arc-length of initial curve. Let u be the curve parametrization variable, $0 \le u \le l$. The arc-length of F is given by

$$s(u) = \int_0^u \left|\frac{\partial F}{\partial u}\right| du \tag{6}$$

where

$$\left|\frac{\partial F}{\partial u}\right| = \left|\left\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u}\right\rangle\right|^{\frac{1}{2}}. \tag{7}$$

The operator of $\dfrac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u},$$

where $v = \left|\dfrac{\partial F}{\partial u}\right|$. The arc-length parameter is $\partial s = v \partial u$.

Any flow of F can be given by

$$\frac{\partial F}{\partial t} = fN_1 + gN_2 + hB, \tag{8}$$

where f,g,h are tangential, principal normal, binormal speeds of the curve in E^3, respectively. We put $s(u,t)=\int_0^u v\,du$, which is called the arc-length variation of curve F. From this, the requirement that the curve not be subject to any elongation or compression can be expressed by condition

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}du=0, \tag{9}$$

for all $u\in[0,l]$.

Definition 3.1. *A curve evolution* $F(u,t)$ *and its flow* $\frac{\partial F}{\partial t}$ *in Euclidean 3- space* E^3 *are said to be inextensible if*

$$\frac{\partial}{\partial t}\left|\frac{\partial F}{\partial u}\right|=0.$$

Theorem 3.1. (*Necessary and Sufficient Conditions for an Inextensible Flow*) *Let* $\frac{\partial F}{\partial t}=fN_1+gN_2+hB$ *be a smooth flow of* F *in* E^3. *The flow is inextensible if and only if*

$$\frac{\partial v}{\partial t}=\frac{\partial f}{\partial u}+hvk_1. \tag{10}$$

Proof. Suppose that $\frac{\partial F}{\partial u}$ be a smooth flow of the curve F. Using definition of F, we get

$$v^2=\left\langle\frac{\partial F}{\partial u},\frac{\partial F}{\partial u}\right\rangle. \tag{11}$$

Since u and t are independent coordinates, $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute. So, by differentiating the formula (11) with respect to t, we have

$$2v\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}\left\langle\frac{\partial F}{\partial u},\frac{\partial F}{\partial u}\right\rangle.$$

On the other hand, changing $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$, we get

$$v\frac{\partial v}{\partial t}=\left\langle \frac{\partial F}{\partial u},\frac{\partial}{\partial u}(\frac{\partial F}{\partial t})\right\rangle .$$

From (8), we obtain

$$v\frac{\partial v}{\partial t}=\left\langle \frac{\partial F}{\partial u},\frac{\partial}{\partial u}(fN_1+gN_2+hB)\right\rangle .$$

By using type-2 Bishop frame, we get

$$\frac{\partial v}{\partial t}=\left\langle T,(\frac{\partial f}{\partial u}+hvk_1)N_1+(\frac{\partial g}{\partial u}+hvk_2)N_2+(\frac{\partial h}{\partial u}-fvk_1-gvk_2)N_2\right\rangle .$$

If N_1 is taken instead of T

$$\frac{\partial v}{\partial t}=\left\langle N_1,(\frac{\partial f}{\partial u}+hvk_1)N_1+(\frac{\partial g}{\partial u}+hvk_2)N_2+(\frac{\partial h}{\partial u}-fvk_1-gvk_2)N_2\right\rangle$$

by using features of inner product and after straightforward calculations from above equation, we get

$$\frac{\partial v}{\partial t}=\frac{\partial f}{\partial u}+hvk_1 .$$

Theorem 3.2. *Let* $\frac{\partial F}{\partial t}=fN_1+gN_2+hB$ *be a smooth flow of* F *in* E^3. *The flow is inextensible if and only if*

$$\frac{\partial f}{\partial s}=-hk_1 . \tag{12}$$

Proof. Now let $\frac{\partial F}{\partial u}$ be extensible. From (9), we have

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u\frac{\partial v}{\partial t}du=\int_0^u\left(\frac{\partial f}{\partial u}+hvk_1\right)du=0. \tag{13}$$

$\forall u\in[0,I]$. Substituting (10) in (13) complete the proof of the theorem.

We now restrict ourselves to arc length parametrized curves. That is, $v=1$ and the local coordinate u corresponds to the curve arc-length s. We require the following lemma.

