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ANALYTICAL SOLUTIONS OF NON-LINEAR EQUATIONS OF POWER-LAW FLUIDS OF SECOND GRADE OVER AN INFINITE POROUS PLATE

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Abstract- The flow of an incompressible fluid of modified second grade past an infinite porous plate subject to either suction or blowing at the plate is studied. The model is a combination of power-law and second grade fluid in which the fluid may exhibit normal stresses, shear thinning or shear thickening behaviors. Equations of motion in dimensionless form are derived. Analytical solutions of the outcoming non-linear differential equations are found by using the homotopy analysis method (HAM), which is a powerful semi-analytical method. Effects of power-law index and second grade coefficient on the boundary layers are shown and solutions are contrasted with the usual second grade fluid solutions.

Key Words- Homotopy Analysis Method, Boundary Layer, Porous Plate, Non-Newtonian Fluids

1.INTRODUCTION

The flow of an incompressible non-Newtonian fluid has important industrial applications, for example in the extrusion of a polymer sheet from a die or in the drawing of plastic films. During the manufacture of these sheets, the melt issues from a slit and is subsequently stretched to achieve the desired thickness. Material manufactured by extrusion processes and heat-treated materials traveling between a feed roll and wind-up roll or on conveyor belts possesses the characteristics of a moving continuous surface. The mechanical properties of the final product strictly depend on the stretching rate and on the rate of cooling in this process. Non-Newtonian fluid mechanics afford an excellent opportunity for studying many of the mathematical techniques which have been developed to solutions of non-linear equations. Several models have been proposed to explain the non-Newtonian behavior of fluids. Among these, the power-law, differential-type and rate-type models gained much acceptance. Boundary layer assumptions were successfully applied to these models and much work has been done on them. Power-law fluids are by far the most widely used model to express non-Newtonian behavior in fluids. The model predicts shear thinning and shear thickening behavior. Normal stress effects can be expressed in second grade fluid model, a special type of Rivlin-Ericksen fluids, but this model is incapable in representing shear thinning/thickening behavior. A fluid model which exhibits all behaviors is deserved and [1] and [2] proposed two models which they called “the power law fluid of grade 2” and “modified second order (grade) fluid”. These models were actually slight modifications of a usual second grade fluid. The below power-law fluid of second grade model is considered in this work

$$\mathbf{T}^* = -p^*\mathbf{I} + \mathbf{\Pi}^{m/2}(\mu\mathbf{A}_1^* + \alpha_1\mathbf{A}_2^* + \alpha_2\mathbf{A}_1^{*2}) \tag{1}$$

The other model proposed is the modified second grade fluid

$$\mathbf{T}^* = -p^*\mathbf{I} + \mu\mathbf{\Pi}^{m/2}\mathbf{A}_1^* + \alpha_1\mathbf{A}_2^* + \alpha_2\mathbf{A}_1^{*2} \tag{2}$$

where $\mathbf{T}^*$ is the Cauchy stress tensor, p^* is the pressure, $\mathbf{I}$ is the identity matrix, $\mathbf{A_1}^*$ and $\mathbf{A_2}^*$are the first and second Rivlin-Ericksen tensors respectively, μ, m, α_1 and α_2 are material moduli that may be constants or depend on temperature. For both models, when m=0, $\alpha_1=\alpha_2=0$, the fluid is Newtonian and hence μ represents the usual viscosity. m=0 corresponds to the second grade fluid, $\alpha_1=\alpha_2=0$ corresponds to the power-law fluid. The tensors are defined as

$$\mathbf{A}_1^* = \mathbf{L}^* + \mathbf{L}^{*T}, \quad \mathbf{A}_2^* = \frac{d\mathbf{A}_1^*}{dt^*} + \mathbf{A}_1^*\mathbf{L}^* + \mathbf{L}^{*T}\mathbf{A}_1^*, \quad \Pi = \frac{1}{2}\mathrm{tr}(\mathbf{A}_1^{*2}), \quad \mathbf{L}^* = \nabla^*\mathbf{v}^* \tag{3}$$

