*Mathematical and Computational Applications*

MDPI

*Article*

# An Improved Interval-Valued Hesitant Fuzzy Multi-Criteria Group Decision-Making Method and Applications

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Academic Editor: Mehmet Ali Ilgın
Received: 11 March 2016; Accepted: 30 May 2016; Published: 8 June 2016

**Abstract:** The Bonferroni mean (BM) can be used in situations where the aggregated arguments are correlated. BM is very useful for solving decision-making problems. For describing fuzziness and vagueness more accurately, the interval-valued hesitant fuzzy set (IVHFS), which is a generalization of the hesitant fuzzy set (HFS), can be used to describe the membership degrees with interval numbers. The aim of this paper is to propose the interval-valued hesitant fuzzy Bonferroni mean (IVHFBM) for aggregating interval-valued hesitant fuzzy information. Furthermore, the weighted form of IVHFBM (IVHFWBM) is forwarded and, hereby, a multi-criteria group decision-making (MCGDM) method is established. A case study on the problem of evaluating research funding applications in China is analyzed. A comparison between the proposed method and existing ones demonstrates its practicability.

**Keywords:** Bonferroni mean; HFS; IVHFS; MCGDM

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## 1. Introduction

The Bonferroni mean (BM) is a typical class of average mean, which was first proposed by Bonferroni [1]. It has attracted much attention from scholars in recent years [2–4]. Yager [5] introduced a series of generalized Bonferroni mean operators. The BM operator has been further developed by other scholars [6–8] since then. Xu and Yager [9] combined the BM operators with the intuitionistic fuzzy set [10–12]. Xia *et al.* [13] introduced a series of generalized intuitionistic fuzzy BM operators. Xu and Chen [14] extended the BM operator to an interval-valued intuitionistic fuzzy environment. Zhu and Xu [15] introduced a series of hesitant fuzzy BM operators, while Zhu *et al.* [16] presented some geometric hesitant fuzzy BM operators.

Undeniably, much progress in BM has been made recently. However, a lack of knowledge or insufficient information is frequently encountered in the specific decision problems, and, therefore, it may be helpful for decision makers to express their preference with several interval numbers within [0, 1]. The described phenomenon cannot be effectively processed by a fuzzy set, linguistic fuzzy set, intuitionistic fuzzy set or hesitant fuzzy set. The interval-valued hesitant fuzzy set (IVHFS) [17] is a useful technique for dealing with this situation. Some aggregation operators for aggregating interval-valued hesitant fuzzy information have also been proposed, such as the interval-valued hesitant fuzzy weighted averaging (IVHFWA) operator, IVHFWG, GIVHFWA, GIVHFWG and their ordered and induced forms [17–19].

The existing interval-valued hesitant fuzzy aggregation operators do not consider the relationships between the aggregated arguments. However, the aggregated arguments are correlative, especially in

MCGDM [20–23]. Therefore, it is difficult to use the existing operators in real applications. The aim of this paper is to introduce some new interval-valued hesitant fuzzy aggregation operators such as the interval-valued hesitant fuzzy BM (IVHFBM) and the weighted forms of IVHFBM (IVHFWBM). We also focus on comparing our method with existing ones. We then apply our proposed method to the problem of evaluating research fund applications in China.

The remainder of this paper is organized as follows. Some basic concepts are briefly reviewed in Section 2. Section 3 studies the BM and weighted BM operators in an interval-valued hesitant fuzzy environment. The case of evaluating research fund applications in China is studied in Section 4. A comparison of the proposed method with existing ones is provided in Section 5. In Section 6, we summarize the main contributions of this paper.

## 2. Preliminaries

The IVHFS introduced by Chen *et al.* [17] represents an extension of the hesitant fuzzy set theory [24]. It is chiefly characterized by its membership degree which is represented by several sub-intervals of [0, 1]. It was defined as follows

**Definition 1.** [17] *Let X be a referenced set. An IVHFS on X can be represented as the following mathematical form:*

$$E = \{< x, \tilde{f}_E(x) > | x \in X\} \tag{1}$$

*where $f_E(x)$ is a set of several intervals belonging to the interval [0, 1], which represents the membership degree range of the element $x \in X$ to the set E.*

For any interval-valued hesitant fuzzy elements (IVHFEs), Chen *et al.* [17] defined the operations and gave the comparison rules.