Lemma 3.1. *Let* $\dfrac{\partial F}{\partial t}= fN_1+gN_2+hB$ *be a smooth flow of* F *in* E^3. *Then,*

$$\frac{\partial N_1}{\partial t}=(\frac{\partial g}{\partial s}+hk_2)N_2+(\frac{\partial h}{\partial s}-fk_1-gk_2)B$$

$$\frac{\partial N_2}{\partial t}=-(\frac{\partial g}{\partial s}+hk_2)N_1+\psi B \tag{14}$$

$$\frac{\partial B}{\partial t}=(-\frac{\partial h}{\partial s}+fk_1+gk_2)N_1-\psi N_2$$

where $\psi=\left\langle \dfrac{\partial N_2}{\partial t},B\right\rangle$.

Proof. Using definition of F, we have

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial t}\frac{\partial F}{\partial s}=\frac{\partial}{\partial s}(fN_1+gN_2+hB).$$

Using the type-2 Bishop frame and after straightforward calculations, we get

$$\frac{\partial T}{\partial t}=(\frac{\partial f}{\partial s}+hk_1)N_1+(\frac{\partial g}{\partial s}+hk_2)N_2+(\frac{\partial h}{\partial s}-fk_1-gk_2)B. \tag{15}$$

Substituting (12) in (15) and If N_1 take instead of T, we have

$$\frac{\partial N_1}{\partial t}=(\frac{\partial g}{\partial s}+hk_2)N_2+(\frac{\partial h}{\partial s}-fk_1-gk_2)B.$$

Now differentiate the 2-type Bishop frame by t, we obtain

$$0=\frac{\partial}{\partial t}\langle N_1,N_2\rangle=\left\langle \frac{\partial N_1}{\partial t},N_2\right\rangle+\left\langle N_1,\frac{\partial N_2}{\partial t}\right\rangle=\frac{\partial g}{\partial s}+hk_2+\left\langle N_1,\frac{\partial N_2}{\partial t}\right\rangle,$$

$$0=\frac{\partial}{\partial t}\langle N_1,B\rangle=\left\langle \frac{\partial N_1}{\partial t},B\right\rangle+\left\langle N_1,\frac{\partial B}{\partial t}\right\rangle=\frac{\partial h}{\partial s}-fk_1-gk_2+\left\langle N_1,\frac{\partial B}{\partial t}\right\rangle,$$

$$0=\frac{\partial}{\partial t}\langle N_2,B\rangle=\left\langle\frac{\partial N_2}{\partial t},B\right\rangle+\left\langle N_2,\frac{\partial B}{\partial t}\right\rangle=\psi+\left\langle N_2,\frac{\partial B}{\partial t}\right\rangle.$$

From the above and using $\left\langle\frac{\partial N_2}{\partial t},N_2\right\rangle=\left\langle\frac{\partial B}{\partial t},B\right\rangle=0,$

we obtain

$$\begin{aligned}\frac{\partial N_2}{\partial t}&=-(\frac{\partial g}{\partial s}+hk_2)N_1+\psi B\\ \frac{\partial B}{\partial t}&=(-\frac{\partial h}{\partial s}+fk_1+gk_2)N_1-\psi N_2.\end{aligned}$$

where $\psi=\left\langle\frac{\partial N_2}{\partial t},B\right\rangle.$

The following theorem states the conditions on the curvature and the torsion for the curve flow $F(u,t)$ to be inextensible.

Theorem 3.3. *(Equations for Inextensible Evolution) If the curve flow* $\frac{\partial F}{\partial t}=fN_1+gN_2+hB$ *is inextensible, then the following system of partial differential equations holds:*

$$\frac{\partial k_2}{\partial t}=\frac{\partial^2 g}{\partial s^2}-\frac{\partial}{\partial s}(hk_2)-\psi k_1.$$

Proof. Using (14), we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial N_1}{\partial t}&=\frac{\partial}{\partial s}\left[(\frac{\partial g}{\partial s}+hk_2)N_2+(\frac{\partial h}{\partial s}-fk_1-gk_2)B\right]\\ &=\left(\frac{\partial^2 g}{\partial s^2}+\frac{\partial}{\partial s}(hk_2)\right)N_2+(\frac{\partial g}{\partial s}+hk_2)(-k_2B)+\left(\frac{\partial^2 h}{\partial s^2}+\frac{\partial}{\partial s}(-fk_1-gk_2)\right)B+(\frac{\partial h}{\partial s}-fk_1-gk_2)(k_1N_1+k_2N_2).\end{aligned}$$