where $\mathbf{v}^*$is the velocity vector. The stars over the symbols indicate that the quantities are dimensional. Models (1) and (2) satisfy the principle of material frame-indifference. Man and Sun [1] first proposed the constitutive relations (1) and (2). Later Man [2] considered the unsteady channel flow of model (2) and existence, uniqueness and asymptotic stability of the solutions are exploited. Franchi and Straughan [3] presented a stability analysis of the modified model for a special viscosity function which depends linearly on the temperature. Gupta and Massoudi [4] investigated the flow of this fluid with temperature dependent viscosity between heated plates. Massoudi and Phuoc [5] studied the flow down a heated inclined plane. Massoudi and Phuoc [6] analyzed the pipe flow with Reynolds temperature dependent viscosity model. Aksoy et al. [7] derived the two dimensional equations of motion as well as boundary layer equations for the model. Stretching sheet problem is considered for the boundary layer equations. Detailed thermodynamic and stability analysis exist for second grade [8] and third grade [9] fluids. Dunn and Rajagopal [10] presented a critical review and thermodynamic analysis for fluids of differential type including the model considered here. Many issues regarding the applicability of such non-Newtonian models to real fluids, thermodynamic restrictions imposed on the constitutive equations and doubts raised in the previous literature on these models were addressed in detail. Flow over a porous plate was considered previously for closely related models. Contrary to the Newtonian flow which does not permit solutions for the blowing case, [11] showed that solutions exist both for the suction and blowing case for a second grade fluid if material parameters meet certain criteria. Rajagopal and Szeri [12] studied a third grade fluid past a porous plate. Regular perturbation solutions up to arbitrary orders of approximation, as well as numerical solutions were presented for the problem. Maneschy et al. [13] extended the numerical solutions of [12] by considering heat transfer also. A close study to this work is due to [14]. They solved the porous plate problem using homotopy analysis method and presented analytical solutions for the integer values of power law index (i.e. m=1,2,3). Symmetry analysis for the boundary layer equations of the modified second grade fluid has been presented much recently [15]. Pakdemirli et al. [16] solved the porous plate problem using perturbation method and presented analytical and numerical solutions of model (2).

In this study, the flow of an incompressible fluid of modified second grade fluid past a porous plate is governed by a non-linear ordinary differential equation in a reasonably simple structure. The analytical solution of the ordinary differential equation has been found by HAM [17,18]. It is shown that HAM solutions agree very well with the numerical solutions. Effect of power- law index *m*, suction and injection parameter v_0, power-law parameter ε, and the second grade coefficient ε_1 on the solutions are investigated. The numerical solutions of the ordinary differential equations have been computed by a collocation method, yielding a high degree of accuracy. In numerical computations, Matlab package sbvp4c, which uses the three-stage Lobatto formula, is employed. The formula is implemented as an implicit Runge-Kutta formula.

2. EQUATIONS OF MOTION

The non-dimensional form of the equations of motion of a modified second grade fluid over an infinite porous plate was derived by [7] and [16]

$$v_0 \frac{du}{dy} + \varepsilon(m+1)\left(\frac{du}{dy}\right)^m \frac{d^2u}{dy^2} - \varepsilon_1 v_0 m \left(\frac{du}{dy}\right)^{m-1}\left(\frac{d^2u}{dy^2}\right)^2 - \varepsilon_1 v_0 \left(\frac{du}{dy}\right)^m \frac{d^3u}{dy^3} = 0 \tag{4}$$

The boundary conditions for the problem are

$$u(0) = 0, \quad u(\infty) = 1, \quad \frac{du}{dy}(\infty) = 0 \tag{5}$$

The last condition implies that there is no shear stress at infinity. Detailed discussions of the boundary conditions can be seen at reference [12]. The equation of the flow of power-law fluid of second grade over an infinite porous plate is derived first in this paper. This non-linear equation is original, both analytical and numerical solutions will be given in this paper.

