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Article

New Analytic Solutions for the (N + 1)-Dimensional Generalized Boussinesq Equation

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Abstract: In this paper, the generalized Jacobi elliptic functions expansion method with computerized symbolic computation are employed to investigate explicitly analytic solutions of the (N + 1)-dimensional generalized Boussinesq equation. The exact solutions to the equation are constructed analytically under certain circumstances, some of these solutions are degenerated to soliton-like solutions and trigonometric function solutions in the limit cases when the modulus of the Jacobi elliptic function solutions tends to 0 and 1, which shows that the applied method is more powerful and will be used in further works to establish more entirely new exact solutions for other kinds of higher-dimensional nonlinear partial differential equations in mathematical physics.

Keywords: generalized Jacobi elliptic functions expansion method; generalized Boussinesq equation; analytic solutions; soliton-like solutions; Jacobi elliptic function solutions

1. Introduction

In recent years, due to the wide applications of soliton theory in natural science, searching for exact soliton solutions of nonlinear evolution equations plays an important and significant role in the study on the dynamics of those phenomena [1,2]. Particularly, various powerful methods have been presented, such as inverse scattering transformation, Cole-Hopf transformation, sinecosine method, Painlevé method, Lie group analysis, similarity reduced method, Hirota bilinear method, homogeneous balance method, Bäcklund transformation, Darboux transformation, the extended tanh-function method, the extended F-expansion method, projective Riccati equations method, the Jacobi elliptic function expansion method and so on. In this paper, we would like to discuss an (N + 1)-dimensional generalized Boussinesq equation by our generalized Jacobi elliptic functions expansion method [3] proposed recently. As a result, more new exact solutions are obtained. The character feature of our method is that, without much extra effort, we can get series of exact solutions using a uniform way. Another advantage of our method is that it also applies to general higher-dimensional nonlinear partial differential equations.

This paper is arranged as follows. In Section 2, we briefly describe the generalized Jacobi elliptic function expansion method. In Section 3, several families of solutions to the higher-dimensional generalized Boussinesq equation are obtained. In Section 4, some conclusions are given.

2. Summary of the Generalized Jacobi Elliptic Functions Expansion Method

For a given partial differential equation in $N+1$ variables x,t and y_j ($j = 1, \ldots, N-1$)

$$P(u, u_t, u_x, u_{y_1}, u_{y_2}, \cdots, u_{y_{N-1}}, u_{tt}, u_{xx}, \cdots) = 0 \tag{1}$$

We seek the following formal solutions of the given system by a new intermediate transfoumation:

$$u(\xi)=\sum_{i=0}^{k}A_iF^i(\xi)+\sum_{i,j=1;i\leqslant j\leqslant k}^{k}[B_iF^{j-i}(\xi)E^i(\xi)+C_iF^{j-i}(\xi)G^i(\xi)+D_iF^{j-i}(\xi)H^i(\xi)] \tag{2}$$

where A_0, A_i, B_i, C_i, D_i, (i = 1,2, ... ,k) are time-dependent functions to be determined later. $\xi=\xi(x,t,y_1,\ldots,y_{N-1})$ are arbitary functions with the variables x,t and y_j (j = 1, ... ,N − 1). The parameter k can be determined by balancing the highest order derivative terms with the nonlinear terms in Equation (2). And E(ξ)F(ξ)G(ξ)H(ξ) are an arbitrary array of the four functions $e=e(\xi)f=f(\xi)g=g(\xi)$ and $h=h(\xi)$, the selection obey the principle which makes the calculation more simple. Here we ansatz

$$\begin{cases} e=\frac{1}{p+qsn\xi+rcn\xi+ldn\xi}, f=\frac{sn\xi}{p+qsn\xi+rcn\xi+ldn\xi} \\ g=\frac{cn\xi}{p+qsn\xi+rcn\xi+ldn\xi}, h=\frac{dn\xi}{p+qsn\xi+rcn\xi+ldn\xi} \end{cases} \tag{3}$$

