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DERIVATION OF A SIX-STEP BLOCK METHOD FOR DIRECT SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract- A new six-step block method for solving second order initial value problems of ordinary differential equations is proposed using interpolation and collocation strategies. In developing this method, the power series adopted as an approximate solution is employed as interpolation equation while its second derivative is used as collocation equation. In addition, the stability properties of the developed method are also established. The numerical results reveal that the new method produces better accuracy if compared to existing methods when solving the same problems.

Key Words- Power Series, Interpolation, Collocation, Block Method, Ordinary Differential Equations

1. INTRODUCTION

This paper considers the development of numerical method for the direct solution of second order initial value problems of ordinary differential equations (ODEs) of the form

$$y'' = f(x, y, y') \quad y(a) = y_0, y'(a) = y_1 \quad x \in [a, b] \tag{1}$$

where f is continuous in the given interval of integration. Direct solution of higher order ODEs has been found to be more accurate than when they are reduced to their equivalent system of first order ODEs (see [4], [8], [3], [9]). A lot of scholars such as Awoyemi [4-5] and Omar & Suleiman [11-13] amongst others have worked on the derivation of direct method for solving higher order ordinary differential equations.

Adesanya et. al. [2] proposed two steps block method for the direct solution of (1). An improved parallel method with a step-length of three was developed by Yayaha [10] for solving (1) directly. Osilagun et. al.[6] increased the step-length by developing four-step implicit method for solving (1) without reduction process. Furthermore, Yahaya and Badmus [1] developed an accurate uniform order six block method for the direct solution of (1) where the step-length of five was considered but the accuracy of the method is low.

In order to improve the accuracy of the existing methods, a new block method of six step-length for direct solution of second order ordinary differential equations is proposed.

2. DERIVATION OF THE METHOD

Power series approximate solution of the form

$$y(x)=\sum_{j=0}^{s+r-1}a_j x^j \tag{2}$$

is considered as an interpolation polynomial where r and s are the number of interpolation and collocation points respectively. Differentiating (2) once and twice gives

$$y'(x)=\sum_{j=1}^{s+r-1}ja_j x^{j-1} \tag{3}$$

$$y''(x)=\sum_{j=2}^{s+r-1}j(j-1)a_j x^{j-2} \tag{4}$$

Interpolating equation (2) $x=x_{n+i}, i=3(1)4$ and collocating equation (4) at $x=x_{n+i}, i=0(1)6$ produces nonlinear system of equations of the form

$$\begin{aligned}\sum_{j=0}^{s+r-1}a_j x^j &= y_{n+i}\\ \sum_{j=2}^{s+r-1}j(j-1)a_j x^{j-2} &= f_{n+i}\end{aligned} \tag{5}$$

In order to determine the values of a_j, Gaussian elimination method is employed. The values of a_j are then substituted into the interpolation polynomial (2) which yields a continuous linear multistep method of the form.

$$y(z)=\sum_{j=3}^{k=2}\alpha_j(z)y_{n+j}+h^2\sum_{j=0}^{k}\beta_j(z)f_{n+j} \tag{6}$$

where the step-length $k=6$. The coefficients of $\alpha_j(z)$ *and* $\beta_j(z)$ are given as

$$\begin{pmatrix}\alpha_3(z)\\ \alpha_4(z)\end{pmatrix}=\begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}z^0\\ z^1\end{pmatrix}$$

$$\begin{pmatrix} \beta_0(z) \\ \beta_1(z) \\ \beta_2(z) \\ \beta_3(z) \\ \beta_4(z) \\ \beta_5(z) \\ \beta_6(z) \end{pmatrix} = \frac{1}{C} \begin{pmatrix} 62 & 351 & 0 & -672 & -364 & 126 & 140 & 36 & 3 \\ 372 & -2586 & 0 & 5040 & 2604 & -1008 & -1008 & -240 & -18 \\ 426 & 8085 & 0 & -16800 & -7980 & 3654 & 3108 & 660 & 45 \\ 10856 & -5772 & 0 & 33600 & 13160 & -8064 & -5152 & -960 & -60 \\ 98706 & 135561 & 0 & -50400 & -7140 & 10458 & 4788 & 780 & 45 \\ 11724 & 47238 & 60480 & 25872 & -4116 & -7056 & -2352 & -336 & -18 \\ 442 & -1437 & 0 & 3360 & 3836 & 1890 & 476 & 60 & 3 \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \\ z^2 \\ z^3 \\ z^4 \\ z^5 \\ z^6 \\ z^7 \\ z^8 \end{pmatrix} \tag{7}$$

where $C = 120960$, $z = \dfrac{x - x_{n+k-1}}{h}$.