3. ANALYTIC SOLUTIONS BY THE HOMOTOPY ANALYSIS METHOD

For real power-law index $m(|m|<1)$, the function *u(y)* can be expressed by a set of base functions

$$\{y^k \exp(-n\lambda y) \mid k \ge 0, n \ge 0\} \tag{6}$$

in the form

$$u(y) = \sum_{k,n=0}^{+\infty} u_{k,n} y^k \exp(-n\lambda y) \tag{7}$$

where $u_{k,n}$ are the coefficients to be determined and we have freedom to choose a positive value for λ. Now from the above expression and the boundary conditions (5), it is straightforward to choose $u_0(y) = 1 - \exp(-\lambda y)$ as the initial guess of $u(y)$. According to (7) and the governing equation (4), we choose the auxiliary linear operator

$$L[\phi(y;p)] = \frac{\partial^3 \phi(y;p)}{\partial y^3} - \lambda^2 \frac{\partial \phi(y;p)}{\partial y} \tag{8}$$

with the property, $L[C_1 + C_2 \exp(-\lambda y) + C_3 \exp(\lambda y)]$ and where p is an embedding parameter. From (4), we define the nonlinear operator

$$N[\phi(y;p)] = v_0 \left[\frac{\partial \phi(y;p)}{\partial y}\right]^2 + \varepsilon(m+1)\left[\frac{\partial \phi(y;p)}{\partial y}\right]^{mp+1} \frac{\partial^2 \phi(y;p)}{\partial y^2}$$
$$- \varepsilon_1 v_0 m \left[\frac{\partial \phi(y;p)}{\partial y}\right]^{mp} \left[\frac{\partial^2 \phi(y;p)}{\partial y^2}\right]^2 - \varepsilon_1 v_0 \left[\frac{\partial \phi(y;p)}{\partial y}\right]^{mp+1} \frac{\partial^3 \phi(y;p)}{\partial y^3}$$

Like in [19,20]. Using the above definition, we construct the zero-order deformation equation

$$(1-p)L[\phi(y;p) - u_0(y)] = p\hbar N[\phi(y;p)]$$

subject to the boundary conditions

$$\varphi(0;p) = 0, \ \ \phi(+\infty;p) = 1, \ \ \left.\frac{\partial \phi(y;p)}{\partial y}\right|_{y=+\infty} = 0 \tag{9}$$

Therefore, as the embedding parameter p increases from 0 to 1, $\phi(y;p)$ varies from the initial guess $u_0(y)$ to the solution $u(y)$. Expanding $\phi(y;p)$ in Taylor series with respect to p, one has

$$\phi(y;p) = u_0(y) + \sum_{n=1}^{+\infty} u_n(y) p^n$$

where $u_n(y) = \frac{1}{n!} \left.\frac{\partial^n \phi(y;p)}{\partial p^n}\right|_{p=0}$ If the auxiliary linear operator, the initial guess, and the auxiliary parameter $\hbar$ are so properly chosen, the above series converges at $p=1$, and one has

$$u(y) = u_0(y) + \sum_{n=1}^{+\infty} u_n(y) \tag{10}$$

which must be one of the solutions of the original nonlinear equation (4), as proved by Liao [16].