where p, q, r, l are arbitrary constants, the four function e, f, g, h satisfy the following restricted relation:

$$\begin{cases} e'=-qgh+rfh+lm^2fg, f'=pgh+reh+leg, \\ g'=-pfh-qeh+l(m^2-1)ef, h'=-m^2pfg-r(m^2-1)ef-qeg \end{cases} \tag{4}$$

where "$'$" denotes $\frac{d}{d\xi}$.m is the modulus of the Jacobi elliptic function($0\leqslant m\leqslant 1$), and e, f, g, h satisfy one of the following relation at the same time.

Family 1: when p = 0, we can select $F(\xi)=f(\xi)$ or $g(\xi)$, using the following iterative restrictions

$$\begin{cases} lh=1-qf-rg, e^2=f^2+g^2, \\ (l^2-r^2)g^2=1-2(qf+rg-qrfg)+(l^2m^2-l^2+q^2)f^2 \end{cases} \tag{5a}$$

Family 2: when q = 0, we can select $F(\xi)=g(\xi)$ or $h(\xi)$, using the following iterative restrictions

$$\begin{cases} pe=1-rg-lh, (m^2-1)f^2=g^2-h^2, \\ (l^2(m^2-1)+p^2)h^2=1-m^2+2(m^2-1)(lh+rg-rlgh)+(p^2m^2+r^2-m^2r^2)g^2 \end{cases} \tag{5b}$$

Family 3: when r = 0, we can select $F(\xi)=h(\xi)$ or $e(\xi)$, using the following iterative restrictions

$$\begin{cases} qf=1-pe-lh, m^2g^2=h^2+(m^2-1)e^2, \\ (q^2-m^2p^2)e^2=m^2-2m^2(lh+pe-pleh)+(l^2m^2+q^2)h^2 \end{cases} \tag{5c}$$

Family 4: when l = 0, we can select $F(\xi)=e(\xi)$ or $f(\xi)$, using the following iterative restrictions

$$\begin{cases} rg=1-pe-qf, h^2=e^2-m^2f^2, \\ (q^2+r^2)f^2=-1+2(pe+qf-pqef)+(r^2-p^2)e^2 \end{cases} \tag{5d}$$

Substituting (4) along with (5a–5d) into Equation (1) separately yields four families of polynomial equations for $E(\xi)F(\xi)G(\xi)H(\xi)$. Setting the coefficients of $F^i(\xi)E^{j_1}(\xi)G^{j_2}(\xi)H^{j_3}(\xi)(i=0,1,2,\cdots;j_{1,2,3}=0,1;j_1j_2j_3=0)$ to zero yields a set of over-determined differential equations(ODEs) in $A_0A_iB_iC_iD_i$, (i = 1,2, ... ,k) and ξ, solving the ODEs by Mathematica and Wu elimination, we can obtain many exact solutions of Equation (1) accroding to (2)–(3) and (5a)–(5d).

3. Exact Solutions of the Equation

Let us consider the following $(N+1)$-dimensional generalized Boussinesq equation

$$u_{tt} = \beta u_{xx} + \lambda (u^n)_{xx} + \gamma u_{xxxx} + \alpha \sum_{j=1}^{N-1} u_{y_j y_j} \tag{6}$$

where $u = u(x, y_1, y_2, \cdots, y_{N-1}, t)$, $\beta \neq 0, \lambda \neq 0, \gamma \neq 0, n \neq 1 > 0$ is a real number and $N > 1$ is an integer. The semi-travelling wave similarity transformation was used in [4] to investigate Equation (6) with $\alpha = \beta = \gamma = 1$ and acquired many types of its exact solutions. The auxiliary differential equation approach is employed in [5] to investigate some new exact solutions of Equation (6) under the same circumstance. Equation (6) includes a class of $(1+1)$-dimensional and $(2+1)$-dimensional modified Boussinesq equations.

