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STABILITY AND BIFURCATION ANALYSIS OF A PIPE CONVEYING PULSATING FLUID WITH COMBINATION PARAMETRIC AND INTERNAL RESONANCES

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Abstract- The stability and bifurcations of a hinged-hinged pipe conveying pulsating fluid with combination parametric and internal resonances are studied with both analytical and numerical methods. The system has geometric cubic nonlinearity. Three types of critical points for the bifurcation response equations are considered. These points are characterized by a double zero and two negative eigenvalues, double zero and a pair of purely imaginary eigenvalues, and two pairs of purely imaginary eigenvalues, respectively. With the aid of normal form theory, the expressions for the critical bifurcation lines leading to incipient and secondary bifurcations are obtained. Possible bifurcations leading to 2-D tori are also investigated. Numerical simulations confirm the analytical results.

Keywords- Pipe with Pulsating Fluid, Nonlinear Vibration, Perturbation Methods, Parametric Resonances, Stability, Bifurcation;

1. INTRODUCTION

The linear and nonlinear dynamics of pipes conveying fluid has been studied widely during the last decades. Detailed review and extensive bibliography on this flow-induced vibrations and instabilities of piping and cylindrical structures were provided by Padoussis et al [1-3]. The parametric instabilities depending on the amplitude and frequency of flow fluctuation will occur when the flow velocity has a harmonically fluctuating component over a mean value. A lot of investigations based on linearized analytical models of these parametric instability problems for simply supported pipes were done by Chen [4], Padoussis and Issid[5], Padoussis and Sundararajan [6], Ginsberg [7] and Ariaratnam and Namachchivaya [8], Jayaraman and Tien [9]. They studied the parametric and combination resonances and evaluated instability with numerical methods. In [10], Panda and Kar studied the nonlinear dynamics of a hinged-hinged pipe conveying pulsating fluid subjected to combination and principle parametric resonance in the presence of internal resonance with the method of multiple scales and numerical methods. Using the method of multiple scales, Panda and Kar [11] studied the nonlinear planar vibration of a pipe conveying pulsating fluid subjected to principal parametric resonance in the presence of internal resonance. By considering the effect of motion constraints modeled as cubic springs, the nonlinear dynamics of simply supported pipes conveying pulsating fluid was further investigated, and some new dynamical behaviors including quasi-periodic and chaotic motions were obtained [12]. Jin and Song [13] investigated the stability and parametric resonances of supported pipes conveying pulsating with numerical methods. The post-divergence behavior of extensible fluid-conveying pipes supported at both ends was studied by Modarres-Sadeghi and Padoussis [14], and a supercritical pitchfork bifurcation was obtained. Using numerical methods, Wang and Ni [15] investigated the stability and chaotic motions of a standing pipe conveying fluid. A spectral element model was developed for the uniform straight pipelines conveying internal unsteady fluid [16]. By using the Melnikov method, the global dynamics of parametrically excited conveying fluid near 0:1 resonance was studied, and chaotic dynamics may exist in the system [17]. The stability and dynamics of a cantilevered pipe conveying fluid with motion-limiting constraints and a linear spring support were investigated [18]. Using the Euler-Bernoulli beam theory and nonlinear Lagrange strain theory, a new nonlinear model of a straight pipe conveying fluid was presented [19]. The vibration was analyzed with the Galerkin method. Using Hamilton's principle and Galerkin method, Sina [20] investigated the non-linear vibrations of slightly curved pipes conveying fluid with constant velocity. Periodic and chaotic motions were observed in the transverse vibrations.

In this paper, the stability and bifurcations of a hinged-hinged pipe conveying pulsating fluid with combination parametric and internal resonances are studied both analytically and numerically. Three types of critical points for the bifurcation response equations are discussed. These points are characterized by a double zero and two negative eigenvalues, double zero and a pair of purely imaginary eigenvalues, and two pairs of purely imaginary eigenvalues, respectively. With the aid of normal form theory, the expressions for the critical bifurcation lines leading to incipient and secondary bifurcations are obtained. Possible bifurcation solutions and their stability are investigated. Numerical simulations are also given, which verify the analytical results.

2. FORMULATION OF THE PROBLEM

Consider a uniform horizontal pipe hinged at both ends conveying fluid with a flow-velocity having harmonically pulsating component superimposed over a steady one (Fig 1). Assume that the motion is planar and the uniform cross-section remains plane during the motion and the tube behaves like an Euler-Bernoulli beam in transverse vibration. It is also assumed that the fluid is incompressible and has plug flow conditions. The equation of transverse motion of the pipe including the nonlinearity due to midline stretching is

$$\begin{aligned} & EI\frac{\partial^4 y}{\partial x^4}+E^*I\frac{\partial^5 y}{\partial x^4\partial t}+2MU\frac{\partial^2 y}{\partial x\partial t}+(M+m)\frac{\partial^2 y}{\partial t^2}+c\frac{\partial y}{\partial t} \\ & +[MU^2-\bar{T}+M\frac{\partial U}{\partial t}(L-x)-\frac{EA}{2L}\int_0^L (y')^2\mathrm{d}x-\frac{E^*A}{L}\int_0^L (y'\dot{y}')\mathrm{d}x]\frac{\partial^2 y}{\partial x^2}=0 \end{aligned} \tag{1}$$

with the boundary conditions

$$y(0,t)=y(L,t)=\frac{\partial^2 y}{\partial x^2}(0,t)=\frac{\partial^2 y}{\partial x^2}(L,t)=0 \tag{2}$$

where x is the longitudinal coordinate, y is the transverse deflection, $\bar{T}$ is the externally imposed axial tension, m and M are the mass per unit length of pipe and fluid materials, respectively, A is the cross sectional area of the pipe, L is the length, EI is the flexural stiffness of the pipe material, E^* is the coefficient of internal dissipation of the pipe material which is assumed to be viscoelastic and of the Kelvin-Voigt type and c is the external damping factor, U is the fluid velocity which has the following form

$$U = U_0(1+\nu\sin\omega t) \tag{3}$$

where U_0 is the mean flow velocity, ν and ω are the amplitude and frequency of the flow-velocity fluctuation, which may lead to parametric instabilities.

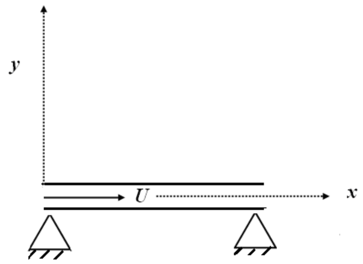


Figure 1. Schematics of the model.