Differentiating the zero-order deformation equation and boundary conditions (9) n times with respect to p and dividing them by $n!$ and finally setting $p = 0$, we have the nth-order deformation equation

$$L[u_n(y) - \chi_n u_{n-1}(y)] = \hbar R_n(y) \tag{11}$$

subject to the boundary conditions

$$u_n(0), \ u_n(+\infty) = 0, \ u_n'(+\infty) = 0 \tag{12}$$

where $R_n(y) = \frac{1}{(n-1)!} \frac{\partial^{n-1} N[\phi(y;p)]}{\partial p^{n-1}}\Big|_{p=0}$ and $\chi_n = \begin{cases} 0, & n \le 1 \\ 1, & n > 1 \end{cases}$

It can be found that

$$R_n(y) = v_0 \sum_{i=0}^{n-1} u_i'(y) u_{n-i-1}'(y) + \sum_{i=0}^{n-1} w_i(y) z_{n-i-1}(y)$$

where

$$z_k(y) = \sum_{j=0}^{k-1} \left[\varepsilon(m+1) u_j'(y) u_{k-j-1}''(y) - \varepsilon_1 v_0 m u_j''(y) u_{k-j-1}''(y) - \varepsilon_1 v_0 u_j'(y) u_{k-j-1}'''(y) \right]$$

and

$$w_k(y) = \frac{1}{k!} \left(\frac{\partial^k}{\partial p^k} \left[\frac{\partial \phi(y;p)}{\partial y} \right]^{mp} \right)_{p=0}$$

It can be verified that

$$w_0(y) = 1$$

$$w_1(y) = m \ln(u_0'(y))$$

$$w_2(y) = \frac{m u_1'(y)}{u_0'(y)} + \frac{m^2}{2} (\ln(u_0'(y)))^2$$

and so on, where $\ln(u_0'(y)) = \ln(\lambda) - \lambda y$ Now, the solution of the deformation Eq. (11) for n≥1 becomes

$$u_n(y) = \hat{u}_n(y) + C_1 + C_2 e^{-\lambda y} + C_3 e\lambda y$$

where $\hat{u}_n(y)$ is a special solution of Eq. (11) and the coefficients C_1, C_2 and C_3 are determined by the boundary conditions (12). Obviously, $C_3 = 0$ and the unknowns C_1 and C_2 are obtained by

$$\hat{u}_n(0) + C_1 + C_2 = 0, \quad \hat{u}_n(+\infty) + C_1 = 0$$

At the *N*th-order approximation, the analytic solution is $u(y) \approx U_N(y) = \sum_{n=0}^{N} u_n(y)$. The convergence of the solution also depends on the choice of the auxiliary parameters $\hbar$ and λ. It can be found that, we have

$$U_0(y) = 1 - e^{-\lambda y}$$

$$U_1(y) = 1 - e^{-\lambda y}\left(1 - \frac{\hbar v_0 y}{2\lambda} + \frac{\hbar\varepsilon(1+m)y}{2} + \frac{\hbar\lambda v_0 \varepsilon_1 (m+3y)}{6}\right) + e^{-2\lambda y}\frac{\hbar\lambda v_0 \varepsilon_1 m}{6}$$

and so on. In this work, we find the best values of $\hbar$ and λ by minimizing the exact square residual error of Eq. (4) at the *N*th-order. This quantity is given by

$$\hat{E}_N(\hbar,\lambda) = \int_0^{+\infty} \left(N[U_N(y)]\right)^2 dy \tag{13}$$

In practice the evaluation of $\hat{E}_N(\hbar,\lambda)$ tends to be time-consuming. A simpler way is calculating the averaged square residual error, which is the discretization of Eq. (13), i.e.,

$$E_N(\hbar,\lambda) = \frac{1}{1+L}\sum_{i=0}^{L}(N[U_N(y_i)])^2$$

where $y_i = i\Delta y = \frac{i}{10}$ for i=0,1,2,..L. Hereafter, a value of L0100 will be used for the purpose of optimization.

