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BERNSTEIN COLLOCATION METHOD FOR SOLVING THE FIRST ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH THE MIXED NON-LINEAR CONDITIONS

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Abstract: In this study, we present the Bernstein matrix method to solve the first order nonlinear ordinary differential equations with the mixed non-linear conditions. By using this method, we obtain the approximate solutions in form of the Bernstein polynomials [1,2,16,17]. The method reduces the problem to a system of the nonlinear algebraic equations by means of the required matrix relations of the solutions form. By solving this system, the approximate solution is obtained. Finally, the method will be illustrated on the examples.

Keywords: Nonlinear ordinary differential equations, Riccati equation, Bernstein polynomials, Collocation points.

1. INTRODUCTION

In this study, modifying and developing methods in [3-17] and using matrix relations between Bernstein polynomials and their derivatives, we present an approach to numerical solutions of the first order nonlinear ordinary differential equation in the form

$$P(x)y(x)+Q(x)y'(x)+R(x)y^2(x)+S(x)y(x)y'(x)+T(x)\left(y'(x)\right)^2+U(x)y^3(x)=g(x)\ , \quad 0\le x\le R \tag{1}$$

given together with the mixed non-linear conditions defined as follows

$$\alpha y(a)+\beta y(b)+\gamma\left(y(a)\right)^2+\tau\left(y(b)\right)^2+\psi y(a)y(b)+\xi y^3(a)+\zeta y^3(b)=\lambda\ , \quad 0\le a\le x\le b<R \tag{2}$$

where $P(x),Q(x),R(x),S(x),T(x),U(x)$ and $g(x)$ are known functions defined on the interval $0\le x\le R$; $\alpha,\beta,\gamma,\tau,\psi,\xi,\zeta$ and λ are appropriate constants; $y(x)$ is an unknown function to be determined.

The equation defined by (1) is a class of the first order nonlinear differential equation. This is an important branch of modern mathematics and arises frequently in many applied areas which include engineering, ecology, economics, biology and astrophysics. That is why these methods are important to Engineers and scientists.
Our purpose in this study is to develop a new matrix method, obtain the approximate solution of the problem (1)-(2) in the Bernstein polynomial form,

$$y(x) \cong \sum_{n=0}^{N} y_n B_{n,N}(x) \ , \tag{3}$$

$$B_{n,N}(x) = \sum_{k=0}^{N-n} \frac{(-1)^k}{R^{n+k}} \binom{N}{n} \binom{N-n}{k} x^{n+k} \quad , (n = 0,1,...,N), \quad 0 \le x \le R$$

where y_n, $(n = 0,1,...,N)$ are the coefficients to be determined and $B_{n,N}(x)$ is the Bernstein polynomial of degree N.

2. FUNDAMENTAL MATRIX RELATIONS

Let us consider the nonlinear differential equation (1) and find the matrix forms of each term in these equations. Firstly, we consider the solution $y(x)$ defined by a truncated series (3) and then we can convert it to the matrix form

$$y(x) = \mathbf{B}(x)\mathbf{Y} \tag{4}$$

where

$$\mathbf{B}(x) = \begin{bmatrix} B_{0,N}(x) & B_{1,N}(x) & \cdots & B_{N,N}(x) \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} y_0 & y_1 \cdots y_N \end{bmatrix}^T .$$

If we differentiate equation (4) with respect to *x*, we obtain

$$y'(x) = \mathbf{B}'(x)\mathbf{Y} = \mathbf{B}(x)\mathbf{HY} \tag{5}$$

so that

$$\mathbf{H} = (\mathbf{M}^T)^{-1} \mathbf{\Pi} \mathbf{M}^T$$

$$\mathbf{M} = \begin{bmatrix} m_{00} & m_{01} & \cdots & m_{0N} \\ m_{10} & m_{11} & \cdots & m_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ m_{N0} & m_{N1} & \cdots & m_{NN} \end{bmatrix} \quad , \ m_{ij} = \begin{cases} \dfrac{(-1)^{j-i} \binom{N}{i} \binom{N-i}{j-i}}{R^j} & , \quad i \le j \\ 0 & , \quad i > j \end{cases}$$