**Definition 2.** *Suppose that $\tilde{h} = \bigcup_{\tilde{\gamma}\in\tilde{h}} \{[\tilde{\gamma}^L, \tilde{\gamma}^U]\}$, $\tilde{h}_1 = \bigcup_{\tilde{\gamma}_1\in\tilde{h}_1} \{[\tilde{\gamma}_1^L, \tilde{\gamma}_1^U]\}$ and $\tilde{h}_2 = \bigcup_{\tilde{\gamma}_2\in\tilde{h}_2} \{[\tilde{\gamma}_2^L, \tilde{\gamma}_2^U]\}$ be three IVHFEs. $\lambda$ is a real number bigger than 0. Then the operations are defined as follows.*

(1) $\tilde{h}^\lambda = \bigcup_{\tilde{\gamma}\in\tilde{h}} \left\{\left[(\tilde{\gamma}^L)^\lambda, (\tilde{\gamma}^U)^\lambda\right]\right\}$

(2) $\lambda\tilde{h} = \bigcup_{\tilde{\gamma}\in\tilde{h}} \left\{\left[1-(1-\tilde{\gamma}^L)^\lambda, 1-(1-\tilde{\gamma}^U)^\lambda\right]\right\}$

(3) $\tilde{h}_1 \oplus \tilde{h}_2 = \bigcup_{\tilde{\gamma}_1\in\tilde{h}_1,\tilde{\gamma}_2\in\tilde{h}_2} \{[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L\tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U\tilde{\gamma}_2^U]\}$

(4) $\tilde{h}_1 \oplus \tilde{h}_2 = \bigcup_{\tilde{\gamma}_1\in\tilde{h}_1,\tilde{\gamma}_2\in\tilde{h}_2} \{[\tilde{\gamma}_1^L\tilde{\gamma}_2^L, \tilde{\gamma}_1^U\tilde{\gamma}_2^U]\}$

**Definition 3.** *For an IVHFE $\tilde{h} = \bigcup_{\tilde{\gamma}\in\tilde{h}} \{[\tilde{\gamma}^L, \tilde{\gamma}^U]\}$, $\#\tilde{h}$ is the number of the elements in $\tilde{h}$.*

$$S(\tilde{h}) = \frac{1}{\#\tilde{h}}\sum_{\tilde{\gamma}\in\tilde{h}}\tilde{\gamma} = \frac{1}{\#\tilde{h}}\sum_{\tilde{\gamma}\in\tilde{h}}\left(\tilde{\gamma}^L + \frac{\tilde{\gamma}^U - \tilde{\gamma}^L}{2}\right) = \frac{1}{\#\tilde{h}}\sum_{\tilde{\gamma}\in\tilde{h}}\left(\frac{\tilde{\gamma}^L + \tilde{\gamma}^U}{2}\right) \tag{2}$$

*is defined as the score function of IVHFE $\tilde{h}$. For two IVHFEs $\tilde{h}_1$ and $\tilde{h}_2$, if $S(\tilde{h}_1) > S(\tilde{h}_2)$, then $\tilde{h}_1 > \tilde{h}_2$; if $S(\tilde{h}_1) = S(\tilde{h}_2)$, then $\tilde{h}_1 \sim \tilde{h}_2$.*

## 3. Interval-Valued Hesitant Fuzzy Information Aggregation with BM Techniques

In order to further enrich the IVHFS theory and expand its range of applications, we develop some new information aggregation techniques for IVHFS with the assistance of BM.

**Definition 4.** [1] *Let $e_i (i = 1, 2, ..., n)$ be a group of real numbers, where all $e_i \geqslant 0$, and $p, q > 0$, then*

$$\mathrm{BM}^{p,q}(e_1, e_2, ..., e_n) = \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} e_i^p e_j^q \right)^{\frac{1}{p+q}} \tag{3}$$

*is called the Bonferroni mean (BM).*

**Definition 5.** *Let $\tilde{h}_j = \bigcup_{\tilde{\gamma}_j \in \tilde{h}_j} \left\{ \left[ \tilde{\gamma}_j^L, \tilde{\gamma}_j^U \right] \right\} (j = 1, 2, ..., n)$ be a group of IVHFEs. If*

$$\mathrm{IVHFBM}\left( \tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n \right) = \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \tilde{h}_i^p \otimes \tilde{h}_j^q \right)^{\frac{1}{p+q}} \tag{4}$$

*then IVHFBM is called the interval-valued hesitant fuzzy BM operator (IVHFBM).*

Based on the operations given in Definition 2, the IVHFBM operator can be changed and Theorem 1 can be obtained.