Introducing the following dimensionless quantities

$$\xi = \frac{x}{L}\,, \quad w = \frac{y}{L}\,, \quad \tau = \frac{1}{L^2}\left(\frac{EI}{M+m}\right)^{1/2} t = \omega_n t\,, \quad u = \left(\frac{M}{EI}\right)^{1/2} UL\,, \quad \beta = \frac{M}{M+m}\,, \quad \Gamma = \frac{\bar{T}L^2}{EI}$$

$$,\ \sigma^* = \frac{cL^2}{\sqrt{EI(M+m)}}\,, \quad \alpha^* = \frac{E^*}{L^2}\left\{\frac{I}{(M+m)E}\right\}^{1/2}, \quad k = \frac{AL^2}{2I}\,, \quad \Omega = \frac{\omega}{\omega_n}\,,$$

$$\delta^* = \frac{E^* A}{\sqrt{EI(M+m)}}\,, \tag{4}$$

the equation of motion becomes [10]

$$\alpha^* \dot{w}'''' + w'''' + \{u^2 - \Gamma + \beta^{1/2}\dot{u}(1-\xi)\}w'' + 2\beta^{1/2}u\dot{w}' + \sigma^*\dot{w} + \ddot{w}$$
$$-k\int_0^1 (w')^2 \mathrm{d}\xi w'' - \delta^* \int_0^1 w'\dot{w}'\mathrm{d}\xi w'' = 0\,, \tag{5}$$

$$u = u_0(1+\nu\sin\Omega\tau)\,, \tag{6}$$

The primes and dots represent differentiation with respect to non-dimensional longitudinal coordinate ξ and non-dimensional time τ . To express the smallness of the amplitude of motion w, we scale it with the factor $\varepsilon^{1/2}$ as in [10], where the small parameter ε is a measure of amplitude and is also used as a book keeping device in the subsequent perturbation analysis. Introducing this scaling factor and using Eq.(6) for pulsating flow velocity, the non-dimensional equation of motion can be written as

$$w'''' + (u_0^2 + 2\varepsilon u_0 u_1 \sin\Omega\tau - \Gamma)w'' + \sqrt{\beta}\varepsilon u_1 \Omega\cos\Omega\tau(1-\xi)w'' + 2\sqrt{\beta}(u_0 + \varepsilon u_1\sin\Omega\tau)\dot{w}'$$
$$+2\varepsilon\mu\dot{w} + 2\varepsilon\alpha\dot{w}'''' + \ddot{w} = \varepsilon k\int_0^1 (w')^2 \mathrm{d}\xi w'' + O(\varepsilon^2)\,, \tag{7}$$

with the associated boundary conditions

$$w(0,\tau) = w(1,\tau) = w''(0,\tau) = w''(1,\tau) = 0\,, \tag{8}$$

where $\mu = \sigma^* / 2\varepsilon$, $\alpha = \alpha^* / 2\varepsilon$, $u_0 \nu = \varepsilon u_1$ (9)

Using the method of multiple scales, introducing the time scale $T_n = \varepsilon^n \tau, n = 0,1,\cdots$,
and the time derivatives $\frac{\mathrm{d}}{\mathrm{d}\tau} = D_0 + \varepsilon D_1 + \cdots$, $\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots$, $D_n = \frac{\partial}{\partial T_n}$,
$n = 0,1,\cdots$, we write the expansion of $w(\xi,\tau)$ in the form

$$w(\xi,\tau) = w_0(T_0,T_1,\xi) + \varepsilon w_1(T_0,T_1,\xi) + \cdots \tag{10}$$

Substituting Eq.(10) into (7) and (8), and equating coefficients of like powers of ε on both sides, one can obtain

$$O(\varepsilon^0): D_0^2 w_0 + 2\sqrt{\beta} u_0 D_0 w_0' + (u_0^2 - \Gamma) w_0'' + w_0'''' = 0,$$
$$w_0(0,\tau) = w_0(1,\tau) = w_0''(0,\tau) = w_0''(1,\tau) = 0, \tag{11}$$

$$O(\varepsilon^1): D_0^2 w_1 + 2\sqrt{\beta} u_0 D_0 w_1' + (u_0^2 - \Gamma) w_1'' + w_1''''$$
$$= -2D_0 D_1 w_0 - 2\alpha D_0 w_0'''' - 2\mu D_0 w_0 - 2\sqrt{\beta} u_0 D_1 w_0' - 2\sqrt{\beta} u_1 \sin\Omega T_0 \times D_0 w_0'$$
$$-\sqrt{\beta} u_1 \Omega \cos\Omega T_0 (1-\xi) w_0'' - 2\sqrt{\beta} u_0 u_1 \sin\Omega T_0 \times D_0 w_0' + k w_0'' \int_0^1 w_0'^2 \mathrm{d}x,$$
$$w_1(0,\tau) = w_1(1,\tau) = w_1''(0,\tau) = w_1''(1,\tau) = 0 \tag{12}$$

According to (11), we can write

$$w_0(T_0,T_1,\xi) = A_1(T_1)\phi_1(\xi) e^{\mathrm{i}\omega_1 T_0} + A_2(T_1)\phi_2(\xi) e^{\mathrm{i}\omega_2 T_0} + cc, \tag{13}$$

where the complicate expressions of $\phi_m(\xi)(m = 1,2)$ are given in reference [10]. Substituting (13) into (12), considering the case of the internal resonance and combination parametric resonance, i.e.,

$$\omega_2 = 3\omega_1 + \varepsilon\sigma_1, \quad \Omega = \omega_1 + \omega_2 + \varepsilon\sigma_2, \tag{14}$$

the modulation equations can be written as

$$2A_1' + 2\mu C_1 A_1 + 2\alpha e_1 A_1 + 8S_1 A_1^2 \bar{A}_1 + 8S_2 A_1 A_2 \bar{A}_2 + 8G_1 \bar{A}_1^2 A_2 e^{\mathrm{i}\sigma_1 T_1} + 2H_4 \bar{A}_2 e^{\mathrm{i}\sigma_2 T_2} = 0, \tag{15}$$
$$2A_2' + 2\mu C_2 A_2 + 2\alpha e_2 A_2 + 8S_4 A_2^2 \bar{A}_2 + 8S_3 A_1 A_2 \bar{A}_1 + 8G_2 \bar{A}_1^3 e^{-\mathrm{i}\sigma_1 T_1} + 2H_5 \bar{A}_1 e^{\mathrm{i}\sigma_2 T_2} = 0, \tag{16}$$

The coefficients which are very complicated and can be seen in the appendix of reference [10] are omitted here.

Letting $A_n = \frac{1}{2}[p_n(T_1) - \mathrm{i}q_n(T_1)] e^{i\lambda_n(T_1)}$, $(n = 1,2)$, (17)

substituting it into Eqs.(15) and (16), carrying out algebraic manipulations and separating real and imaginary parts, we can obtain the normalized reduced equations as follows [10]

$$p_1' = -\mu C_{1R} p_1 - \mu C_{1I} q_1 - \alpha e_{1R} p_1 - \alpha e_{1I} q_1 - S_{1R}(p_1^3 + p_1 q_1^2) - S_{1I}(p_1^2 q_1 + q_1^3)$$
$$-S_{2R}(p_1 p_2^2 + p_1 q_2^2) - S_{2I}(q_1 p_2^2 + q_1 q_2^2) - \vartheta_1 q_1 - H_{4R} p_2 + H_{4I} q_2$$
$$-G_{1R}(p_1^2 p_2 - p_2 q_1^2 + 2p_1 q_1 q_2) + G_{1I}(2p_1 q_1 p_2 - p_1^2 q_2 + q_1^2 q_2), \tag{18a}$$

$$q_1' = -\mu C_{1R} q_1 + \mu C_{1I} p_1 - \alpha e_{1R} q_1 + \alpha e_{1I} p_1 + S_{1I}(p_1^3 + p_1 q_1^2) - S_{1R}(p_1^2 q_1 + q_1^3)$$
$$-S_{2R}(q_1 p_2^2 + q_1 q_2^2) + S_{2I}(p_1 p_2^2 + p_1 q_2^2) + \vartheta_1 p_1 + H_{4R} q_2 + H_{4I} p_2$$
$$+G_{1R}(-p_1^2 q_2 + q_1^2 q_2 + 2p_1 q_1 p_2) + G_{1I}(2p_1 q_1 q_2 + p_1^2 p_2 - p_2 q_1^2), \tag{18b}$$