$$\mathbf{\Pi} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} .$$

On the other hand, the matrix form of the equation $y^2(x)$ is obtained as

$$y^2(x) = \begin{bmatrix} B_{0,N}(x) & B_{1,N}(x) & \cdots & B_{N,N}(x) \end{bmatrix} \begin{bmatrix} \mathbf{B}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{B}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}(x) \end{bmatrix} \begin{bmatrix} y_0\mathbf{Y} \\ y_1\mathbf{Y} \\ \vdots \\ y_N\mathbf{Y} \end{bmatrix}$$

or briefly

$$y^2(x) = \mathbf{B}(x)\bar{\mathbf{B}}(x)\bar{\mathbf{Y}} \tag{6}$$

where

$$\bar{\mathbf{B}}(x) = \begin{bmatrix} \mathbf{B}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{B}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}(x) \end{bmatrix}_{(N+1)\times(N+1)^2}, \qquad \bar{\mathbf{Y}} = \begin{bmatrix} y_0\mathbf{Y} & y_1\mathbf{Y} & \cdots & y_N\mathbf{Y} \end{bmatrix}^T_{(N+1)^2\times 1}.$$

By using the equations (4), (5) and (6) we obtain

$$y(x)y'(x) = \mathbf{B}(x)\bar{\mathbf{B}}(x)\bar{\mathbf{H}}\bar{\mathbf{Y}}. \tag{7}$$

Following a similar way to (6), we have

$$\left(y'(x)\right)^2 = \mathbf{B}(x)\mathbf{H}\bar{\mathbf{B}}(x)\bar{\mathbf{H}}\bar{\mathbf{Y}} \tag{8}$$

where

$$\bar{\mathbf{H}} = \begin{bmatrix} \mathbf{H} & 0 & \cdots & 0 \\ 0 & \mathbf{H} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H} \end{bmatrix}_{(N+1)^2\times(N+1)^2}.$$

On the other hand, the matrix form of expression $y^3(x)$ is obtained as

$$y^3(x)=\begin{bmatrix} B_{0,N}(x) & B_{1,N}(x) & \cdots & B_{N,N}(x)\end{bmatrix}\begin{bmatrix}\mathbf{B}(x) & 0 & \cdots & 0\\ 0 & \mathbf{B}(x) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{B}(x)\end{bmatrix}\begin{bmatrix}\bar{\mathbf{B}}(x) & 0 & \cdots & 0\\ 0 & \bar{\mathbf{B}}(x) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \bar{\mathbf{B}}(x)\end{bmatrix}\begin{bmatrix} y_0\bar{\mathbf{Y}}\\ y_1\bar{\mathbf{Y}}\\ \vdots\\ y_N\bar{\mathbf{Y}}\end{bmatrix}$$

or briefly

$$y^3(x)=\mathbf{B}(x)\bar{\mathbf{B}}(x)\bar{\bar{\mathbf{B}}}(x)\bar{\bar{\mathbf{Y}}} \tag{9}$$

where

$$\bar{\bar{\mathbf{B}}}(x)=\begin{bmatrix}\bar{\mathbf{B}}(x) & 0 & \cdots & 0\\ 0 & \bar{\mathbf{B}}(x) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \bar{\mathbf{B}}(x)\end{bmatrix}_{(N+1)^2\times(N+1)^3} \quad , \quad \bar{\bar{\mathbf{Y}}}=\begin{bmatrix} y_0\bar{\mathbf{Y}} & y_1\bar{\mathbf{Y}} & \cdots & y_N\bar{\mathbf{Y}}\end{bmatrix}^T_{(N+1)^3\times 1}$$

3. MATRIX RELATIONS BASED ON COLLOCATION POINTS

Let us use the collocation points defined by

$$x_i=a+\frac{b-a}{N}i \quad , \quad i=0,1,\ldots,N \tag{10}$$

in order to

$$a=x_0<x_1<\cdots<x_N=b$$

By using the collocation points (10) into Eq. (1), we obtain the system

$$P(x_i)y(x_i)+Q(x_i)y'(x_i)+R(x_i)y^2(x_i)+S(x_i)y(x_i)y'(x_i)+T(x_i)\left(y'(x_i)\right)^2+U(x_i)y^3(x_i)=g(x_i)$$

$$i=0,1,\ldots,N \quad ; \quad 0\le x_i\le R \tag{11}$$

By using the relations (4), (5), (6), (7) and (8); the system (11) can be written in the matrix form

$$\left[P(x_i)\mathbf{B}(x_i)\mathbf{I}+Q(x_i)\mathbf{B}(x_i)\mathbf{H}\right]\mathbf{Y}+\left[R(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)+S(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}+T(x_i)\mathbf{B}(x_i)\mathbf{H}\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}\right]\bar{\mathbf{Y}}$$

$$+\left[U(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)\bar{\bar{\mathbf{B}}}(x_i)\right]\bar{\bar{\mathbf{Y}}}=g(x_i) \tag{12}$$