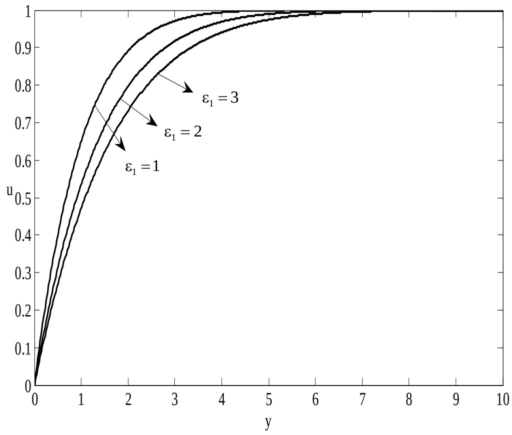

Figure 1. Effect of second grade parameter on the velocity profiles ($\in$=0.1, v_0=1, m=0.2)

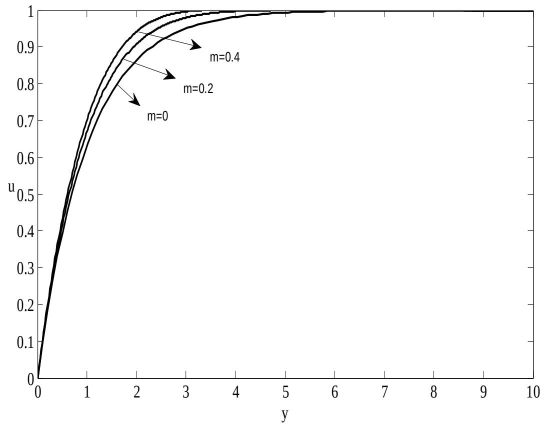

Figure 2. Effect of power-law parameter on the velocity profiles ($\in_1$=0.1, v_0=1, m=0.2)

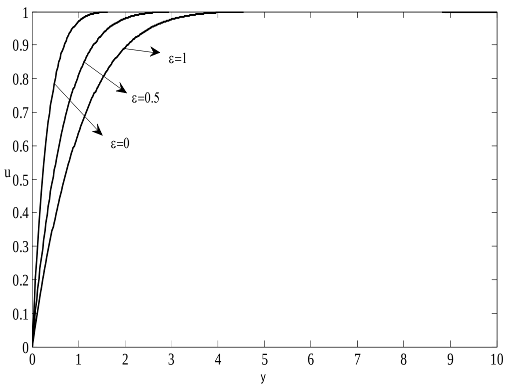

Figure 3. Effect of power-law index (m>0) on the velocity profiles ($\in$=0.5,$\in_1$=0.5, v_0=1)

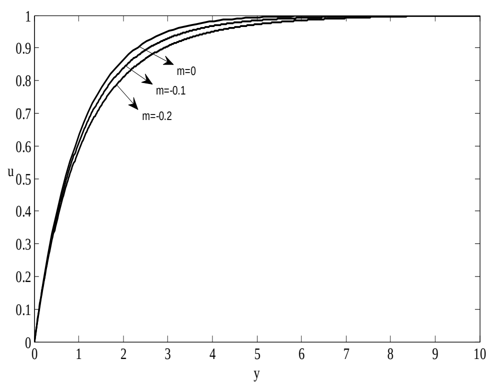

Figure 4. Effect of power-law index (m<0) on the velocity profiles ($\in$=0.5,$\in_1$=0.5, v_0=1)

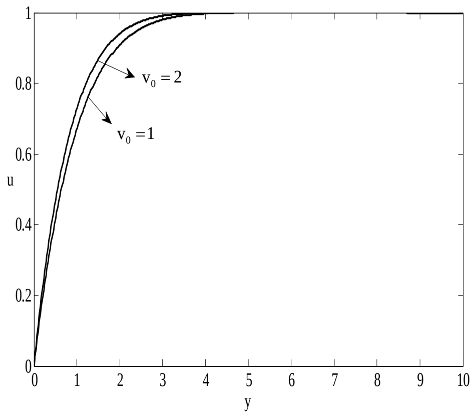

Figure 5. Effect of suction on the velocity profiles ($\in$=0.5,$\in_1$=0.5, m=0.2)