$$p_2' = -\mu C_{2R} p_2 - \mu C_{2I} q_2 - \alpha e_{2R} p_2 - \alpha e_{2I} q_2 - S_{4R}(p_2^3 + p_2 q_2^2) - S_{4I}(q_2^3 + p_2^2 q_2)$$
$$-S_{3R}(p_1^2 p_2 + p_2 q_1^2) - S_{3I}(p_1^2 q_2 + q_1^2 q_2) - \vartheta_2 q_2 - H_{5R} p_1 + H_{5I} q_1$$
$$-G_{2R}(p_1^3 - 3p_1 q_1^2) + G_{2I}(q_1^3 - 3p_1^2 q_1)\,, \tag{18c}$$
$$q_2' = -\mu C_{2R} q_2 + \mu C_{2I} p_2 - \alpha e_{2R} q_2 - \alpha e_{2I} p_2 - S_{4R}(q_2^3 + p_2^2 q_2) + S_{4I}(p_2^3 + p_2 q_2^2)$$
$$-S_{3R}(p_1^2 q_2 + q_1^2 q_2) + S_{3I}(p_1^2 p_2 + q_1^2 q_2) + \vartheta_2 p_2 + H_{6R} q_2 + H_{6I} p_2$$
$$+G_{2R}(q_1^3 - 3p_1^2 q_1) + G_{2I}(p_1^3 - 3p_1 q_1^2)\,, \tag{18d}$$

where $\vartheta_1 = (\sigma_1 + \sigma_2)/4$, $\vartheta_2 = (3\sigma_2 - \sigma_1)/4$. (19)

The characteristic equation of the Jacobi matrix evaluated at the initial equilibrium point $(p_1, q_1, p_2, q_2) = (0,0,0,0)$ for Eq.(18) is

$$\lambda^4 + R_1\lambda^3 + R_2\lambda^3 + R_3\lambda + R_4 = 0\,, \tag{20}$$

where R_1, R_2, R_3, R_4 are complicated functions of the parameters and omitted here.

According to the Routh-Hurwitz criterion, the initial equilibrium point $(p_1, q_1, p_2, q_2) = (0,0,0,0)$ is stable if the following conditions are satisfied.

$$R_1 > 0\,,\quad R_1R_2 - R_3 > 0\,,\quad R_3(R_1R_2 - R_3) - R_1^2 R_4 > 0\,,\quad R_4 > 0\,. \tag{21}$$

3. BIFURCATION ANALYSIS

Conditions (21) imply that all the eigenvalues of the Jacobi matrix have negative real parts. When (21) are not satisfied, this is not the case. Three cases will be discussed here.

3.1. Case 1: Double zero and two negative eigenvalues

Taking $R_1 = 4$, $R_2 = 3$, $R_3 = R_4 = 0$, Eq.(20) has a double zero and two negative eigenvalues $\lambda_{1,2} = -1$, $\lambda_4 = -3$. One choice of parameters that satisfy these conditions is

$$\mu = 1\,,\quad C_{1R} = 0\,,\quad C_{2R} = 2\,,\quad C_{1I} = \frac{\sqrt{6\sqrt{13}+15}}{3}\,,\quad C_{2I} = \frac{\sqrt{6\sqrt{13}+6}}{3}\,,\quad H_{4I} = 1\,,$$
$$H_{5I} = \frac{4\sqrt{13}+10}{3}\,,\quad \alpha = 0\,,\quad \sigma_1 = \sigma_2 = 0\,,\quad H_{4R} = H_{3R} = H_{6R} = H_{6I} = 0\,.$$

Let us consider σ_1, σ_2 as perturbation parameters. Using the parameter transformations $\sigma_1 = \tilde{\sigma}_1$, $\sigma_2 = \tilde{\sigma}_2$ and the state variable transformation

$$\begin{bmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{22}$$

where $J_{ij}(i, j = 1,2,3,4)$ are given in the appendix (A.1), one may transformation Eq.(18) into a new system as follows

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + Nf_1 \tag{23a}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + Nf_2 \tag{23b}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + Nf_3 \tag{23c}$$

$$\frac{\mathrm{d}x_4}{\mathrm{d}t} = a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + Nf_4 \tag{23d}$$

where $a_{ij}(i, j = 1,2,3,4)$ are given in the appendix (A.2), $Nf_i(i = 1,2,3,4)$ are third order nonlinear terms whose expressions are very complicated and omitted here.

The Jacobi matrix of Eq.(23) evaluated on the initial equilibrium solution $(x_1, x_2, x_3, x_4) = (0,0,0,0)$ at the critical point $P_c(\tilde{\sigma}_{1c} = \tilde{\sigma}_{2c} = 0)$ is now in the canonical form

$$J_{(x_i=0)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \tag{24}$$

The local dynamical behavior of system (23) is characterized by the critical variables x_1 and x_2 . Further more, the bifurcation solutions for the non-critical variables x_3 and x_4 may be determined from Eq.(23) up to leading order terms [21]. Therefore one may verify that neglecting x_3 and x_4 (i.e., setting $x_3 = x_4 = 0$) in the first two equations of Eq.(23) does not effect the results of the bifurcation solution (x_1, x_2) and their stability conditions up to leading order terms. So, in order to consider the bifurcation and stability properties of system (23) in the vicinity of the critical point P_c, one only needs to analyze the following two-dimensional system:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}x_1 + a_{12}x_2 + Nff_1 \,,$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}x_1 + a_{22}x_2 + Nff_2 \,, \tag{25}$$

where Nff_i are complicated third order nonlinear terms and omitted. Now based on the reduced system (25), the results and formulae obtained in [22] can be applied here. Using these methods, we can study the stability and bifurcations of this model analytically. Applying the general formula yields the following results.

The stability conditions for the initial equilibrium solution $x_i = 0$ are described by

$$131.03\tilde{\sigma}_1 - 16.86\tilde{\sigma}_2 < 0 \quad \text{and} \quad 10.61\tilde{\sigma}_1 + 5.03\tilde{\sigma}_2 > 0 \,, \tag{26}$$

which leads to two critical lines. One of these is

$$L_1 : 10.61\tilde{\sigma}_1 + 5.03\tilde{\sigma}_2 = 0 \,,\ (131.03\tilde{\sigma}_1 - 16.86\tilde{\sigma}_2 < 0) \,, \tag{27}$$

along which a static bifurcation solution takes place from the initial equilibrium solution and the solution is expressed by

$$x_1^2 = -(0.028\tilde{\sigma}_1 + 0.0015\tilde{\sigma}_2) \,,$$
$$x_2 = (-0.70\tilde{\sigma}_1 + 1.11\tilde{\sigma}_2)x_1 \,, \tag{28}$$

It is called a pitchfork bifurcation.

On the other hand, the second critical line

$$L_2 : 131.03\tilde{\sigma}_1 - 16.86\tilde{\sigma}_2 = 0, \quad (10.61\tilde{\sigma}_1 + 5.03\tilde{\sigma}_2 > 0) \tag{29}$$

describes a dynamic boundary where the initial equilibrium solution loses its stability and bifurcates a family of limit cycles. Again from Table 1 of reference [22], one may find the stability condition for the family of limit cycles, given by

$$\gamma_{11} - \gamma_{22} = -11.2 < 0 \tag{30}$$

so the family of limit cycles bifurcating from the initial equilibrium solution is stable.