Consequently, the fundamental matrix equations of (12) can be written in the following compact form

$$\mathbf{W}(x_i)\mathbf{Y}+\mathbf{V}(x_i)\bar{\mathbf{Y}}+\mathbf{Z}(x_i)\bar{\bar{\mathbf{Y}}}=g(x_i)$$

where

$$\mathbf{W}(x_i)=P(x_i)\mathbf{B}(x_i)\mathbf{I}+Q(x_i)\mathbf{B}(x_i)\mathbf{H}$$

$$\mathbf{V}(x_i)=R(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)+S(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}+T(x_i)\mathbf{B}(x_i)\mathbf{H}\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}$$

and

$$\mathbf{Z}(x_i)=U(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)\bar{\bar{\mathbf{B}}}(x_i).$$

Or it can be written shortly as

$$\mathbf{WY}^*+\mathbf{V}\bar{\mathbf{Y}}^*+\mathbf{Z}\bar{\bar{\mathbf{Y}}}^*=\mathbf{G} \tag{13}$$

where

$$\mathbf{W}=\begin{bmatrix}\mathbf{W}(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{W}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{W}(x_N)\end{bmatrix}_{(N+1)\times(N+1)^2};$$

$$\mathbf{V}=\begin{bmatrix}\mathbf{V}(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{V}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{V}(x_N)\end{bmatrix}_{(N+1)\times(N+1)^3}$$

$$\mathbf{Z}=\begin{bmatrix}\mathbf{Z}(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{Z}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{Z}(x_N)\end{bmatrix}_{(N+1)\times(N+1)^4}; \qquad \mathbf{G}=\begin{bmatrix}g(x_0)\\ g(x_1)\\ \vdots\\ g(x_N)\end{bmatrix}_{(N+1)\times 1}$$

$$\mathbf{Y}^{*}=\begin{bmatrix}\mathbf{Y}\\ \mathbf{Y}\\ \vdots\\ \mathbf{Y}\end{bmatrix}_{(N+1)^{2}\times 1}\quad;\quad \bar{\mathbf{Y}}^{*}=\begin{bmatrix}\bar{\mathbf{Y}}\\ \bar{\mathbf{Y}}\\ \vdots\\ \bar{\mathbf{Y}}\end{bmatrix}_{(N+1)^{3}\times 1}\quad;\quad \bar{\bar{\mathbf{Y}}}^{*}=\begin{bmatrix}\bar{\bar{\mathbf{Y}}}\\ \bar{\bar{\mathbf{Y}}}\\ \vdots\\ \bar{\bar{\mathbf{Y}}}\end{bmatrix}_{(N+1)^{4}\times 1}\quad;\quad \mathbf{I}=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}_{(N+1)\times(N+1)}$$

.

4. METHOD OF SOLUTION

The fundamental matrix equation (13) corresponding to Eq. (1) can be written as

$$\mathbf{W}\mathbf{Y}^{*}+\mathbf{V}\bar{\mathbf{Y}}^{*}+\mathbf{Z}\bar{\bar{\mathbf{Y}}}^{*}=\mathbf{G}$$

or

$$\left[\mathbf{W};\mathbf{V};\mathbf{Z}:\mathbf{G}\right] \tag{14}$$

We can find the corresponding matrix equation for the condition (2), using the relation (4) and (6), as follows:

$$\left\{\alpha\mathbf{B}(a)+\beta\mathbf{B}(b)\right\}\mathbf{Y}+\left\{\gamma\mathbf{B}(a)\bar{\mathbf{B}}(a)+\tau\mathbf{B}(b)\bar{\mathbf{B}}(b)+\psi\mathbf{B}(a)\bar{\mathbf{B}}(b)\right\}\bar{\mathbf{Y}}+\left\{\xi\mathbf{B}(a)\bar{\mathbf{B}}(a)\bar{\bar{\mathbf{B}}}(a)+\zeta\mathbf{B}(b)\bar{\mathbf{B}}(b)\bar{\bar{\mathbf{B}}}(b)\right\}\bar{\bar{\mathbf{Y}}}=\lambda$$