In fact, if one takes $\beta = \gamma = \lambda = 1$, $\alpha = 0$, $n = 2$, Equation (6) represents the well-known Boussinesq equation [6]

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx} \tag{7}$$

which describes the propagation of long waves on the surface of water with a small amplitude and plays an important role in fluid mechanics [7].

In fact, if one takes $\beta = \gamma = \lambda = 1$, $\alpha = 0$, $n = 3$, Equation (6) represents the modified Boussinesq equation

$$u_{tt} = u_{xx} + (u^3)_{xx} + u_{xxxx} \tag{8}$$

which can be regarded as the continuous limit of a FPU dynamical system with cubic nonlinearity [8] and some similarity reductions of (8) were obtained [9].

If one takes $\beta = \gamma = \lambda = 1$, $\alpha = 1$, $n = 2$, $N = 2$, Equation (6) represents the classical $(2+1)$-dimensional Boussinesq equation [10,11]

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx} + u_{yy} \tag{9}$$

El-Sayed and Kaya [12] considered the approximate solution of Equation (9) with initial value. If one takes $\lambda = 8$, $\alpha = \beta = \gamma = 1$, $n = 3$, $N = 2$, Equation (6) represents $(2+1)$-dimensional generalized Boussinesq equation

$$u_{tt} = u_{xx} + 8(u^3)_{xx} + u_{xxxx} + u_{yy} \tag{10}$$

Matsukawa and Watanabe [13] used the bilinear method to obtained several N-soliton solutions of Equation (10). Some other research about Equation (6) can be seen in [14–18]. In the following, we construct exact solutions of Equation (6).

Making the gauge transformation

$$\xi = \tau(x + \sum_{j=1}^{N-1} l_j y_j + ct) \tag{11}$$

where τ, l_j, c are constants to be determined later

We have

$$(\alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2) u_{\xi\xi} + \lambda (u^n)_{\xi\xi} + \gamma \tau^2 u_{\xi\xi\xi\xi} = 0 \tag{12}$$

Integrating (12) about ξ and ignoring the constant of integration give rise to

$$(\alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2) u_{\xi} + \lambda (u^n)_{\xi} + \gamma \tau^2 u_{\xi\xi\xi} = 0 \tag{13}$$

Using transformation $u = v^{\frac{1}{n-1}}$ yield

$$(\alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2)v^2 v_\xi + n\lambda v^3 v_\xi + \gamma\tau^2[\frac{(n-2)(2n-3)}{(n-1)^2}(v_\xi)^3 + \frac{3(2-n)}{n-1} v v_\xi v_{\xi\xi} + v^2 v_{\xi\xi\xi}] = 0 \quad (14)$$

where $v = v(x, y_1, y_2, \cdots, y_{N-1}, t) v_\xi = \frac{dv}{d\xi}, v_{\xi\xi} = \frac{d^{(2)}v}{d\xi^2}, v_{\xi\xi\xi} = \frac{d^{(3)}v}{d\xi^3}$.

By balancing the term $n\lambda v^3 v_\xi$ and $\gamma\tau^2 v^2 v_{\xi\xi\xi}$ in (14), we obtain $N = 2$, thus we assume that the solutions of Equation (14) is expressed in the form

$$\begin{aligned} v = c_0 + c_1 e + c_2 f + c_3 g + c_4 h + d_1 e^2 + d_2 f^2 + d_3 g^2 + d_4 h^2 \\ + d_5 fg + d_6 fh + d_7 gh + d_8 ef + d_9 eg + d_{10} eh \end{aligned} \quad (15)$$

where $v = v(\xi), e = e(\xi), f = f(\xi), g = g(\xi), h = h(\xi), c_i, d_j (i = 0, \cdots, 4; j = 1, \cdots, 10)$ are constants to be determined later and e, f, g, h satisfy (4) and (5a–5d) .