The static bifurcation solution (28) becomes unstable on the third critical line

$$L_3 : 1.11\tilde{\sigma}_1 + 111.20\tilde{\sigma}_2 = 0 \tag{31}$$

from which another family of limit cycles which is usually called secondary Hopf bifurcations occurs. The frequency of this family of limit cycles is

$$\omega_c = \sqrt{-(0.77\tilde{\sigma}_1 + 0.12\tilde{\sigma}_2)} > 0 \tag{32}$$

where $0.77\tilde{\sigma}_1 + 0.12\tilde{\sigma}_2 < 0$ since the secondary Hopf bifurcation solution exists in the region located on the right of the critical line L_1 (see Fig.2). The stability condition for this family of limit cycles is given as follows:

$$\gamma_{12} + \gamma_{21} = -\frac{0.57}{\omega_c} < 0 \tag{33}$$

Therefore the secondary Hopf bifurcation solution is stable. The critical bifurcation lines are illustrated in the parameter space in Fig 2. From Fig 2 it is seen that there may exist stable trivial and non-trivial equilibrium solutions, periodic motions in this case.

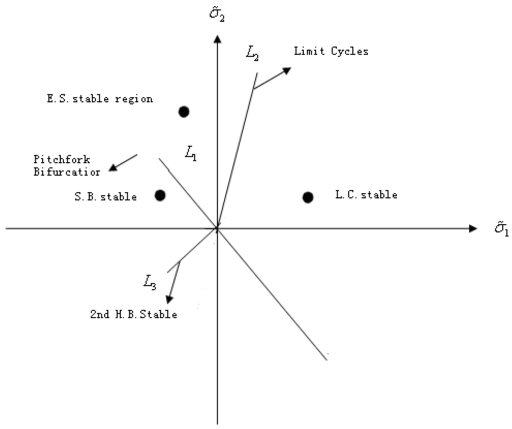


Figure 2. The bifurcation diagram in the case of double zero and two negative eigenvalues.

3.2. Case 2:Double zero and a pair of purely imaginary eigenvalues

Taking the parameters

$$\mu = \frac{1}{2}, \quad C_{1R} = 1, \quad C_{2R} = 0, \quad C_{1I} = \sqrt{\frac{5}{3}}, \quad C_{2I} = \sqrt{\frac{20}{3}}, \quad H_{4I} = 1, \quad H_{5I} = \frac{4}{3},$$

$$\alpha = 0, \quad \sigma_1 = \sigma_2 = 0, \quad H_{4R} = H_{3R} = H_{6R} = H_{6I} = 0,$$

in Eq.(20) yields $R_1 = R_3 = R_4 = 0$, $R_2 = 1$, so the eigenvalues of the Jacobian are

$\lambda_{1,2} = 0$, and $\lambda_{3,4} = \pm \mathrm{i}$, where $\mathrm{i} = \sqrt{-1}$. Choosing σ_1 and σ_2 as parameters, and using the parameter transformation $\sigma_1 = \tilde{\sigma}_1$, $\sigma_2 = \tilde{\sigma}_2$, introducing the following state variable transformation

$$\begin{bmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{15} & 0 & \frac{\sqrt{15}}{5} \\ 2 & 3 & 1 & 0 \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{15}}{3} & \sqrt{15} & -\frac{\sqrt{15}}{30} & \frac{\sqrt{15}}{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{34}$$

one may obtain the following equations

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{\sqrt{15}}{12}(\tilde{\sigma}_1 + \tilde{\sigma}_2)x_1 + (1 - \frac{\sqrt{15}}{6}\tilde{\sigma}_2)x_2 + \sqrt{15}(\frac{1}{15}\tilde{\sigma}_1 + \frac{7}{60}\tilde{\sigma}_2)x_3 + \frac{\sqrt{15}}{60}\tilde{\sigma}_2 x_4 + Nm_1, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= \frac{\sqrt{15}}{18}\tilde{\sigma}_2 x_1 + \sqrt{15}(\frac{1}{12}\tilde{\sigma}_1 + \frac{1}{4}\tilde{\sigma}_2)x_2 - \frac{\sqrt{15}}{60}(\tilde{\sigma}_1 + \tilde{\sigma}_2)x_3 + \frac{\sqrt{15}}{60}(\tilde{\sigma}_1 + 2\tilde{\sigma}_2)x_4 + Nm_2, \\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= -\frac{\sqrt{15}}{6}(\tilde{\sigma}_1 + 2\tilde{\sigma}_2)x_1 - \frac{\sqrt{15}}{6}\tilde{\sigma}_2 x_2 - \frac{\sqrt{15}}{12}(\tilde{\sigma}_1 + 2\tilde{\sigma}_2)x_3 + (1 - \frac{\sqrt{15}}{12}\tilde{\sigma}_2)x_4 + Nm_3, \\
\frac{\mathrm{d}x_4}{\mathrm{d}t} &= -\sqrt{15}(\frac{1}{6}\tilde{\sigma}_1 + \frac{4}{9}\tilde{\sigma}_2)x_1 - \sqrt{15}(\frac{2}{3}\tilde{\sigma}_1 + \frac{3}{2}\tilde{\sigma}_2)x_2 + (-1 + \frac{\sqrt{15}}{36}\tilde{\sigma}_2)x_3 \\
&\quad - \frac{\sqrt{15}}{12}(\tilde{\sigma}_1 + 2\tilde{\sigma}_2)x_4 + Nm_4,
\end{aligned} \tag{35}$$

where the third order nonlinear terms Nm_i (i = 1,2, 3,4) are omitted here.

Using an intrinsic method of harmonic analysis [23], we obtain the normal form of Eq.(35) as follows

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_2, \\
\frac{\mathrm{d}y_2}{\mathrm{d}t} &= \frac{\sqrt{15}}{18}\tilde{\sigma}_2 y_1 + \frac{\sqrt{15}}{6}(\tilde{\sigma}_1 + 2\tilde{\sigma}_2)y_2 + (\frac{2\sqrt{15}}{27} - \frac{10}{27})y_1^3 - (\frac{6}{5} + \frac{\sqrt{15}}{15})y_1\rho^2, \\
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\frac{\sqrt{15}}{12}(\tilde{\sigma}_1 + 2\tilde{\sigma}_2)\rho - (\frac{1}{12} + \frac{\sqrt{15}}{5})\rho^3 - (64 + 40\sqrt{15})y_1^2\rho, \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= 1 - \frac{\sqrt{15}}{18}\tilde{\sigma}_2 - (\frac{7}{60} + \frac{\sqrt{15}}{10})\rho^2 + (\frac{8\sqrt{15}}{9} - \frac{14}{3})y_1^2,
\end{aligned} \tag{36}$$

where y_1, y_2, ρ, θ are the variables that are transformational systems which are topology equivalent to the original systems. The transformational systems can display the dynamical behaviors of the original ones. System (36) has the following equilibrium solutions.

(i) The initial equilibrium solution (E.S.) $y_1 = y_2 = \rho = 0$.

Evaluating the Jacobian at the initial equilibrium solution, we obtain the stability conditions as follows

$$\tilde{\sigma}_1 + 2\tilde{\sigma}_2 < 0\,,\quad \tilde{\sigma}_2 > 0\,,\ (15\tilde{\sigma}_1^2 + 60\tilde{\sigma}_1\tilde{\sigma}_2 + 8\sqrt{15}\tilde{\sigma}_2 + 60\tilde{\sigma}_2^2) < 0\,, \tag{37}$$

So the initial equilibrium solution is unstable.