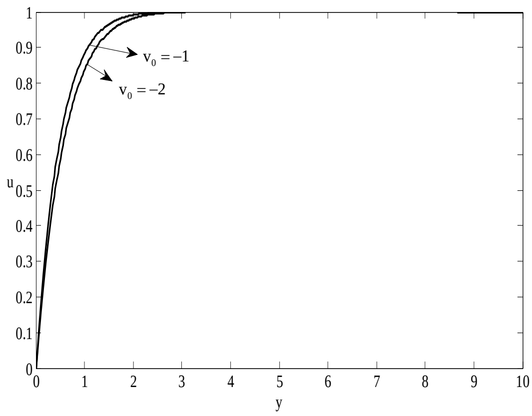

Figure 6. Effect of injection on the velocity profiles ($\in$=0.5,$\in_1$=0.5, m=0.2)

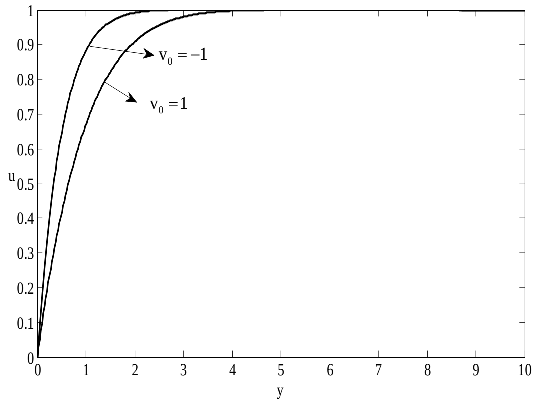

Figure 7. Comparison of effect of suction with effect of injection ($\in_1$=0.5, $\in$=0.5, m=0.2)

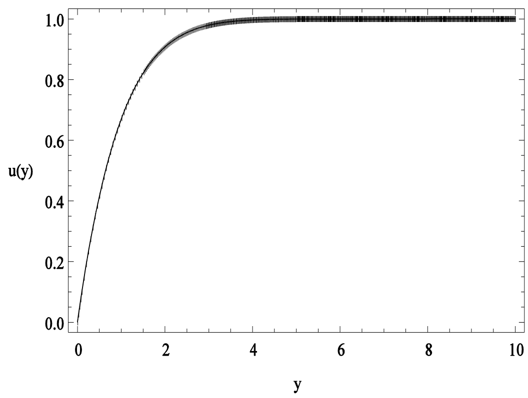

Figure 8. Curve of u(y) function versus y solid curve 10^{th} order approximation by the HAM; dotted curve, numerical solution. ($\in_1$=0.5, $\in$=0.5, m=0.2, v_0=1)

Table 1. Numerical values of $u'(0)$ various values of *m* for $\varepsilon = 0.5,\ \varepsilon_1 = 0.5,\ v_0 = 1$ for 10^{th}-order of HAM and numerical solutions (NS).

m	u′(0) (HAM)	u′(0) (NS)	$\hbar$	λ	$E_{10}(\hbar,\lambda)$
0	0	1.000068104928405	1	1	0
0.2	1.038041914561423	1.03718602225475	0.779142016163386	0.9	0.0000209693
0.4	1.079056757660077	1.063032519144386	0.477351325409284	1.0	0.000466182
-0.1	0.976527350140495	0.97652069624041	3.525371697615499	0.8	0.0000404276
-0.2	0.947327019989428	0.95094176915943	2.8518296424699128	0.6	0.0000678953

Table 2. Numerical values of $u'(0)$ various values of v_0 for $\varepsilon = 0.5,\ \varepsilon_1 = 0.5,\ m = 0.2$ for 10^{th}-order of HAM and numerical solutions (NS).

v_0	u′(0) (HAM)	u′(0) (NS)	$\hbar$	λ	$E_{10}(\hbar,\lambda)$
1	1.038041914561423	1.037186022254751	0.779142016163386	0.9	0.0000209693
2	1.2144122253521075	1.213581607070671	0.4139964141665231	1.0	0.000029861
-1	1.9955806355423007	1.995516150900364	-0.955595591158211	1.9	0.0000046345
-2	1.6924298093780985	1.692076455206649	-0.44670043958587	1.5	0.0000116429

Table 3. Numerical values of $u'(0)$ various values of $\varepsilon$for $v_0 = 1,\ \varepsilon_1 = 0.1,\ m = 0.2$ for 10^{th}-order of HAM and numerical solutions (NS).