Substituting (4) and (5a–5d) separately along with (11) into (14) and setting the coefficients of $F^i(\xi)E^{j_1}(\xi)G^{j_2}(\xi)H^{j_3}(\xi)(i = 0, 1, 2, \cdots)(j_{1\ldots3} = 0, 1, j_1 j_2 j_3 = 0)$ to zero yields an ODEs with respect to the unknowns $c_i(i = 0, \cdots, 4), d_j(j = 1, \cdots, 10), \tau, c, p, q, r, l, m$. After solving the ODEs by Mathematica and Wu elimination we could determine the following solutions:

State 1 $n = 2$

Case 1

$$\begin{aligned} & p = 0, r = l = 1, q = \pm 1, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(7 - 8m^2), \\ & c_2 = \pm\tfrac{3\gamma\tau^2(m^2-2)}{\lambda}, c_4 = \tfrac{6\gamma\tau^2(m^2-1)}{\lambda}, d_2 = \tfrac{3\gamma\tau^2(m^2-2)^2}{2\lambda}, d_4 = -\tfrac{6\gamma\tau^2}{\lambda} \end{aligned}$$

Case 2

$$\begin{aligned} & q = 0, p = \sqrt{1-m^2}, l = 1, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(m^2 - 2 + 3\varepsilon\sqrt{1-m^2}), \\ & r = \pm(\sqrt{1-m^2} - \varepsilon), \varepsilon = \pm 1, c_3 = \pm\tfrac{3\gamma\tau^2((m^2-1)\varepsilon + \sqrt{1-m^2})}{\lambda} \end{aligned}$$

Case 3

$$\begin{aligned} & q = 0, p = \sqrt{1-m^2}, l = 1, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(m^2 - 2 + 3\varepsilon\sqrt{1-m^2}), \\ & r = \mp m, \varepsilon = \pm 1, c_1 = \tfrac{6\gamma\tau^2 m^2\sqrt{1-m^2}}{\lambda}, d_1 = \tfrac{6\gamma\tau^2 m^2(1-m^2)}{\lambda}, \end{aligned}$$

Case 4

$$\begin{aligned} & r = 0, p = l = 1, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(m^2 - 2 - 3\varepsilon\sqrt{1-m^2}), \\ & q = \pm\varepsilon(1 + \varepsilon\sqrt{1-m^2}), \varepsilon = \pm 1, c_2 = \pm\tfrac{3\gamma\tau^2(\varepsilon(1-m^2) + \sqrt{1-m^2})}{\lambda} \end{aligned}$$

Case 5

$$\begin{aligned} & r = 0, p = l = 1, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(1 + 4m^2), \\ & q = \mp mi, i = \sqrt{-1}, c_2 = \pm\tfrac{6\gamma\tau^2 mi}{\lambda}, d_2 = -\tfrac{6\gamma\tau^2 m^2}{\lambda} \end{aligned}$$

Case 6

$$l = 0, p = q = 1, r = \pm i, i = \sqrt{-1}, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(1 - 3m + m^2), c_3 = \pm\frac{3\gamma\tau^2 mi}{\lambda}$$

Case 7

$$l = 0, p^2 = 1, q^2 = 1, r = \pm 1, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(4m^2 - 5), c_3 = \pm\frac{6\gamma\tau^2}{\lambda}, d_3 = -\frac{6\gamma\tau^2}{\lambda}$$

Case 8

$$p = l = 0, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(m^2 - 18\sqrt{1-m^2} - 2),$$
$$d_5 = \frac{6\gamma\tau^2 r^2 \sqrt[4]{1-m^2}(m^2 - 2\sqrt{1-m^2} - 2)}{\lambda}, q = \pm r\sqrt[4]{1-m^2}$$