**Theorem 1.** *Let $p$, $q > 0$, and $\tilde{h}_j = \bigcup_{\tilde{\gamma}_j \in \tilde{h}_j} \left\{ \left[ \tilde{\gamma}_j^L, \tilde{\gamma}_j^U \right] \right\} (j = 1, 2, ..., n)$ be a group of IVHFEs. Using IVHFBM operator, the aggregated IVHFE is obtained as follows:*

$$\mathrm{IVHFBM}\left( \tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n \right) = \bigcup_{\tilde{\gamma}_i \in \tilde{h}_i, \tilde{\gamma}_j \in \tilde{h}_j}$$

$$\left\{ \left[ \left( 1 - \prod_{\substack{i=1, j=1 \\ i \neq j}}^{n} \left( 1 - (\tilde{\gamma}_i^L)^p (\tilde{\gamma}_j^L)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \left( 1 - \prod_{\substack{i=1, j=1 \\ i \neq j}}^{n} \left( 1 - (\tilde{\gamma}_i^U)^p (\tilde{\gamma}_j^U)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right] \right\} \tag{5}$$

**Proof.** According to the operations defined in Definition 2, we know,

$$\tilde{h}_i^p = \bigcup_{\tilde{\gamma}_i \in \tilde{h}_i} \left\{ \left[ \left( \tilde{\gamma}_i^L \right)^p, \left( \tilde{\gamma}_i^U \right)^p \right] \right\} \tag{6}$$

$$\tilde{h}_j^q = \bigcup_{\tilde{\gamma}_j \in \tilde{h}_j} \left\{ \left[ \left( \tilde{\gamma}_j^L \right)^q, \left( \tilde{\gamma}_j^U \right)^q \right] \right\} \tag{7}$$

and

$$\tilde{h}_i^p \otimes \tilde{h}_j^q = \bigcup_{\tilde{\gamma}_i \in \tilde{h}_i, \tilde{\gamma}_j \in \tilde{h}_j} \left\{ \left[ \left( \tilde{\gamma}_i^L \right)^p \left( \tilde{\gamma}_j^L \right)^q, \left( \tilde{\gamma}_i^U \right)^p \left( \tilde{\gamma}_j^U \right)^q \right] \right\} \tag{8}$$

Then

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \tilde{h}_i^p \otimes \tilde{h}_j^q = \bigcup_{\tilde{\gamma}_i \in \tilde{h}_i, \tilde{\gamma}_j \in \tilde{h}_j} \left\{ \left[ 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left( 1 - (\tilde{\gamma}_i^L)^p \left( \tilde{\gamma}_j^L \right)^q \right), 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left( 1 - (\tilde{\gamma}_i^U)^p \left( \tilde{\gamma}_j^U \right)^q \right) \right] \right\} \tag{9}$$

and

$$\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \widetilde{h}_i^p \otimes \widetilde{h}_j^q = \bigcup_{\widetilde{\gamma}_i \in \widetilde{h}_i, \widetilde{\gamma}_j \in \widetilde{h}_j} \left\{ \left[ 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (\widetilde{\gamma}_i^L)^p \left(\widetilde{\gamma}_j^L\right)^q\right)^{\frac{1}{n(n-1)}}, 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (\widetilde{\gamma}_i^U)^p \left(\widetilde{\gamma}_j^U\right)^q\right)^{\frac{1}{n(n-1)}} \right] \right\} \tag{10}$$

Therefore, we have

$$\mathrm{IVHFBM}\left(\widetilde{h}_1, \widetilde{h}_2, \cdots, \widetilde{h}_n\right) = \bigcup_{\widetilde{\gamma}_i \in \widetilde{h}_i, \widetilde{\gamma}_j \in \widetilde{h}_j} \left\{ \left[ \left( 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (\widetilde{\gamma}_i^L)^p \left(\widetilde{\gamma}_j^L\right)^q\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \left( 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (\widetilde{\gamma}_i^U)^p \left(\widetilde{\gamma}_j^U\right)^q\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right] \right\} \tag{11}$$

which completes the proof of Theorem 1.

Through the study of the IVHFBM above, we find that it can adequately reflect the correlations between the aggregated IVHFEs. However, they assign the same importance to the aggregated arguments. Take group decision-making for example: different experts may assign different weights and different criteria, which may make affect the final decision result. Therefore, it is important to study the weighted aggregation operators in aggregating IVHFEs. In the following, we will propose the weighted IVHFBM operator.