(ii) The static bifurcation solution (S.B.)

$$y_1^2 = (\sqrt{15} + 3)\tilde{\sigma}_2 / 4\,,$$
$$y_2 = \rho = 0\,, \tag{38}$$

The stability conditions for this solution are as follows

$$\tilde{\sigma}_1 + 2\tilde{\sigma}_2 < 0\,,\quad \tilde{\sigma}_2 > 0\,,\ (15\tilde{\sigma}_1^2 + 60\tilde{\sigma}_1\tilde{\sigma}_2 + 32\sqrt{15}\tilde{\sigma}_2 + 60\tilde{\sigma}_2^2) > 0\,, \tag{39}$$

(iii) Hopf bifurcation solution (H.B.)

$$y_1 = y_2 = 0\,,$$
$$\rho^2 = -5\sqrt{15}(\tilde{\sigma}_1 + 2\tilde{\sigma}_2)/(5 + 12\sqrt{15}) \approx -0.38(\tilde{\sigma}_1 + 2\tilde{\sigma}_2)\,, \tag{40}$$

The stability conditions for this solution are as follows

$$\tilde{\sigma}_1 + 2\tilde{\sigma}_2 < 0\,,\quad -16804.8\tilde{\sigma}_1 - 40201\tilde{\sigma}_2 > 0\,,$$
$$25620\tilde{\sigma}_2^2 - 10086.33\tilde{\sigma}_2 + 25620\tilde{\sigma}_1\tilde{\sigma}_2 - 4216.28\tilde{\sigma}_1 + 6405\tilde{\sigma}_1^2 > 0\,, \tag{41}$$

(iv) Hopf bifurcation solution 2 (H.B.2)

$$y_1^2 = -(0.0015\tilde{\sigma}_1 + 0.0035\tilde{\sigma}_2)\,,$$
$$y_2 = 0\,,$$
$$\rho^2 = 0.000084\tilde{\sigma}_1 + 0.153836\tilde{\sigma}_2\,, \tag{42}$$

Obviously, when

$$0.0015\tilde{\sigma}_1 + 0.0035\tilde{\sigma}_2 < 0 \quad \text{and} \quad 0.000084\tilde{\sigma}_1 + 0.153836\tilde{\sigma}_2 > 0 \tag{43}$$

there exists H.B.2 solution.

The stability conditions for this solution are

$$-0.65\tilde{\sigma}_1 - 1.04\tilde{\sigma}_2 > 0\,,$$
$$0.34\tilde{\sigma}_2^3 + 0.38\tilde{\sigma}_1\tilde{\sigma}_2^2 + 0.11\tilde{\sigma}_1^2\tilde{\sigma}_2 - 0.66\tilde{\sigma}_2^2 - 0.28\tilde{\sigma}_1\tilde{\sigma}_2 > 0\,,$$
$$(0.0015\tilde{\sigma}_1 + 0.0035\tilde{\sigma}_2)(0.000084\tilde{\sigma}_1 + 0.153836\tilde{\sigma}_2) > 0\,, \tag{44}$$

so this bifurcation solution is unstable.

The bifurcation diagram is shown as in Fig 3. Here the dashed lines just mark the regions of different bifurcation solutions. From Fig 3 we can see that there exist stale non-trivial equilibrium solution and periodic motion under certain conditions.

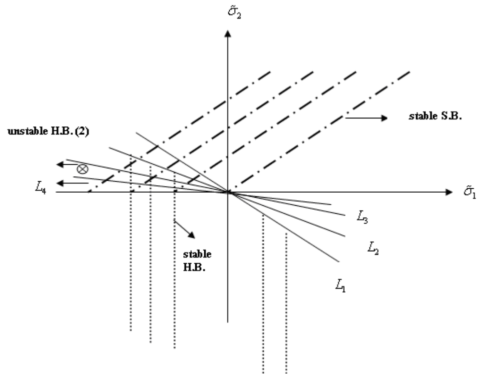


Figure 3. The bifurcation diagram in the case of double zero and a pair of purely imaginary eigenvalues.

3.3 Case 3: two pairs of purely imaginary eigenvalues

If the values of parameters are chosen as

$$\mu=(\frac{1}{2})^{1/4}\,,\quad C_{1R}=1\,,\quad C_{2R}=-1\,,\quad C_{1I}=1\,,\quad C_{2I}=1\,,\quad H_{4I}=1\,,\quad H_{5I}=-3\,,$$

$$\alpha=0\,,\quad \sigma_1=\sigma_2=0\,,\quad H_{4R}=H_{3R}=H_{6R}=H_{6I}=0\,,$$

then we have $R_1=R_3=0$ in Eq.(20) and the eigenvalues are

$$\lambda_{1,2}=\pm\mathrm{i}\,,\quad \lambda_{3,4}=\pm\sqrt{2}\mathrm{i} \tag{45}$$

Considering σ_1, σ_2 as parameters, using the parameter transformation $\sigma_1=\tilde{\sigma}_1$, $\sigma_2=\tilde{\sigma}_2$ and the state variable transformation

$$\begin{bmatrix} p_1\\ q_1\\ p_2\\ q_2\end{bmatrix}=\begin{bmatrix} 0 & \sqrt{2} & 0 & \frac{\sqrt{2}}{2}\\ 0 & -1 & 0 & -1\\ 1 & -2^{-1/4}-2^{/4} & \sqrt{2} & -2^{-1/4}-2^{-3/4}\\ -\sqrt{2} & 2^{1/4}-2^{-1/4} & -1 & 2^{-3/4}-2^{-1/4}\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \tag{46}$$

Eq.(18) becomes

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}=(\frac{\sqrt{2}}{2}\tilde{\sigma}_1-\frac{3\sqrt{2}}{2}\tilde{\sigma}_2)x_1+(1+(2^{-1/4}+2^{5/4})\tilde{\sigma}_2)x_2+(\frac{3}{4}\tilde{\sigma}_1-\frac{9}{4}\tilde{\sigma}_2)x_3$$
$$+3\cdot 2^{-3/4}\tilde{\sigma}_2 x_4+Nh_1\,,$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t}=-x_1+\frac{\sqrt{2}}{2}(\tilde{\sigma}_1+\tilde{\sigma}_2)x_2+\frac{3\sqrt{2}}{8}(\tilde{\sigma}_1+\tilde{\sigma}_2)x_4+Nh_2\,,$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t}=(\frac{9}{4}\tilde{\sigma}_2-\frac{3}{4}\tilde{\sigma}_1)x_1-3\bullet 2^{-1/4}\tilde{\sigma}_2 x_2+(\frac{3\sqrt{2}}{2}\tilde{\sigma}_2-\frac{\sqrt{2}}{2}\tilde{\sigma}_1)x_3$$
$$+(\sqrt{2}+(2^{-3/4}-2^{3/4})\tilde{\sigma}_2 x_4+Nh_3\,,$$

$$\frac{\mathrm{d}x_4}{\mathrm{d}t}=-\frac{3\sqrt{2}}{4}(\tilde{\sigma}_1+\tilde{\sigma}_2)x_2-\sqrt{2}x_3-\frac{\sqrt{2}}{2}(\tilde{\sigma}_1+\tilde{\sigma}_2)x_4+Nh_4\,, \tag{47}$$

The third order nonlinear terms $Nh_i\,(i=1,2,3,4)$ are also omitted here.