$$0\le a\le x\le b<R \tag{15}$$

so that

$$\mathbf{B}(a)=\left[B_{0,N}(a)\quad B_{1,N}(a)\quad\cdots\quad B_{N,N}(a)\right]$$

$$\mathbf{B}(b)=\left[B_{0,N}(b)\quad B_{1,N}(b)\quad\cdots\quad B_{N,N}(b)\right]$$

$$\bar{\mathbf{B}}(a)=\begin{bmatrix}\mathbf{B}(a) & 0 & \cdots & 0\\ 0 & \mathbf{B}(a) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{B}(a)\end{bmatrix}_{(N+1)\times(N+1)^{2}}\quad;$$

$$\bar{\mathbf{B}}(b)=\begin{bmatrix}\mathbf{B}(b) & 0 & \cdots & 0\\ 0 & \mathbf{B}(b) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{B}(b)\end{bmatrix}_{(N+1)\times(N+1)^{2}}$$

$$\bar{\bar{\mathbf{B}}}(a)=\begin{bmatrix}\bar{\mathbf{B}}(a) & 0 & \cdots & 0\\ 0 & \bar{\mathbf{B}}(a) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \bar{\mathbf{B}}(a)\end{bmatrix}_{(N+1)^2\times(N+1)^3};$$

$$\bar{\bar{\mathbf{B}}}(b)=\begin{bmatrix}\bar{\mathbf{B}}(b) & 0 & \cdots & 0\\ 0 & \bar{\mathbf{B}}(b) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \bar{\mathbf{B}}(b)\end{bmatrix}_{(N+1)^2\times(N+1)^3}.$$

We can write the corresponding matrix form (15) for the mixed condition (2) in the augmented matrix form as

$$[\mathbf{K};\mathbf{L};\mathbf{N}:\boldsymbol{\lambda}] \tag{16}$$

where

$$\mathbf{K}=[k_0 \quad k_1 \quad \cdots \quad k_N]_{1\times(N+1)}=\alpha\mathbf{B}(a)+\beta\mathbf{B}(b)$$

$$\mathbf{L}=[l_0 \quad l_1 \quad \cdots \quad l_N]_{1\times(N+1)^2}=\gamma\mathbf{B}(a)\bar{\mathbf{B}}(a)+\tau\mathbf{B}(b)\bar{\mathbf{B}}(b)+\psi\mathbf{B}(a)\bar{\mathbf{B}}(b)$$

$$\mathbf{N}=[n_0 \quad n_1 \quad \cdots \quad n_N]_{1\times(N+1)^3}=\xi\mathbf{B}(a)\bar{\mathbf{B}}(a)\bar{\bar{\mathbf{B}}}(a)+\zeta\mathbf{B}(b)\bar{\mathbf{B}}(b)\bar{\bar{\mathbf{B}}}(b)\ .$$

To obtain the approximate solution of Eq. (1) with the mixed condition (2) in the terms of Bernstein polynomials, by replacing the row matrix (16) by the last row of the matrix (13), we obtain the required augmented matrix:

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{V}}:\tilde{\mathbf{G}}\right]=\left[\begin{array}{cccccccccccccc}\mathbf{W}(x_0) & 0 & \cdots & 0\,; & \mathbf{V}(x_0) & 0 & \cdots & 0\,; & \mathbf{Z}(x_0) & 0 & \cdots & 0\,: & g(x_0)\\ 0 & \mathbf{W}(x_1) & \cdots & 0\,; & 0 & \mathbf{V}(x_1) & \cdots & 0\,; & 0 & \mathbf{Z}(x_1) & \cdots & 0\,: & g(x_1)\\ \vdots & \vdots & \ddots & \vdots\,; & \vdots & \vdots & \ddots & \vdots\,; & \vdots & \vdots & \ddots & \vdots\,: & \vdots\\ 0 & 0 & \cdots & \mathbf{K}\,; & 0 & 0 & \cdots & \mathbf{L}\,; & 0 & 0 & \cdots & \mathbf{N}: & \lambda\end{array}\right]$$

or the corresponding matrix equation

$$\tilde{\mathbf{W}}\mathbf{Y}^*+\tilde{\mathbf{V}}\bar{\mathbf{Y}}^*+\tilde{\mathbf{Z}}\bar{\bar{\mathbf{Y}}}^*=\tilde{\mathbf{G}} \tag{17}$$

where

$$\tilde{\mathbf{W}} = \begin{bmatrix} \mathbf{W}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{W}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{K} \end{bmatrix} \quad ; \quad \tilde{\mathbf{V}} = \begin{bmatrix} \mathbf{V}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{V}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{L} \end{bmatrix}$$

$$\tilde{\mathbf{Z}} = \begin{bmatrix} \mathbf{Z}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{Z}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{N} \end{bmatrix} \quad ; \quad \tilde{\mathbf{G}} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ \lambda \end{bmatrix} \quad .$$