Case 9

$$r = l = 0, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + 2\gamma\tau^2(1 + m^2), d_3 = \frac{3\gamma\tau^2(1-m^2)p^2}{2\lambda}, q = \pm p$$

Case 10

$$r = l = 0, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(1 - 18m + m^2), d_8 = \pm\frac{6\gamma\tau^2 p^2 \beta(1-m)^2\sqrt{m}}{\lambda}, q = \mp p\sqrt{m}$$

Case 11

$$r = l = 0, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(1 - 5m^2), c_2 = \mp\frac{3\gamma\tau^2 m(1-m^2)p}{\lambda}, q = \pm mp$$

Case 12

$$q = l = 0, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(m^2 - 2), c_4 = \pm\frac{3\gamma\tau^2 r}{2\lambda}, c_3 = \frac{3\gamma\tau^2 r}{2\lambda}, p = \varepsilon r, \varepsilon = \pm 1$$

Case 13

$$p = q = 0, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(1 - 18m + m^2), d_7 = \pm\frac{6\gamma\tau^2 l^2 \beta(1-m)^2\sqrt{m}}{\lambda}, r = \mp l\sqrt{m}$$

Case 14

$$p = q = 0, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = -2\lambda c_0 + \gamma\tau^2(m^2 - 2), r = \varepsilon l, \varepsilon = \pm 1,$$
$$c_2 = \pm\frac{3\gamma\tau^2 l(m^2-1)}{2\lambda}, c_4 = -\frac{3\gamma\tau^2 l(m^2-1)}{2\lambda}$$

where c $\neq 0$, τ $\neq 0$, l_j are arbitrary constants in Case 1–Case 14. c_i, d_j don't mention in all above cases is zero. So do the following situations. Therefore, from (3), (11), (15), Cases 1–14 and $u = v^{\frac{1}{n-1}}$, we obtain the Jacobi elliptic wave-like solutions to Equation (6):

$$u_{1.1} = c_0 + \frac{\pm\frac{3\gamma\tau^2(m^2-2)}{\lambda} sn\xi_1 + \frac{6\gamma\tau^2(m^2-1)}{\lambda} dn\xi_1}{\pm sn\xi_1 + cn\xi_1 + dn\xi_1} + \frac{\frac{3\gamma\tau^2(m^2-2)^2}{2\lambda} sn^2\xi_1 - \frac{6\gamma\tau^2}{\lambda} dn^2\xi_1}{(\pm sn\xi_1 + cn\xi_1 + dn\xi)^2}$$

$$u_{1.2} = c_0 \pm \frac{\frac{3\gamma\tau^2((m^2-1)\varepsilon + \sqrt{1-m^2})}{\lambda} cn\xi_2}{\sqrt{1-m^2} \pm (\sqrt{1-m^2} - \varepsilon)cn\xi_2 + dn\xi_2}$$

$$u_{1.3} = c_0 + \frac{\frac{6\gamma\tau^2 m^2\sqrt{1-m^2}}{\lambda}}{\sqrt{1-m^2} \mp mcn\xi_3 + dn\xi_3} + \frac{\frac{6\gamma\tau^2 m^2(1-m^2)}{\lambda}}{(\sqrt{1-m^2} \mp mcn\xi_3 + dn\xi_3)^2}$$

$$u_{1.4} = c_0 + \frac{\pm\frac{3\gamma\tau^2(\varepsilon(1-m^2)+\sqrt{1-m^2})}{\lambda}sn\xi_4}{1 \pm \varepsilon(1+\varepsilon\sqrt{1-m^2})sn\xi_4 + dn\xi_4}$$

$$u_{1.5} = c_0 \pm \frac{\frac{6\gamma\tau^2 mi}{\lambda}sn\xi_5}{1 \mp misn\xi_5 + dn\xi_5} - \frac{\frac{6\gamma\tau^2 m^2}{\lambda}sn^2\xi_5}{(1 \mp misn\xi_5 + dn\xi_5)^2}$$