With the method of multiple scales and computer algebra [24], we get the normal form of Eq.(47) in polar coordinates as follows:

$$
\begin{aligned}
\dot{r}_1 &= r_1(\alpha_{11}\tilde{\sigma}_1 + \alpha_{12}\tilde{\sigma}_2 + a_{20}r_1^2 + a_{02}r_2^2)\ ,\\
\dot{r}_2 &= r_2(\alpha_{21}\tilde{\sigma}_1 + \alpha_{22}\tilde{\sigma}_2 + b_{20}r_1^2 + b_{02}r_2^2)\ ,\\
\dot{\theta}_1 &= \omega_{1c} + \beta_{11}\tilde{\sigma}_1 + \beta_{12}\tilde{\sigma}_2 + c_{20}r_1^2 + c_2 r_2^2\ ,\\
\dot{\theta}_2 &= \omega_{2c} + \beta_{21}\tilde{\sigma}_1 + \beta_{22}\tilde{\sigma}_2 + d_{20}r_1^2 + d_{02}r_2^2\ ,
\end{aligned}
\tag{48}
$$

where $\alpha_{11} = \alpha_{22} = \dfrac{\sqrt{2}}{2}$, $\alpha_{12} = \alpha_{21} = -\dfrac{\sqrt{2}}{2}$, $\beta_{11} = \beta_{21} = 0$, $\beta_{12} = 2^{-5/4} + 2^{1/4}$,

$\beta_{22} = 2^{-7/4} - 2^{-1/4}$, $a_{20} = \dfrac{35}{8} - \dfrac{9}{8}\bullet 2^{3/4} + \dfrac{9}{2}\bullet 2^{1/4} - \dfrac{7\sqrt{2}}{8}$, $a_{02} = \dfrac{9}{2}\bullet 2^{1/4} - \dfrac{3\sqrt{2}}{4} - \dfrac{7}{8}\bullet 2^{3/4} + \dfrac{33}{8}$,

$b_{20} = -7 - \dfrac{61}{8}\bullet 2^{3/4} - \dfrac{81}{16}\sqrt{2} + \dfrac{5}{8}\bullet 2^{3/4}$, $b_{02} = -\dfrac{31}{16} + \dfrac{9}{32}\bullet 2^{3/4} - \dfrac{111\sqrt{2}}{64} - \dfrac{71}{32}\bullet 2^{1/4}$,

$c_{20} = \dfrac{9}{2} + \dfrac{11}{2}\bullet 2^{1/4} - \dfrac{27\sqrt{2}}{8} + \dfrac{115}{15}\bullet 2^{3/4}$, $c_{02} = \dfrac{63}{8}\bullet 2^{1/4} + \dfrac{45}{8}\bullet 2^{3/4} - \dfrac{13\sqrt{2}}{8} + \dfrac{13}{2}$,

$d_{20} = -\dfrac{13}{8} - \dfrac{7}{8}\bullet 2^{1/4} - \dfrac{55\sqrt{2}}{16} - \dfrac{41}{16}\bullet 2^{3/4}$, $d_{02} = -\dfrac{33}{64} - \dfrac{69}{64}\bullet 2^{3/4} - \dfrac{99\sqrt{2}}{64}$,

and r_1, r_2, θ_1 and θ_2 are the variables that are transformational systems which are topology equivalent to the original systems.

We now discuss the main types of equilibrium states and their stability for system (48). These fall into four categories.

(i) Trivial state: The initial equilibrium solution (E.S.): $r_1 = r_2 = 0$.

Evaluating the Jacobi matrix at the initial equilibrium solution results in the stability conditions for the E.S. as $\alpha_{11}\tilde{\sigma}_1 + \alpha_{12}\tilde{\sigma}_2 < 0$ and $\alpha_{21}\tilde{\sigma}_1 + \alpha_{22}\tilde{\sigma}_2 < 0$,i.e., $\tilde{\sigma}_1 - \tilde{\sigma}_2 < 0$ and $\tilde{\sigma}_2 - \tilde{\sigma}_1 < 0$, so it is unstable.

(ii) Pure mode 1: Hopf bifurcation solution (i.e., a self-sustained oscillation, H.B.(I) with frequency ω_1):

$$
\begin{aligned}
r_1^2 &= -\frac{1}{a_{20}}(\alpha_{11}\tilde{\sigma}_1 + \alpha_{12}\tilde{\sigma}_2) = -0.11\tilde{\sigma}_1 + 0.11\tilde{\sigma}_2\ ,\\
r_2 &= 0\ ,\\
\omega_1 &= \omega_{1c} + \beta_{11}\tilde{\sigma}_1 + \beta_{12}\tilde{\sigma}_2 + c_{20}r_1^2 = 1 - 2.02\tilde{\sigma}_1 + 3.63\tilde{\sigma}_2
\end{aligned}
\tag{49}
$$

Obviously, when $\tilde{\sigma}_2 - \tilde{\sigma}_1 > 0$ there exists H.B.(I) solution.

Evaluating the Jacobi matrix at the H.B.(I) solution results in the following stability conditions $\alpha_{11}\tilde{\sigma}_1 + \alpha_{12}\tilde{\sigma}_2 > 0$ and $\alpha_{21}\tilde{\sigma}_1 + \alpha_{22}\tilde{\sigma}_2 - \frac{b_{20}}{a_{20}}(\alpha_{11}\tilde{\sigma}_1 + \alpha_{12}\tilde{\sigma}_2) < 0$. The condition $\alpha_{11}\tilde{\sigma}_1 + \alpha_{12}\tilde{\sigma}_2 > 0$ implies that $\tilde{\sigma}_1 - \tilde{\sigma}_2 > 0$, So H.B.(I) solution is unstable.

(iii) Pure mode 2: Hopf bifurcation solution (H.B.(II) with frequency ω_2)

$$r_1 = 0\ ,$$

$$r_2^2 = -\frac{1}{b_{02}}(\alpha_{21}\tilde{\sigma}_1 + \alpha_{22}\tilde{\sigma}_2) = -0.11\tilde{\sigma}_1 + 0.11\tilde{\sigma}_2 \,,$$
$$\omega_2 = \omega_{2c} + \beta_{21}\tilde{\sigma}_1 + \beta_{22}\tilde{\sigma}_2 + d_{02}r_2^2 = \sqrt{2} + 0.49\tilde{\sigma}_1 - 1.03\tilde{\sigma}_2 \,, \tag{50}$$

Obviously, when $\tilde{\sigma}_2 - \tilde{\sigma}_1 > 0$ there exists H.B.(II) solution.

Evaluating the Jacobi matrix at the H.B.(II) solution results the following stability conditions $\alpha_{21}\tilde{\sigma}_1 + \alpha_{22}\tilde{\sigma}_2 > 0$ and $\alpha_{11}\tilde{\sigma}_1 + \alpha_{12}\tilde{\sigma}_2 - \frac{a_{02}}{b_{02}}(\alpha_{21}\tilde{\sigma}_1 + \alpha_{22}\tilde{\sigma}_2) < 0$,i.e., $\tilde{\sigma}_2 - \tilde{\sigma}_1 > 0$ and $\tilde{\sigma}_2 - \tilde{\sigma}_1 < 0$, so H.B.(II) solution is also unstable.

(iv) Mixed modes: quasi-periodic solution (2D tori with frequency ω_1, ω_2):

$$r_1^2 = \frac{a_{02}(\alpha_{21}\tilde{\sigma}_1 + \alpha_{22}\tilde{\sigma}_2) - b_{02}(\alpha_{11}\tilde{\sigma}_1 + \alpha_{12}\tilde{\sigma}_2)}{a_{20}b_{02} - a_{02}b_{20}} = -0.0025\tilde{\sigma}_1 + 0.0025\tilde{\sigma}_2 \,,$$
$$r_2^2 = \frac{b_{20}(\alpha_{21}\tilde{\sigma}_1 + \alpha_{12}\tilde{\sigma}_2) - a_{20}(\alpha_{21}\tilde{\sigma}_1 + \alpha_{22}\tilde{\sigma}_2)}{a_{20}b_{02} - a_{02}b_{20}} = 0.18\tilde{\sigma}_1 + 0.1\tilde{\sigma}_2 \,,$$
$$\omega_1 = \omega_{1c} + \beta_{11}\tilde{\sigma}_1 + \beta_{12}\tilde{\sigma}_2 + c_{20}r_1^2 + c_{02}r_2^2 = 1 + 4.18\tilde{\sigma}_1 + 3.95\tilde{\sigma}_2 \,,$$
$$\omega_2 = \omega_{2c} + \beta_{21}\tilde{\sigma}_1 + \beta_{22}\tilde{\sigma}_2 + d_{20}r_1^2 + d_{02}r_2^2 = \sqrt{2} + 4.25\tilde{\sigma}_1 + 1.73\tilde{\sigma}_2 \tag{51}$$

Obviously, when $\tilde{\sigma}_2 - \tilde{\sigma}_1 > 0$ and $0.18\tilde{\sigma}_1 + 0.1\tilde{\sigma}_2 > 0$, there exists 2D tori.