ε	u′(0) (HAM)	u′(0) (NS)	$\hbar$	λ	$E_{10}(\hbar,\lambda)$
0	2.9747911333495445	2.974134198386145	4.34017164092326	1.6	0.0000126613
0.5	1.4762408475845579	1.476246655151934	2.28956699726366	1.3	0.0000445683
1	0.9395121585016436	0.938516553007344	1.484524892746743	1.2	0.0000747265

Table 4. Numerical values of $u'(0)$ various values of ε_1 for $v_0 = 1,\ \varepsilon = 0.1,\ m = 0.2$ for 10^{th}-order of HAM and numerical solutions (NS).

ε_1	u′(0) (HAM)	u′(0) (NS)	$\hbar$	λ	$E_{10}(\hbar,\lambda)$
1	0.9987352577608877	0.997646830829734	0.5971944235837703	0.9	0.0000191029
2	0.7394190129269838	0.738555386449868	0.2523587244495018	0.7	0.0000208333
3	0.6189382568222634	0.618170243930872	0.1652479590863146	0.55	0.0000220588

Effect of second grade parameter on the velocity profiles is depicted in Figure 1. As the second grade effect increases, the boundary layer thickens. A similar trend, as dominant in the previous case, is also observed for the power-law parameter (ε) in Figure 2. An increase in this coefficient results in a decrease in velocity. The boundary layer thickens as ε increases. In Figure 3, power law index m has an appreciable influence on the velocity profiles (i.e. m > 0). The velocity is observed to increase with increasing m. In the Figures, the data is contrasted with the usual second grade fluid (i.e. m = 0). For the shear thinning values (i.e. m < 0), the velocity profiles are given in Figure 4. The velocity is observed to decrease with decreasing m. The boundary layer thickness decreases a big amount when suction increases (See Fig. 5). A reverse effect is observed when injection increases (See Fig. 6). For m = 0.2, v_0 =1 that means suction boundary condition and v_0 =-1 that means injection are contrasted in Figure 7. It is observed that the boundary layer thickness for v_0 =1 is much thicker than the boundary layer thickness for v_0 =-1. Figure 8 shows finite difference solution and HAM solution obtained for $\in_1$=0.5,$\in$=0.5, m=0.2, v_0=1. A good agreement is observed between the methods. Tables 1- 4 show the HAM solutions and numerical solutions of $u'(0)$ for various values of m, ϵ, v_0 and ε_1. In these numerical computations, Matlab package sbvp4c, which uses the three-stage Lobatto formula, is employed.

4. CONCLUDING REMARKS

In this paper, HAM is employed to obtain the approximate analytic solutions for non-linear differential equations in engineering. The behavior of a non-Newtonian modified second-grade fluid past a porous plate was examined. Two dimensional equations of motion are derived for a power-law fluid of second grade which can exhibit shear thinning/thickening behavior as well as normal stresses. For m=0, the equations reduce to those of second grade fluid and for ε_1= ε_2=0, the equations reduce to those of power-law fluid. For the special case of flow over a porous plate, the velocity components are assumed to be dependent only on y coordinate. With this assumption, equation of motion is obtained. An analytical method (HAM) is employed which is particularly suited for the problem under consideration. The increase in power-law parameter and second grade parameter results in a thicker boundary layer. For shear-thickening case (i.e. m > 0), the velocity is observed to increase with increasing m. For the shear thinning values (i.e. m < 0), the velocity is observed to decrease with decreasing m.

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