The unknown coefficients $\{y_0, y_1, \cdots, y_N\}$ can be determined from the nonlinear system (17). As a result, we can obtain approximate solution in the truncated series form (3).

5. ACCURACY OF SOLUTION

We can check the accuracy of the method. The truncated Benstein series in (3) have to be approximately satisfying Eq. (1). For each $x = x_i \in [0, R]$, $i = 0,1,\ldots,N$

$$E(x_i) = \left| P(x_i)y(x_i) + Q(x_i)y'(x_i) + R(x_i)y^2(x_i) + S(x_i)y(x_i)y'(x_i) + T(x_i)\left(y'(x_i)\right)^2 + U(x_i)y^3(x_i) - g(x_i) \right| \cong 0$$

and $E(x_i) \le 10^{-k_i}$ (k_i is any positive integer).

If $\mathbf{max}\left(10^{-k_i}\right) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k} [3-17].

6. NUMERICAL EXAMPLES

In this section, three numerical examples are given to illustrate the accuracy and efficiency of the presented method.

Example 6.1. Let us first consider the nonlinear differential equation

$$y'(x) - xy(x)y'(x) + y^3(x) = x^6 - 2x^4 + 2x \tag{18}$$

with the non-linear initial condition

$$y(0) - y^2(0) + y^3(0) = 0 \quad , \quad 0 \le x \le 1 \ .$$

The approximate solution $y(x)$ by the truncated Bernstein polynomial

$$y(x)=\sum_{n=0}^{2} y_n B_{n,2}(x) \qquad , \qquad 0\le x\le 1$$

where

$Q(x)=1$, $S(x)=-x$, $U(x)=1$, $P(x)=R(x)=T(x)=0$,

$g(x)=x^6-2x^4+2x$.

For $N=2$ the collocation points become

$$x_0=0\ ,\ x_1=\frac{1}{2}\ ,\ x_2=1\ .$$

From the fundamental matrix equations for the given equation and condition respectively are obtained as

$$\left[Q(x_i)\mathbf{B}(x_i)\mathbf{H}\right]\mathbf{Y}+\left[S(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)\bar{\mathbf{H}}\right]\bar{\mathbf{Y}}+\left[U(x_i)\mathbf{B}(x_i)\bar{\mathbf{B}}(x_i)\bar{\bar{\mathbf{B}}}(x_i)\right]\bar{\bar{\mathbf{Y}}}=g(x_i)$$

We can find the matrix equations from

$$\mathbf{W}(x_i)\mathbf{Y}+\mathbf{V}(x_i)\bar{\mathbf{Y}}+\mathbf{Z}(x_i)\bar{\bar{\mathbf{Y}}}=g(x_i)\,.$$

The fundamental matrix equation

$$\mathbf{WY^*}+\mathbf{V}\bar{\mathbf{Y}}^*+\mathbf{Z}\bar{\bar{\mathbf{Y}}}^*=\mathbf{G}$$

where

$$\mathbf{W}=\begin{bmatrix}\mathbf{W}(0) & 0 & 0\\ 0 & \mathbf{W}\left(\frac{1}{2}\right) & 0\\ 0 & 0 & \mathbf{W}(1)\end{bmatrix} \quad , \quad \mathbf{V}=\begin{bmatrix}\mathbf{V}(0) & 0 & 0\\ 0 & \mathbf{V}\left(\frac{1}{2}\right) & 0\\ 0 & 0 & \mathbf{V}(1)\end{bmatrix}$$

$$\mathbf{Z}=\begin{bmatrix}\mathbf{Z}(0) & 0 & 0\\ 0 & \mathbf{Z}\left(\frac{1}{2}\right) & 0\\ 0 & 0 & \mathbf{Z}(1)\end{bmatrix} \qquad , \qquad \mathbf{G}=\begin{bmatrix}g(0)\\ g\left(\frac{1}{2}\right)\\ g(1)\end{bmatrix}\ .$$