**Definition 6.** *Let* $\widetilde{h}_j = \bigcup_{\widetilde{\gamma}_j \in \widetilde{h}_j} \left\{ \left[ \widetilde{\gamma}_j^L, \widetilde{\gamma}_j^U \right] \right\}$ $(j = 1, 2, ..., n)$ *be a group of IVHFEs,* $w = (w_1, w_2, ..., w_n)^T$ *is the weight vector of* $\widetilde{h}_j (j = 1, 2, ..., n)$*, satisfying* $w_i > 0$ $(i = 1, 2, ..., n)$*,* $\sum_{i=1}^{n} w_i = 1$*. If*

$$\mathrm{IVHFWBM}\left(\widetilde{h}_1, \widetilde{h}_2, \cdots, \widetilde{h}_n\right) = \left( \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(w_i \widetilde{h}_i\right)^p \otimes \left(w_j \widetilde{h}_j\right)^p \right)^{1/p+q} \tag{12}$$

*then IVHFWBM is called the interval-valued hesitant fuzzy weighted BM operator.*

**Theorem 2.** *Let* $\widetilde{h}_j = \bigcup_{\widetilde{\gamma}_j \in \widetilde{h}_j} \left\{ \left[ \widetilde{\gamma}_j^L, \widetilde{\gamma}_j^U \right] \right\}$ $(j = 1, 2, ..., n)$ *be a group of IVHFEs,* $w = (w_1, w_2, ..., w_n)^T$ *is the weight vector of* $\widetilde{h}_j (j = 1, 2, ..., n)$*, satisfying* $w_i > 0$ $(i = 1, 2, ..., n)$*, and* $\sum_{i=1}^{n} w_i = 1$*. Then, the IVHFWBM operator can be transformed as follows*

$$IVHFWBM(\widetilde{h}_1, \widetilde{h}_2, ..., \widetilde{h}_n) = \bigcup_{\widetilde{\gamma}_i \in \widetilde{h}_i, \widetilde{\gamma}_j \in \widetilde{h}_j} \left\{ \left[ \left( 1 - \prod_{\substack{i=1,j=1 \\ i \neq j}}^{n} \left(1 - (1 - (1 - \widetilde{\gamma}_i^L)^{w_i})^p (1 - (1 - \widetilde{\gamma}_j^L)^{w_j})^q\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \left( 1 - \prod_{\substack{i=1,j=1 \\ i \neq j}}^{n} \left(1 - (1 - (1 - \widetilde{\gamma}_i^U)^{w_i})^p (1 - (1 - \widetilde{\gamma}_j^U)^{w_j})^q\right)^{\frac{1}{n(n-1)}} \right) \right] \right\} \tag{13}$$

## 4. Research Fund Project Evaluation Problem in China

In this section, we illustrate a new approach with a real example on research fund project evaluation in China based on the idea of MCGDM [25–32].

The main focus and tasks of the National Natural Science Foundation of China are as follows: (1) Foster innovative ideas and enhance the capacity for original innovation in China; (2) Implement a scientific development strategy and promote harmonious development of disciplines; (3) Foster a number of basic and strategic areas. The evaluations conducted by the National Natural Science Foundation of China are very critical. Assume that there are five natural science fund applications that need to be evaluated ($a_1$, $a_2$, $a_3$, $a_4$, $a_5$). According to the assessment guidelines, a group of three experts ($e_1$, $e_2$, $e_3$) take responsibility for this assessment.

When evaluating the five applications, the three experts mainly consider the following four aspects and the weight vector is assigned as (0.2, 0.3, 0.3, 0.2) according to the guidelines set out by the fund board.

$C_1$ = Academic significance and application prospects;
$C_2$ = Innovation;
$C_3$ = Research contents and research objectives;
$C_4$ = Scientific research foundation.

During the evaluation process, the various criteria of the National Natural Science Fund project are interrelated and interacted. Therefore, only when a measurable method is followed to quantify the interrelationship between the criteria can correct and persuasive evaluation results be achieved.

The evaluation undertaken by the National Natural Science Foundation includes many fuzzy factors which are difficult to quantify, including the evaluation criteria and the fuzziness introduced by the the experts. Applying the IVHFS to the evaluation of the National Natural Science Foundation, performance against each criteria is qualitatively and quantitatively assessed. In order to find the best application, two main steps are taken.

Step 1 Obtaining information for the experts to assess

In order to obtain information for evaluation by the experts, Tables 1–3 are designed and sent to the three experts. The blanks in Tables 1–3 should be filled out with numbers between 0 and 1 by the three experts. The following three Tables 1–3 are produced by the three experts.