The stability conditions for the quasi-periodic solutions are obtained from the trace and determinant of the Jacobian, given by

$$Tr = 2(a_{20}r_1^2 + b_{02}r_2^2)$$
$$= \frac{2[a_{20}(a_{02} - b_{02})(\alpha_{21}\delta_1 + \alpha_{22}\delta_2) - b_{02}(a_{20} - b_{20})(\alpha_{11}\delta_1 + \alpha_{12}\delta_2)]}{a_{20}b_{02} - a_{02}b_{20}} < 0 \,,$$
$$Det = 4(a_{20}b_{02} - a_{02}b_{20})r_1^2 r_2^2 > 0 \,, \tag{52}$$

i.e., $-1.21\tilde{\sigma}_1 - 0.64\tilde{\sigma}_2 < 0$. The critical lines are illustrated in Fig 4. Here the dashed lines also just mark the regions of different bifurcation solutions. From Fig 4 one can see that there exist stable mixed modes quasi-periodic motions under certain conditions in this case.

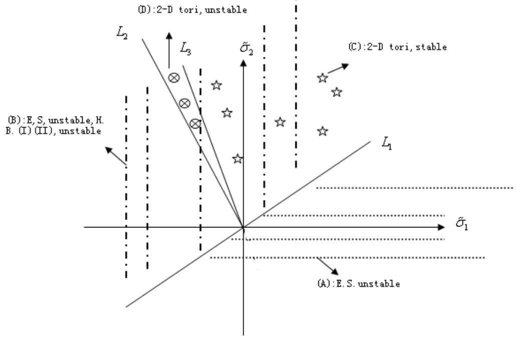


Figure 4. The bifurcation diagram in the case of two pairs of purely imaginary eigenvalues.

4. NUMERICAL SIMULATIONS

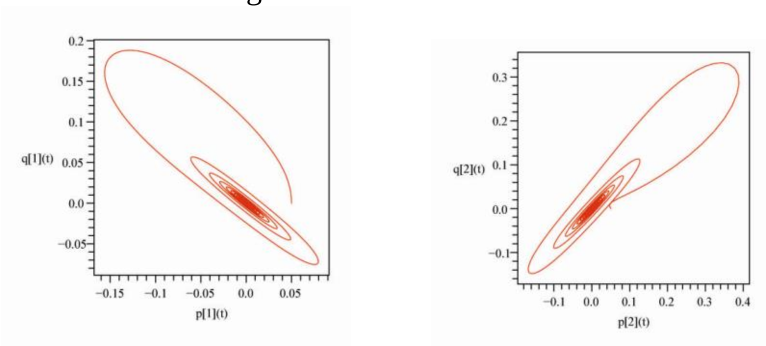
In this section, with the fourth-order Runge-Kutta method, the phase portraits of system (14) are obtained for different values $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$.

For case 1, choosing $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (0.05, 0.15)$, $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (-0.05, 0.1)$, $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (0.05, 0.01)$, respectively, we obtain the phase portraits of system (2.14) as in Fig 5-Fig 7.

Notice that these parameters of $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ are in the stable regions of the initial equilibrium solution, static bifurcation solution and limit cycle, respectively, numerical results agree with the analytical ones.

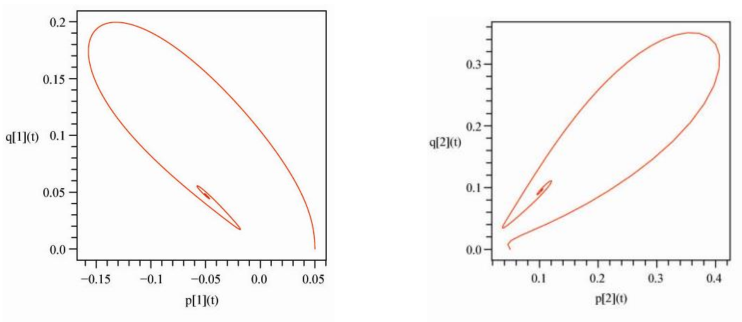
For case 2, choosing $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (0.05, 0.05)$ (in the stable region of the static bifurcation solution), $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (-0.05, -0.1)$ (in the stable region of the Hopf bifurcation solution), respectively, we obtain the phase portraits of system (2.14) as in Fig 8-Fig 9.

For case 3, choosing $(\tilde{\sigma}_1, \tilde{\sigma}_2) = (0.05, 0.05)$ (in the stable region of 2-D tori), the phase portraits are shown as in Fig 10.



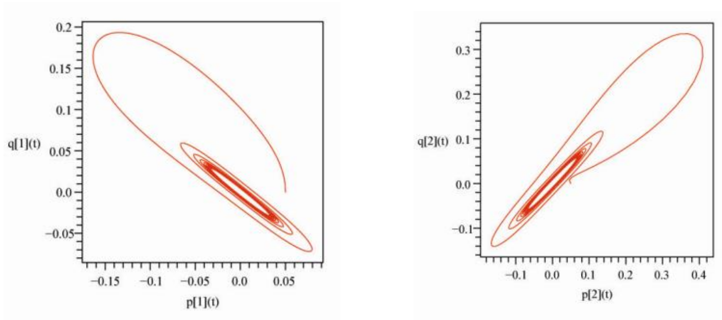
(a) (b)

Figure 5. Trajectory starting from $(p_1, q_1, q_1, q_2) = (0.05, 0, 0.05, 0)$ converges to the E.S. for case 1



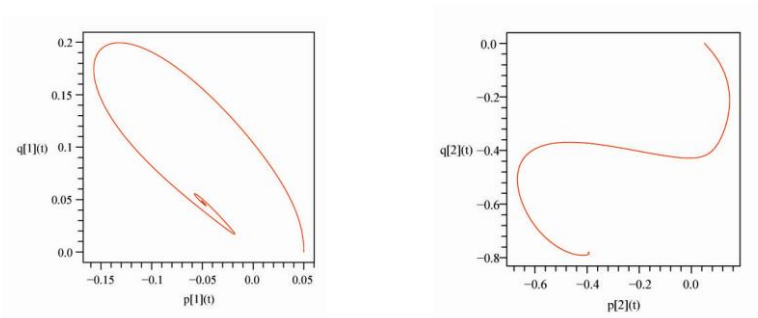
(a) (b)

Figure 6. Trajectory starting from $(p_1, q_1, q_1, q_2) = (0.05, 0, 0.05, 0)$ converges to the S.B. for case 1