$$u_{1.6} = c_0 \pm \frac{\frac{3\gamma\tau^2 mi}{\lambda}cn\xi_6}{1 + sn\xi_6 \pm icn\xi_6}$$

$$u_{1.7} = c_0 \pm \frac{\frac{6\gamma\tau^2}{\lambda}cn\xi_7}{1 \pm sn\xi_7 + \varepsilon cn\xi_7} - \frac{\frac{6\gamma\tau^2}{\lambda}cn^2\xi_7}{(1 \pm sn\xi_7 + \varepsilon cn\xi_7)^2},$$

$$u_{1.8} = c_0 + \frac{6\gamma\tau^2\sqrt[4]{1-m^2}(m^2 - 2\sqrt{1-m^2} - 2)}{\lambda}\frac{sn\xi_8 cn\xi_8}{(\pm\sqrt[4]{1-m^2}sn\xi_8 + cn\xi_8)^2}$$

$$u_{1.9} = c_0 + \frac{3\gamma\tau^2(1-m^2)}{2\lambda}\frac{cn^2\xi_9}{(1 \pm sn\xi_9)^2}$$

$$u_{1.10} = c_0 \pm \frac{6\gamma\tau^2(1-m)^2\sqrt{m}}{\lambda}\frac{sn\xi_{10}}{(1 \mp \sqrt{m}sn\xi_{10})^2}$$

$$u_{1.11} = c_0 \mp \frac{3\gamma\tau^2 m(1-m^2)}{\lambda}\frac{sn\xi_{11}}{1 \pm msn\xi_{11}}$$

$$u_{1.12} = c_0 + \frac{3\gamma\tau^2}{2\lambda}\frac{cn\xi_{12} \pm dn\xi_{12}}{\varepsilon \pm cn\xi_{12}}$$

$$u_{1.13} = c_0 \pm \frac{6\gamma\tau^2(1-m)^2\sqrt{m}}{\lambda}\frac{cn\xi_{13}dn\xi_{13}}{(\mp\sqrt{m}cn\xi_{13} + dn\xi_{13})^2}$$

$$u_{1.14} = c_0 + \frac{3\gamma\tau^2(m^2-1)}{2\lambda}\frac{\pm sn\xi_{14} - dn\xi_{14}}{\varepsilon cn\xi_{14} + dn\xi_{14}}$$

$$\xi_i = \tau(x + \sum_{j=1}^{N-1} l_j y_j \pm t\sqrt{\alpha\sum_{j=1}^{N-1} l_j^2 + \beta + 2\lambda c_0 - \gamma\tau^2\Delta_i}), (i = 1, \cdots, 14)$$

Remark 1: If we let $\beta = \gamma = \alpha = 1, \sum_{j=1}^{N-1} l_j^2 = N-1, c_0 = 0, \tau = \sqrt{\frac{N-c^2}{2(1+m^2)}}$, $u_{1.9}$ is equivalent to the solution $u_{1.23}$ given in [5]. If we select the corresponding parameter, we can get the solutions from $u_{1.1}$ to $u_{1.25}$ given in [5]. Solutions $u_{1.i}(\xi_i)(i = 1, 7, 12)$ are degenerated to soliton-like solutions when the modulus $m \to 1$, and solutions $u_{1.i}(\xi_i)(i = 1, 2, 4, 7, 8, 9, 12, 14)$ are degenerated to trigonometric functions solutions when the modulus $m \to 0$.

Here, $u_{1.1}$ provides us with a compound Jacobi wave solution whose structure are shown in Figure 1. The typical structure of new Jacobi elliptic wave-like solution $u_{1.8}$ is shown in Figure 2.