**Table 1.** Review table for expert $e_1$.

| No. | Criteria | Evaluation Description | Evaluation Results | Minimum Possibility | Maximum Possibility |
|---|---|---|---|---|---|
| 1 | Academic significance and application prospects ($c_1$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_1$ | $a_1$ | 0.3 | 0.7 |
| | | | $a_2$ | 0.4 | 0.6 |
| | | | $a_3$ | 0.3 | 0.6 |
| | | | $a_4$ | 0.5 | 0.8 |
| | | | $a_5$ | 0.2 | 0.4 |
| 2 | Innovation ($c_2$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_2$ | $a_1$ | 0.6 | 0.8 |
| | | | $a_2$ | 0.6 | 0.7 |
| | | | $a_3$ | 0.5 | 0.7 |
| | | | $a_4$ | 0.3 | 0.4 |
| | | | $a_5$ | 0.5 | 0.6 |
| 3 | Research contents and research objectives ($c_3$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_3$ | $a_1$ | 0.6 | 0.8 |
| | | | $a_2$ | 0.2 | 0.4 |
| | | | $a_3$ | 0.1 | 0.3 |
| | | | $a_4$ | 0.4 | 0.6 |
| | | | $a_5$ | 0.3 | 0.5 |
| 4 | Scientific research foundation ($c_4$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_4$ | $a_1$ | 0.5 | 0.9 |
| | | | $a_2$ | 0.4 | 0.5 |
| | | | $a_3$ | 0.6 | 0.7 |
| | | | $a_4$ | 0.3 | 0.5 |
| | | | $a_5$ | 0.5 | 0.6 |

**Table 2.** Review table for expert $e_2$.

| No. | Criteria | Evaluation Description | Evaluation Results | Minimum Possibility | Maximum Possibility |
|---|---|---|---|---|---|
| 1 | Academic significance and application prospects ($c_1$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_1$ | $a_1$<br>$a_2$<br>$a_3$<br>$a_4$<br>$a_5$ | 0.3<br>0.6<br>0.3<br>0.5<br>0.2 | 0.7<br>0.8<br>0.6<br>0.6<br>0.7 |
| 2 | Innovation ($c_2$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_2$ | $a_1$<br>$a_2$<br>$a_3$<br>$a_4$<br>$a_5$ | 0.4<br>0.6<br>0.6<br>0.6<br>0.5 | 0.6<br>0.7<br>0.9<br>0.7<br>0.6 |
| 3 | Research contents and research objectives ($c_3$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_3$ | $a_1$<br>$a_2$<br>$a_3$<br>$a_4$<br>$a_5$ | 0.6<br>0.2<br>0.4<br>0.4<br>0.3 | 0.8<br>0.4<br>0.5<br>0.6<br>0.5 |
| 4 | Scientific research foundation ($c_4$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_4$ | $a_1$<br>$a_2$<br>$a_3$<br>$a_4$<br>$a_5$ | 0.5<br>0.5<br>0.6<br>0.3<br>0.5 | 0.9<br>0.9<br>0.7<br>0.5<br>0.6 |

**Table 3.** Review table for expert $e_3$.

| No. | Criteria | Evaluation Description | Evaluation Results | Minimum Possibility | Maximum Possibility |
|---|---|---|---|---|---|
| 1 | Academic significance and application prospects ($c_1$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_1$ | $a_1$<br>$a_2$<br>$a_3$<br>$a_4$<br>$a_5$ | 0.4<br>0.5<br>0.5<br>0.4<br>0.2 | 0.6<br>0.7<br>0.8<br>0.7<br>0.4 |
| 2 | Innovation ($c_2$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_2$ | $a_1$<br>$a_2$<br>$a_3$<br>$a_4$<br>$a_5$ | 0.5<br>0.6<br>0.6<br>0.6<br>0.3 | 0.6<br>0.7<br>0.7<br>0.7<br>0.6 |
| 3 | Research contents and research objectives ($c_3$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_3$ | $a_1$<br>$a_2$<br>$a_3$<br>$a_4$<br>$a_5$ | 0.6<br>0.5<br>0.4<br>0.4<br>0.3 | 0.8<br>0.6<br>0.5<br>0.6<br>0.5 |
| 4 | Scientific research foundation ($c_4$) | The maximum or minimum possibility that the alternative $a_i$ meets the criterion $c_4$ | $a_1$<br>$a_2$<br>$a_3$<br>$a_4$<br>$a_5$ | 0.5<br>0.5<br>0.6<br>0.4<br>0.3 | 0.9<br>0.9<br>0.7<br>0.4<br>0.6 |

The comprehensive evaluation information is presented in Table 4 as follows.