The matrix forms of the conditions are

$$\mathbf{B}(0)\mathbf{Y}-\{\mathbf{B}(0)\bar{\mathbf{B}}(0)\}\bar{\mathbf{Y}}+\{\mathbf{B}(0)\bar{\mathbf{B}}(0)\bar{\bar{\mathbf{B}}}(0)\}\bar{\bar{\mathbf{Y}}}=0\,.$$

The augmented matrix for this fundamental matrix equation is calculated as

$$\left[\tilde{\mathbf{W}},\tilde{\mathbf{V}},\tilde{\mathbf{Z}}:\tilde{\mathbf{G}}\right]=\begin{bmatrix} \mathbf{W}(0) & 0 & 0 & ;\mathbf{V}(0) & 0 & 0 & ;\mathbf{Z}(0) & 0 & 0 & :g(0) \\ 0 & \mathbf{W}\left(\frac{1}{2}\right) & 0 & ;\;0 & \mathbf{V}\left(\frac{1}{2}\right) & 0 & ;\;0 & \mathbf{Z}\left(\frac{1}{2}\right) & 0 & :g\left(\frac{1}{2}\right) \\ 0 & 0 & \mathbf{K} & ;\;0 & 0 & \mathbf{L} & ;\;0 & 0 & \mathbf{N} & :\;0 \end{bmatrix}$$

where

$$\mathbf{W}(0)=[-2\quad 2\quad 0]\qquad,\qquad \mathbf{W}\left(\frac{1}{2}\right)=[-1\quad 0\quad 1]$$

$$\mathbf{V}(0)=[0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0]\;,$$

$$\mathbf{V}\left(\frac{1}{2}\right)=\left[\frac{1}{8}\quad 0\quad -\frac{1}{8}\quad \frac{1}{4}\quad 0\quad -\frac{1}{4}\quad \frac{1}{8}\quad 0\quad -\frac{1}{8}\right]$$

$$\mathbf{Z}(0)=[1\quad 0]$$

$$\mathbf{Z}\left(\frac{1}{2}\right)=\left[\frac{1}{64}\ \frac{1}{32}\ \frac{1}{64}\ \frac{1}{32}\ \frac{1}{16}\ \frac{1}{32}\ \frac{1}{64}\ \frac{1}{32}\ \frac{1}{64}\ \frac{1}{32}\ \frac{1}{16}\ \frac{1}{32}\ \frac{1}{16}\ \frac{1}{8}\ \frac{1}{16}\ \frac{1}{32}\ \frac{1}{16}\ \frac{1}{32}\ \frac{1}{64}\ \frac{1}{32}\ \frac{1}{64}\ \frac{1}{32}\ \frac{1}{16}\ \frac{1}{32}\ \frac{1}{64}\ \frac{1}{32}\ \frac{1}{64}\right]$$

$$\mathbf{K}=[1\quad 0\quad 0]\qquad,\qquad \mathbf{L}=[-1\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0]$$

$$\mathbf{N}(0)=[1\quad 0]$$

$$g(0)=0,\qquad g\left(\frac{1}{2}\right)=\frac{57}{64},\quad g(1)=1,\qquad \lambda=0\,.$$

From the obtained system, the coefficients y_0, y_1 and y_2 are found as $y_0=0, y_1=0$ and $y_2=1$.

Hence we have the Bernstein polynomial solution

$$y(x)=x^2\,.$$

Example 6.2. Let us now study the nonlinear differential equation

$$y'(x)+y(x)+y^3(x)=1+x+x^3 \tag{19}$$

with the non-linear condition

$$y(0)-y^2(0)=0\,,\quad 0\le x\le 1.$$

Following the previous procedures, we find the coefficients $y_0=0, y_1=\frac{1}{2}$ and $y_2=1$. Substituting the coefficients into equation (3), we obtain the solution

$$y(x) = x$$

which is the exact solution.

Example 6.3. Our last example is the nonlinear differential equation

$$x\left(y'(x)\right)^2 - xy(x)y'(x) + y^3(x) = e^{3x} \tag{20}$$
$$2y(0) - y^2(0) - y^3(0) = 0 \quad , \qquad 0 \le x \le 1 \; .$$

which has the exact solution $y(x) = e^x$.

The solutions obtained for $N = 2, 5, 7$ are compared with the exact solution is e^x, which are given in Figure 1. We compare the numerical solution and absolute errors for N=*2,5,7* in Table 1.