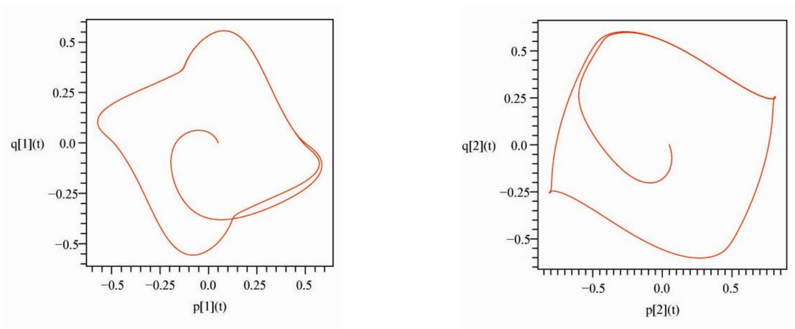
(a) (b)

Figure 7. Trajectory starting from $(p_1, q_1, q_1, q_2) = (0.05, 0, 0.05, 0)$ converges to the L.C. for case 1



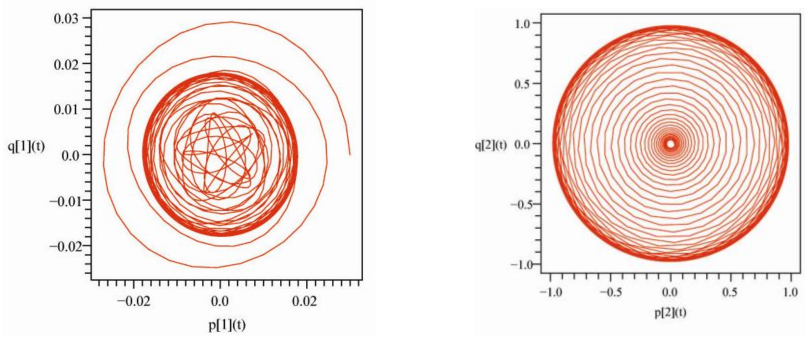
(a) (b)

Figure 8. Trajectory starting from $(p_1, q_1, q_1, q_2) = (0.05, 0, 0.05, 0)$ converges to the S.B. for case 2



(a) (b)

Figure 9. Trajectory starting from $(p_1, q_1, q_1, q_2) = (0.05, 0, 0.05, 0)$ converges to the H.B. for case 2



(a) (b)

Figure 10. Trajectory starting from $(p_1, q_1, q_1, q_2) = (0.05, 0, 0.05, 0)$ converges to the 2-D tori for case 3

5. CONCLUSIONS

With the method of normal forms, the bifurcation solutions and their stability of a hinged-hinged pipe conveying pulsating fluid with combination parametric and internal resonances are studied in detail. When the stability conditions for the initial equilibrium solutions are not satisfied, bifurcations including pitchfork bifurcation, Hopf bifurcation, 2-D tori may occur. Complicated dynamical phenomena of this model are presented here. Numerical simulations agree with the analytical results.

6. ACKNOWLEDGMENTS

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7. APPENDIX

$$J_{11}=J_{12}=J_{14}=J_{23}=1,\quad J_{13}=\frac{\sqrt{6\sqrt{13}+6}}{12}(\sqrt{13}-7),\quad J_{21}=J_{22}=-\frac{\sqrt{6\sqrt{13}+6}}{6},$$

$$J_{24}=J_{31}=-\frac{\sqrt{6\sqrt{13}+6}}{12}(\sqrt{13}+1),\quad J_{32}=-\frac{\sqrt{6\sqrt{13}+6}}{6}-\frac{\sqrt{6\sqrt{13}+6}}{12}(\sqrt{13}+1),$$

$$J_{33}=2,\quad J_{34}=\frac{\sqrt{6\sqrt{13}+6}}{6}(\sqrt{13}+1),\quad J_{41}=-\frac{7}{6}-\frac{\sqrt{13}}{6},\quad J_{43}=\frac{2}{3}\sqrt{6\sqrt{13}+6},$$

$$J_{42}=\frac{5-\sqrt{13}}{12}+\frac{(\sqrt{13}-5)(4+\sqrt{13})}{6},\quad J_{44}=-\frac{2}{3}\sqrt{13}-\frac{14}{3},\tag{A.1}$$

$$a_{11}=\frac{(5\sqrt{13}+113)\tilde{\sigma}_1+(25\sqrt{13}-107)\tilde{\sigma}_2}{4\sqrt{6\sqrt{13}+6}(2\sqrt{13}-5)},\ a_{12}=1+\frac{(63\sqrt{13}+255)\tilde{\sigma}_1-(21\sqrt{13}+117)\tilde{\sigma}_2}{4\sqrt{6\sqrt{13}+6}(2\sqrt{13}-5)},$$

$$a_{13}=\frac{(7\sqrt{13}-32)\tilde{\sigma}_1+(16-9\sqrt{13})\tilde{\sigma}_2}{2(2\sqrt{13}-5)},\quad a_{14}=-\frac{(17\sqrt{13}+35)\tilde{\sigma}_1-(95\sqrt{13}+461)\tilde{\sigma}_2}{2\sqrt{6\sqrt{13}+6}(2\sqrt{13}-5)},$$

$$a_{21}=-\frac{3[(\sqrt{13}+7)\tilde{\sigma}_1+(5\sqrt{13}-13)\tilde{\sigma}_2]}{2\sqrt{6\sqrt{13}+6}(2\sqrt{13}-5)},\quad a_{22}=-\frac{3[(7\sqrt{13}+11)\tilde{\sigma}_1+(3\sqrt{13}-9)\tilde{\sigma}_2]}{4\sqrt{6\sqrt{13}+6}(2\sqrt{13}-5)},$$

$$a_{23}=-\frac{3[(\sqrt{13}-5)\tilde{\sigma}_1+(7-3\sqrt{13})\tilde{\sigma}_2]}{2(2\sqrt{13}-5)},\quad a_{24}=\frac{3[(\sqrt{13}+1)\tilde{\sigma}_1-(7\sqrt{13}+31)\tilde{\sigma}_2]}{\sqrt{6\sqrt{13}+6}(2\sqrt{13}-5)},$$

$$a_{31}=\frac{3[(\sqrt{13}+1)\tilde{\sigma}_1-(\sqrt{13}+1)\tilde{\sigma}_2]}{8(2\sqrt{13}-5)},\quad a_{32}=\frac{9[(\sqrt{13}+3)\tilde{\sigma}_1-(\sqrt{13}+3)\tilde{\sigma}_2]}{16(2\sqrt{13}-5)},$$

$$a_{33}=-1+\frac{(3\sqrt{13}-12)\tilde{\sigma}_1-\tilde{\sigma}_2}{\sqrt{6\sqrt{13}++6}(2\sqrt{13}-5)},\quad a_{34}=\frac{-3[(\sqrt{13}+4)\tilde{\sigma}_1-(7\sqrt{13}+16)\tilde{\sigma}_2]}{4(2\sqrt{13}-5)},$$

$$a_{41} = -\frac{(\sqrt{13}-2)\tilde{\sigma}_1 + (2-\sqrt{13})\tilde{\sigma}_2}{6\sqrt{6\sqrt{13}+6}(2\sqrt{13}-5)}, \quad a_{42} = \frac{-3[(\sqrt{13}+1)\tilde{\sigma}_1 + (3-\sqrt{13})\tilde{\sigma}_2]}{8\sqrt{6\sqrt{13}+6}(2\sqrt{13}-5)},$$
$$a_{43} = \frac{(\sqrt{13}-2)\tilde{\sigma}_1 + (\sqrt{13}-6)\tilde{\sigma}_2}{4(2\sqrt{13}-5)}, \quad a_{44} = -3 + \frac{(\sqrt{13}-8)\tilde{\sigma}_1 + (29-4\sqrt{13})\tilde{\sigma}_2}{\sqrt{6\sqrt{13}+6}(2\sqrt{13}-5)}, \tag{A.2}$$

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