**Table 4.** Interval-valued hesitant fuzzy decision matrix.

| Alternatives | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $a_1$ | {[0.3, 0.7], [0.4, 0.6]} | {[0.6, 0.8], [0.4, 0.6], [0.5, 0.6]} | {[0.6, 0.8]} | {[0.5, 0.9]} |
| $a_2$ | {[0.4, 0.6], [0.6, 0.8], [0.5, 0.7]} | {[0.6, 0.7]} | {[0.2, 0.4], [0.5, 0.6]} | {[0.4, 0.5], [0.5, 0.9]} |
| $a_3$ | {[0.3, 0.6], [0.5, 0.8]} | {[0.5, 0.7], [0.6, 0.9], [0.6, 0.7]} | {[0.1, 0.3], [0.4, 0.5]} | {[0.6, 0.7]} |
| $a_4$ | {[0.5, 0.8], [0.5, 0.6], [0.4, 0.7]} | {[0.3, 0.4], [0.6, 0.7]} | {[0.4, 0.6]} | {[0.3, 0.5], [0.4, 0.4]} |
| $a_5$ | {[0.2, 0.4], [0.2, 0.7]} | {[0.5, 0.6], [0.3, 0.6]} | {[0.3, 0.5]} | {[0.5, 0.6], [0.3, 0.6]} |

Take the evaluation results of alternative $a_1$ against criterion $C_1$ for example (the information provided by three experts in the Tables 1–3 is marked in red), the first expert believes that the minimum possibility that the alternative $a_1$ meets the criterion $C_1$ is 0.3 and the maximum possibility is 0.7. Coincidentally, the second expert has the same opinion as the first expert about this evaluation. The third expert argues that the minimum possibility that the alternative $a_1$ meets the criterion $C_1$ is 0.4 and the maximum possibility is 0.6. In this case, the performance of alternative $a_1$ against criterion $C_1$ can be expressed by an IVHFE {[0.3, 0.7], [0.4, 0.6]}. Similarly, other IVHFEs in Table 4 expressed the corresponding assessment information.

Step 2 Ranking the applications

According to the assessment information provided in Table 4, we use the IVHFWBM operator to determine performance against the four criteria and obtain a comprehensive evaluation result for each alternative. The scores for the comprehensive IVHFEs are shown in Table 5.

**Table 5.** Scores obtained by the IVHFWBM operator and the ranking of alternatives.

| Parameters | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | Ranking |
|---|---|---|---|---|---|---|
| $p = 0.001, q = 10$ | 0.2860 | 0.2538 | 0.2601 | 0.2073 | 0.1741 | $a_1 \succ a_3 \succ a_2 \succ a_4 \succ a_5$ |
| $p = q = 5$ | 0.2579 | 0.2163 | 0.2135 | 0.1792 | 0.1540 | $a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5$ |
| $p = 10, q = 10$ | 0.2738 | 0.2235 | 0.2260 | 0.1595 | 0.1021 | $a_1 \succ a_3 \succ a_2 \succ a_4 \succ a_5$ |

From the scores obtained by the IVHFWBM operator described in Table 5, we find that:

1. Different values of parameters $p$ and $q$ could lead to different scores and a different ranking order. For example, when $p = q = 5$, the second order of the alternative is $a_2$ while this result varies when $p$ and $q$ were set with other values.
2. The scores obtained by the IVHFWBM operator are symmetrical. For example, the scores are not changed between $p = 0.001$ ($q = 10$) and $p = 10$ ($q = 0.001$). This result verified the phenomenon described in Section 3.
3. Although the ranking order varied when $p$ and $q$ was set to different values, the best alternative is always $a_1$. This indicates that the ranking method based on IVHFWBM operators offers good stability.

In the following, we will investigate the changing trend of the scores for the aggregated IVHFEs based on the IVHFWBM operator when one of the two parameters is fixed. As mentioned above, the parameters $p$ and $q$ have same effect on the aggregated results; therefore, we only study the case that the values of parameter p are fixed.

(1) $p = 0.1, q \in [0, 10]$

From Figure 1, we find that

- ♦ When $p = 0.1$ and $q \in (0, 2.963)$, the ranking order of the five alternatives is

$$a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5$$

- ♦ When $p = 0.1$ and $q \in (2.963, 10)$, the ranking order of the five alternatives is