Evaluating (7) at the non-interpolation points gives the discrete schemes. Similarly, evaluating the first derivative of (7) at all the grid points produces the derivative of discrete schemes. Combining the discrete schemes with its derivative in a matrix finite difference equation yields a block method of the form

$$A^{(0)}Y_{N+1} = A^{(1)}Y_N + hB^{(1)}Y_N^{'} + h^2(C^{(0)}F_{N+1} + C^{(1)}F_N) \tag{8}$$

where

$$Y_{N+1} = [y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}]^T, Y_N = [y_{n-5}, y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n]^T$$
$$Y'_N = [y'_{n-5}, y'_{n-4}, y'_{n-3}, y'_{n-2}, y'_{n-1}, y'_n]^T, F_{N+1} = [f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}]^T$$
$$F_N = [f_{n-5}, f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n]^T$$

$$A^{(0)} = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}, A^{(1)} = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{pmatrix}, B^{(1)} = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&2 \\ 0&0&0&0&0&3 \\ 0&0&0&0&0&4 \\ 0&0&0&0&0&5 \\ 0&0&0&0&0&6 \end{pmatrix},$$

$$C^{(0)} = \begin{pmatrix} \frac{57750}{120960} & \frac{-51453}{120960} & \frac{42484}{120960} & \frac{-23109}{120960} & \frac{7254}{120960} & \frac{-995}{120960} \\ \frac{223488}{120960} & \frac{-107520}{120960} & \frac{100864}{120960} & \frac{-55872}{120960} & \frac{17664}{120960} & \frac{-2432}{120960} \\ \frac{14850}{4480} & \frac{-2403}{4480} & \frac{6300}{4480} & \frac{-3267}{4480} & \frac{1026}{4480} & \frac{-141}{4480} \\ \frac{4512}{945} & \frac{-72}{945} & \frac{2624}{945} & \frac{-840}{945} & \frac{288}{945} & \frac{-40}{945} \\ \frac{150750}{24192} & \frac{9375}{24192} & \frac{102500}{24192} & \frac{-5625}{24192} & \frac{11550}{24192} & \frac{-1375}{24192} \\ \frac{1080}{140} & \frac{108}{140} & \frac{816}{140} & \frac{54}{140} & \frac{216}{140} & 0 \end{pmatrix}, \text{ and}$$

$$C^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{28549}{120960} \\ 0 & 0 & 0 & 0 & 0 & \frac{65728}{120960} \\ 0 & 0 & 0 & 0 & 0 & \frac{3795}{4480} \\ 0 & 0 & 0 & 0 & 0 & \frac{1088}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{35225}{24192} \\ 0 & 0 & 0 & 0 & 0 & \frac{176125}{140} \end{pmatrix}$$

The derivative of the block (8) gives

$$\begin{pmatrix} y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y'_{n+4} \\ y'_{n+5} \\ y'_{n+6} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y'_n + h \begin{pmatrix} \frac{19087}{60480} & \frac{65112}{60480} & \frac{-46461}{60480} & \frac{37504}{60480} & \frac{-20211}{60480} & \frac{6312}{60480} & \frac{-863}{60480} \\ \frac{1139}{3780} & \frac{5640}{3780} & \frac{33}{3780} & \frac{1328}{3780} & \frac{-807}{3780} & \frac{264}{3780} & \frac{-37}{3780} \\ \frac{685}{2240} & \frac{3240}{2240} & \frac{1161}{2240} & \frac{2176}{2240} & \frac{-729}{2240} & \frac{216}{2240} & \frac{-29}{2240} \\ \frac{572}{1890} & \frac{2784}{1890} & \frac{768}{1890} & \frac{3008}{1890} & \frac{348}{1890} & \frac{96}{1890} & \frac{-16}{1890} \\ \frac{3715}{12096} & \frac{17400}{12096} & \frac{6375}{12096} & \frac{16000}{12096} & \frac{11625}{12096} & \frac{5640}{12096} & \frac{-275}{12096} \\ \frac{41}{140} & \frac{216}{140} & \frac{27}{140} & \frac{272}{140} & \frac{27}{140} & \frac{216}{140} & \frac{41}{140} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix}$$

3. PROPERTIES OF THE METHOD

This section examines the properties of the new developed block method for solving second order initial value problems of ODEs.