On the other hand, from type-2 Bishop frame we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial N_1}{\partial s}&=\frac{\partial}{\partial t}(-k_1B)\\ &=-\frac{\partial k_1}{\partial t}B-k_1\left(\left(-\frac{\partial h}{\partial s}+fk_1+gk_2\right)N_1-\psi N_2\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial B}{\partial t} &= \frac{\partial}{\partial s}\left[(-\frac{\partial h}{\partial s}+fk_1+gk_2)N_1-\psi N_2\right]\\ &=\left(-\frac{\partial^2 h}{\partial s^2}+\frac{\partial}{\partial s}(fk_1+gk_2)\right)N_1+(-\frac{\partial h}{\partial s}+fk_1+gk_2)\frac{\partial N_1}{\partial s}-\frac{\partial\psi}{\partial s}N_2+\psi\frac{\partial N_2}{\partial s}.\end{aligned} \tag{16}$$

Substituting (14) in (16) and after straightforward calculations, we get

$$\frac{\partial}{\partial s}\frac{\partial B}{\partial t}=\left(-\frac{\partial^2 h}{\partial s^2}+\frac{\partial}{\partial s}(fk_1+gk_2)\right)N_1-\frac{\partial\psi}{\partial s}N_2+(\frac{\partial h}{\partial s}k_1-fk_1^2-gk_1k_2-\psi k_2).$$

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial B}{\partial s} &= \frac{\partial}{\partial t}(k_1N_1+k_2N_2)\\ &=\frac{\partial k_1}{\partial t}N_1+k_1\left((\frac{\partial g}{\partial s}+hk_2)N_2+(\frac{\partial h}{\partial s}-fk_1-gk_2)B\right)+\frac{\partial k_2}{\partial t}N_2+k_2\left(-(\frac{\partial g}{\partial s}+hk_2)N_1+\psi B\right)\end{aligned}$$

After straightforward calculations, we get

$$\frac{\partial}{\partial t}\frac{\partial B}{\partial s}=\left(\frac{\partial k_1}{\partial t}-\frac{\partial g}{\partial s}k_2-hk_2^2\right)N_1+\left(\frac{\partial g}{\partial s}k_1+hk_1k_2+\frac{\partial k_2}{\partial t}\right)N_2+\left(\frac{\partial h}{\partial s}-fk_1-gk_2+\psi k_2\right)B.$$

Similarly

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial N_2}{\partial t} &= \frac{\partial}{\partial s}\left[-(\frac{\partial g}{\partial s}+hk_2)N_1+\psi B\right]\\ &=\left(-\frac{\partial^2 g}{\partial s^2}+\frac{\partial}{\partial s}(hk_2)\right)N_1+(-\frac{\partial g}{\partial s}+hk_2)\frac{\partial N_1}{\partial s}+\frac{\partial\psi}{\partial s}B+\psi\frac{\partial B}{\partial s}.\end{aligned} \tag{17}$$

Substituting (14) in (17) and after straightforward calculations, we get

$$\frac{\partial}{\partial s}\frac{\partial N_2}{\partial t}=\left(-\frac{\partial^2 g}{\partial s^2}+\frac{\partial}{\partial s}(hk_2)+\psi k_1\right)N_1-\psi k_2N_2+(\frac{\partial g}{\partial s}k_1+hk_1k_2+\frac{\partial\psi}{\partial s})B.$$

$$\frac{\partial}{\partial t}\frac{\partial N_2}{\partial s} = \frac{\partial}{\partial t}(-k_2 B)$$
$$= -\frac{\partial k_2}{\partial t}B - k_2\left((-\frac{\partial h}{\partial s} + fk_1 + gk_2)N_1 - \psi N_2\right).$$

Thus, we have

$$\frac{\partial k_2}{\partial t} = \frac{\partial^2 g}{\partial s^2} - \frac{\partial}{\partial s}(hk_2) - \psi k_1.$$

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