$$a_1 \succ a_3 \succ a_2 \succ a_4 \succ a_5$$

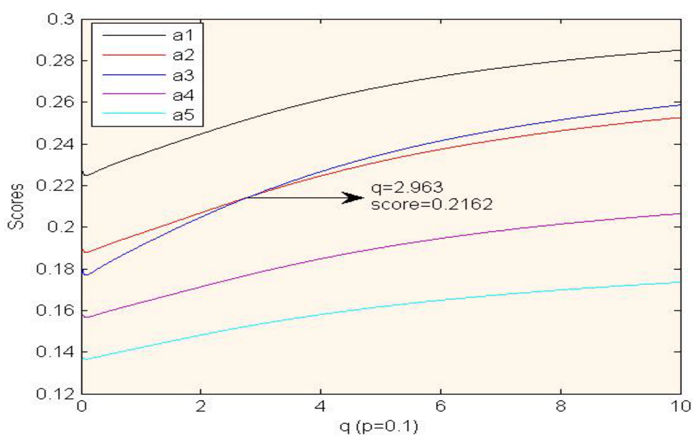


**Figure 1.** Comparison between five alternatives based on IVHFWBM operator.

(2) $p = 10, q \in [0, 10]$

From Figure 2, we find that

- When $p = 10$ and $q \in (0, 4.331]$, the ranking order of the five alternatives is

$$a_1 \succ a_3 \succ a_2 \succ a_4 \succ a_5$$

- When $p = 10$ and $q \in (4.331, 8.911)$, the ranking order of the five alternatives is

$$a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5$$

- When $p = 10$ and $q \in (8.911, 10)$, the ranking order of the five alternatives is

$$a_1 \succ a_3 \succ a_2 \succ a_4 \succ a_5$$

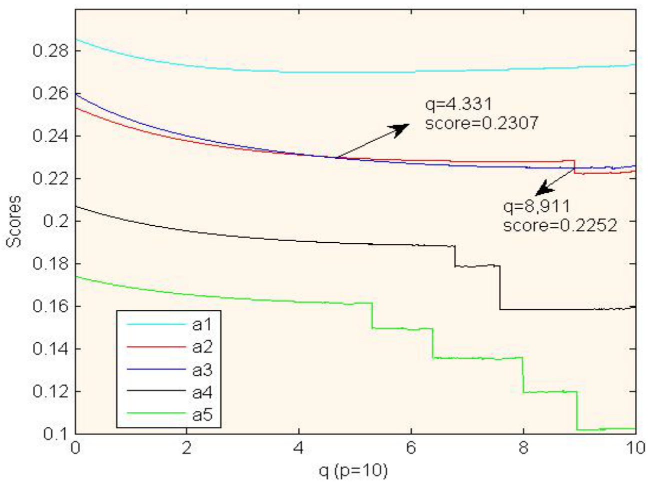


**Figure 2.** Comparison between five alternatives based on IVHFWBM operator.

Furthermore, the scores of the five alternatives can be depicted in Figures 3–7 clearly when the values of two parameters change simultaneously.