3. 1. Order of the Method

Expanding (8) about the point x using Taylor series gives

$$\begin{pmatrix}
\sum_{i=0}^{\infty}\frac{(h)^i}{i!}y_n^i - y_n - hy_n' - \frac{28549}{120960}h^2y_n^{(2)} - \sum_{i=0}^{\infty}\frac{h^{(i+4)}}{120960}y_n^{(i+4)}[-995(6)^i + 7254(5)^i - 23109(4)^i \\
+ 42484(3)^i - 51453(2)^i + 57750(1)^i] \\
\sum_{i=0}^{\infty}\frac{(2h)^i}{i!}y_n^i - y_n - 2hy_n' - \frac{65728}{120960}h^2y_n^{(2)} - \sum_{i=0}^{\infty}\frac{h^{(i+4)}}{120960}y_n^{(i+4)}[-2432(6)^i + 17664(5)^i - 55872(4) \\
+ 100864(3)^i - 107520(2)^i + 223488(1)^i] \\
\sum_{i=0}^{\infty}\frac{(3h)^i}{i!}y_n^i - y_n - 3hy_n' - \frac{3795}{4480}h^2y_n^{(2)} - \sum_{i=0}^{\infty}\frac{h^{(i+4)}}{4480}y_n^{(i+4)}[-141(6)^i + 1026(5)^i - 3267(4) \\
+ 6300(3)^i - 2403(2)^i + 14850(1)^i] \\
\sum_{i=0}^{\infty}\frac{(4h)^i}{i!}y_n^i - y_n - 4hy_n' - \frac{1088}{945}h^2y_n^{(2)} - \sum_{i=0}^{\infty}\frac{h^{(i+4)}}{945}y_n^{(i+4)}[-40(6)^i + 288(5)^i - 840(4) \\
+ 2624(3)^i - 72(2)^i + 4512(1)^i] \\
\sum_{i=0}^{\infty}\frac{(5h)^i}{i!}y_n^i - y_n - 5hy_n' - \frac{35225}{24192}h^2y_n^{(2)} - \sum_{i=0}^{\infty}\frac{h^{(i+4)}}{24192}y_n^{(i+4)}[-1375(6)^i + 11550(5)^i - 5625(4) \\
+ 102500(3)^i + 9375(2)^i + 150750(1)^i] \\
\sum_{i=0}^{\infty}\frac{(6h)^i}{i!}y_n^i - y_n - 6hy_n' - \frac{246}{140}h^2y_n^{(2)} - \sum_{i=0}^{\infty}\frac{h^{(i+4)}}{140}y_n^{(i+4)}[216(5)^i + 54(4) + 816(3)^i + 108(2)^i \\
+ 1080(1)^i]
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Comparing the coefficients of the powers of h makes our method (8) to have order $[7,7,7,7,7,7]^T$ with error constant $[\frac{149}{22413}, \frac{92}{5597}, \frac{9}{350}, \frac{159}{4544}, \frac{148}{3305}, \frac{9}{175}]^T$.

3. 2. Zero Stability

The block method (8) is said to be *zero-stable* if the roots $z_s = 1,2,..., N$ of the first characteristic polynomial $\rho(z) = \det(zA^{(0)} - A^{(1)})$ satisfies $|z| \le 1$ and the root $|z| = 1$ has multiplicity not greater than the order of the differential equation. The first characteristic polynomial of the block method (8) is given by

$$\det[zA^{(0)} - A^{(1)}] = \left| z\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = 0$$

which implies $z = 0,0,0,0,0,1$. Thus, $|z| = 1$ and hence the developed method (8) is zero-stable.

3. 3. Convergence

A method is said to be convergent if it is zero-stable and its order is greater than one (Henrici, 1962). Since the method (8) is zero-stable and its order is seven, it is, therefore, convergent.

3. 4. Region of Absolute Stability

Boundary locus method is adopted in finding the region of absolute stability of this method. Substituting the test equation $y'' = -\lambda^2 y$, $\lambda = \frac{\partial f}{\partial y}$ to equation (8) and after performing some mathematical manipulations, the region of absolute stability $[0,\ 23.47]$ is obtained as depicted in Figure 1 below:

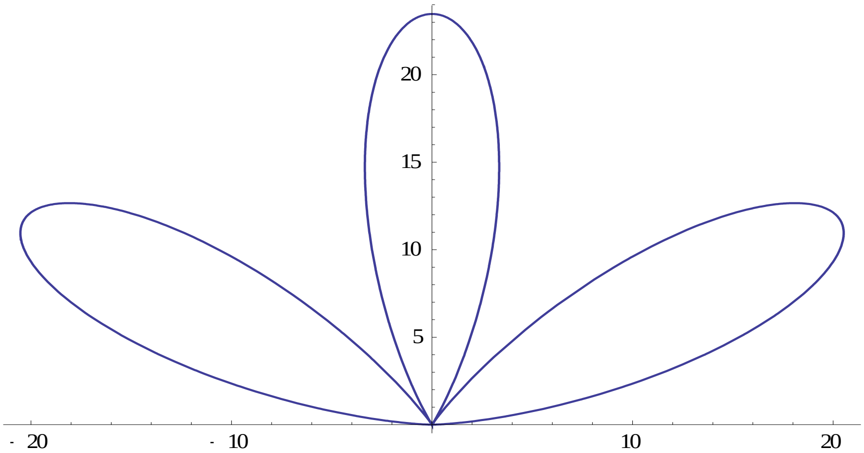

Figure 1**:** *Region of Absolute Stability for a six-step block Method*

4. NUMERICAL PROBLEMS

The following differential problems were solved numerically in order to compare the accuracy of our method with the existing methods.

Problem 1: $y''' = -y + 2\cos x, \quad y(0) = 1, y'(0) = 0, \ 0 \le x \le 1$

Exact solution: $y(x) = \cos x + x \sin x$

The above problem was solved by Omar & Suleiman [13] whereby *k*=5 was considered and maximum errors were selected. Our method was applied to the same problem and the results generated are compared with their results as displayed in Table I.

Table 1. Comparison between the new method and Omar & Suleiman [13] for Problem 1

h	Exact Solution	Numerical Solution	Error in our new method with *k* =6	Error in Omar & Suleiman [13] with *k* =5
0.01	-0.366105114594323600	-0.366105114608609670	1.428607E-11	4.21146E-03
0.001	-0.681709299809421630	-0.681709299809590390	1.687539E-13	4.20825E-04
0.0001	-0.712865615093621450	-0.712865615098958290	5.336842E-12	4.20740E-06
0.00001	-0.715977036435663790	- 0.715977036493128160	5.746437E-11	4.20736E-06

Problem 2: $y'' = y, \quad y(0) = 1, y'(0) = 1, \ 0 \le x \le 1$

Exact solution: $y(x) = e^x$

Omar & Suleiman [13] also solved the above differential problem with their developed method whereby the maximum errors were also selected. The same problem was also considered by our method and the results are compared with their results as shown in Table II below.

Table 2. Comparison between the new method and Omar & Suleiman [13] for Problem 2

h	Exact Solution	Numerical Solution	Error in our new method with *k* =6	Error in Omar & Suleiman [13] with *k* =5
0.01	428.375436859282560000	428.375436859208890000	7.366907E-11	5.87339E-03
0.001	405.856642517224540000	405.856642517222330000	2.216893E-12	5.87625E-05
0.0001	403.670923400927280000	403.670923400538980000	3.882974E-10	5.87604E-05
0.00001	403.452999950629530000	403.452999946529190000	4.100343E-09	5.87601E-06

Problem 3: $y'' = x(y')^2, \quad y(0) = 1, y'(0) = \frac{1}{2}, h = \frac{1}{30}$

Exact solution: $1 + \frac{1}{2} In\left(\frac{2+x}{2-x}\right)$

Badmus & Yahaya [1] applied their method to approximate the solution of the problem above. The same problem was also solved by our method and the comparison of the numerical results is given in Table III.

Table 3. Comparison between the new method and Badmus & Yahaya [1] for Problem 3

x	Exact Solution	Numerical Solution	Error in our new method	Error in Badmus & Yahaya [1] with k =5
0.03	1.016668210133795800	1.016668210133777200	1.865175E-14	....
0.10	1.050041729278491400	1.050041729278346800	1.445510E-13	5.891E-06
0.20	1.100335347731075600	1.100335347693282200	3.779332E-11	8.2399E-05
0.30	1.151140435936466800	1.151140401655123400	3.428134E-08	3.46421E-04
0.40	1.202732554054082100	1.202732484182993400	6.987109E-08	7.52101E-04
0.50	1.255412811882995200	1.255412610176400700	2.017066E-07	1.38028E-03

5. CONCLUSION

A new six-step block method for the direct solution of second order initial value problems of ODEs has been successfully developed. In Tables I and II above, the new developed method produces better accuracy when compared with results generated by Omar and Suleiman [13]. The new method also outperforms the method developed by Badmus and Yahaya [1] in term of accuracy. It is observed that the higher the step-length k, the higher the accuracy of the method. Therefore, in our future work, the value of the step-length k will be increased so that better results will be obtained.

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