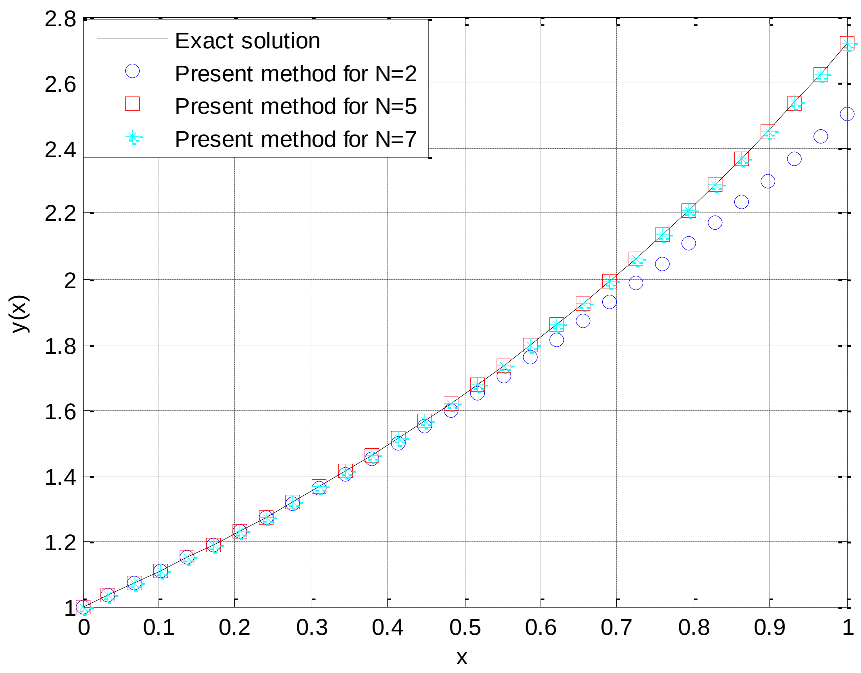

Figure 1. Numerical and exact solution of Example 6.3 for **N = 2, 5, 7**

Table 1. Comparison of the numerical errors of Example 6.3

x_i	Exact Solution	$N=2$ Numerical Solution	$N=5$ Numerical solution	$N=7$ Numerical solution	$N=2$ Error	$N=5$ Error	$N=7$ Error
0	1	1	1	1	0	0	0
0.1	1.105170918	1.105	1.105170917	1.105170918	1.70918E-4	1E-9	0
0.2	1.221402758	1.22	1.221402667	1.221402758	1.40275E-3	9.1E-8	0
0.3	1.349858807	1.345	1.34985775	1.349858806	4.85880E-3	1.058E-6	2E-9
0.4	1.491824697	1.48	1.491818667	1.491824681	1.11824E-2	6.031E-6	1.7E-8
0.5	1.648721271	1.625	1.648697917	1.648721168	2.37212E-2	2.335E-5	1.03E-7
0.6	1.822118801	1.78	1.822048	1.822118354	4.21188E-2	7.08E-5	4.46E-7
0.7	2.013752707	1.945	2.013571417	2.013751158	6.87527E-2	1.812E-4	1.549E-6
0.8	2.225540928	2.12	2.225130667	2.225536366	1.0554E-1	4.102E-4	4.562E-6
0.9	2.459603111	2.305	2.45875825	2.459591263	1.54603E-1	8.448E-4	1.184E-5
1	2.718281828	2.5	2.716666667	2.718253968	2.18281E-1	1.615E-3	2.786E-5

7. CONCLUSION

In this paper, we have presented a suggested method to solve second order nonlinear ordinary differential equations with mixed non-linear conditions using the matrix method based on collocation points on any interval $[0, R]$. The matrix method avoids the difficulties and massive computational work by determining the analytic solution.

On the other hand, the numerical results show that the accuracy improves when N is increased. Table and Figure indicate that as N increases indicate that as N increases, the errors decrease more rapidly; hence for better results, using large number N is recommended. A considerable advantage of the method is that Bernstein coefficients of the solution are found very easily by using the computer programs. Besides, our Nth order approximation gives the exact solution when the solution is polynomial of degree equal to or less than N. If the solution is not polynomial, Bernstein series approximation converges to the exact solution as N increases.

The method can also be extended to the high order nonlinear differential equations with variable coefficients, but some modifications are required.

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