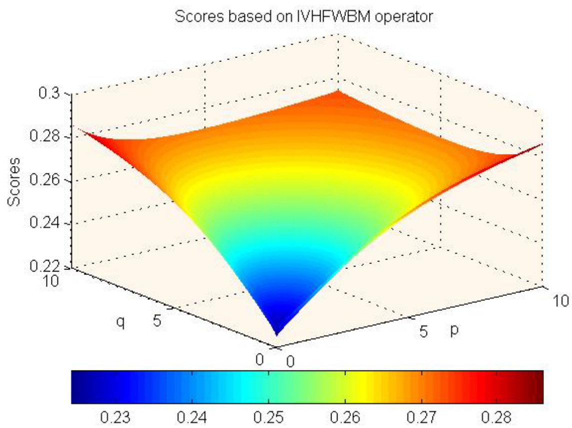


**Figure 3.** Scores for alternative $a_1$ obtained by the IVHFWBM operator.

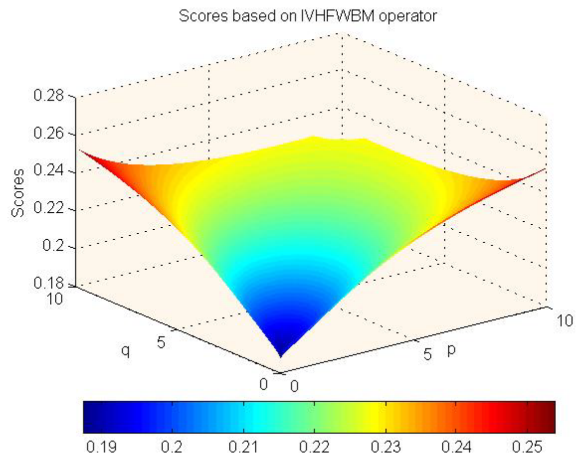


**Figure 4.** Scores for alternative $a_2$ obtained by the IVHFWBM operator.

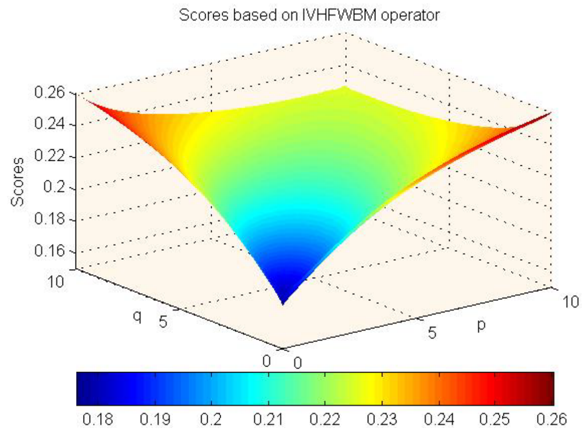


**Figure 5.** Scores for alternative $a_3$ obtained by the IVHFWBM operator.

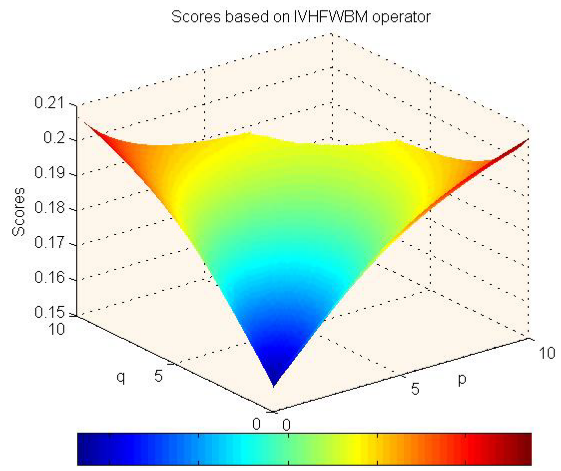


**Figure 6.** Scores for alternative $a_4$ obtained by the IVHFWBM operator.

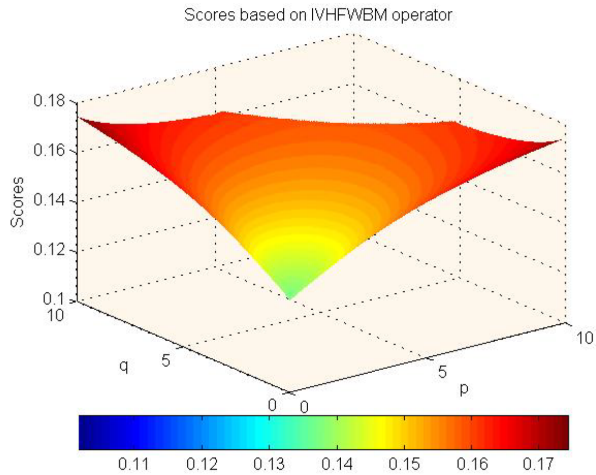


**Figure 7.** Scores for alternative $a_5$ obtained by the IVHFWBM operator.

## 5. Comparison of the Proposed Approach with Other Approaches

In this section, we make a comparison of the assessment methods with another approach proposed by Chen *et al.* [17]. In order to determine the difference between these two methods, we use the IVHFWA operator based method to solve the research project evaluation problem described in Section 4.

Based on the IVHFWA operator, the score functions of each alternative are computed and the results are shown as follows.

$$S_1 = 0.1377; S_2 = 0.1995; S_3 = 0.1950; S_4 = 0.1656; S_5 = 0.1435$$

Based on the score functions, we can rank the five alternatives.

$$a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_1$$

The best alternative is now $a_2$. The main reason for this is that the IVHFWA operator does not consider the correlations between the aggregated arguments which are not perfect. For example, the criterion $C_1$ (academic significance and application prospects) is correlated with $C_3$ (research contents and research objectives) in real cases. If we consider these two criteria as independent, the corresponding result must be considered as doubtful.

## 6. Concluding Remarks

We studied BM in the interval-valued hesitant fuzzy environment. We proposed some BM operators for aggregating IVHFEs, such as IVHFBM and IVHFWBM operators. Furthermore, we applied the IVHFWBM operator to an evaluation problem concerning the assessment of research funding applications in China. We also compared the methods based on the IVHFWBM operator with other methods, such as the approach proposed by Chen *et al.* [17]. In future research, we intend to focus on the application of the developed method to other fields such as supplier evaluation and the selection of financial products.

**Author Contributions:** Ding Zhenhua and Wu Yingyu collectively determined the problem, carried out this study. Ding Zhenhua wrote the manuscript. Wu Yingyu helped to revise the manuscript. All authors read and approved the final manuscript.

**Conflicts of Interest:** The authors declare no conflict of interest.

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