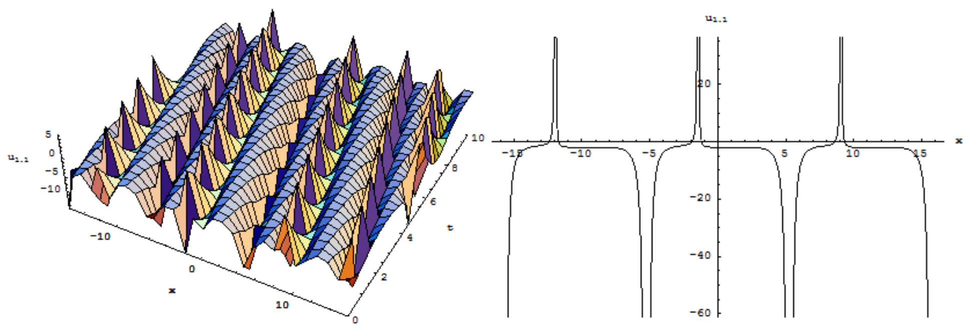


Figure 1. Solution $u_{1.1}$ when $c_0 = 0, \tau = \lambda = l_j = \alpha = \beta = \gamma = N = 1, m = 0.9$ and $t = 0$.

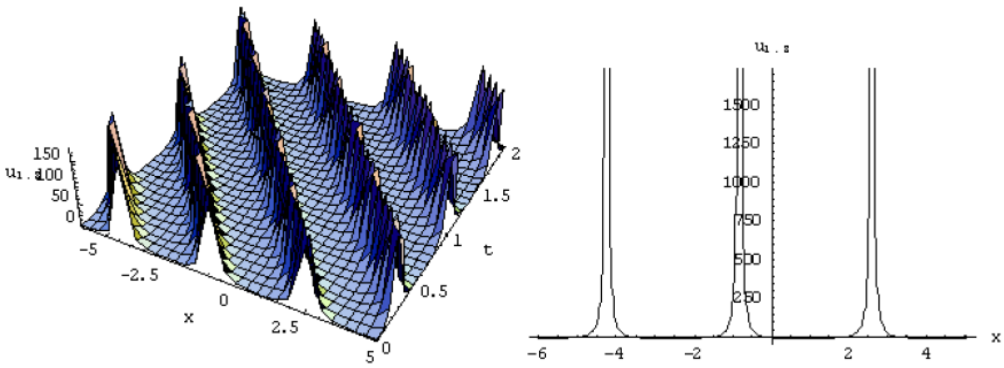


Figure 2. Solution $u_{1.8}$ when $c_0 = 0, \tau = \lambda = l_j = \alpha = \beta = \gamma = N = 1, m = 0.3$ and $t = 0$.

State 2 $n = 3$

Case 1

$$p = c_1 = c_2 = c_4 = 0, r = 1, q^2 = l^2(1-m^2), l^2 \neq 1, c_3 = \pm\frac{\tau}{l}\sqrt{\frac{-\gamma(1+l^4+l^2(2-4m^2))}{2\lambda}},$$
$$c_0 = -(1+l^2(1-2m^2)c_3, \alpha\sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_1, \Delta_1 = \frac{1}{2} + \frac{((1+l^4)(m^2-1)+2l^2(1+m^4))}{1+l^4+l^2(2-4m^2)},$$

Case 2

$$q = c_1 = c_2 = c_3 = 0, p = l = 1, \alpha\sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_2, \Delta_2 = \frac{8-13m^2+6m^4}{2(4-3\mathrm{m}^2)},$$
$$r^2 = \frac{m^2}{m^2-1}, c_4 = \mp\tau m\sqrt{\frac{\gamma(4-3\mathrm{m}^2)}{2\lambda(1-m^2)}}, c_0 = \pm\tau m\sqrt{\frac{\gamma(1-m^2)}{2\lambda(4-3\mathrm{m}^2)}}$$

Case 3

$$r = c_1 = c_3 = c_4 = 0, q = 1, p = \pm l \neq \frac{1}{m}, c_2 = \pm\frac{\tau}{l}\sqrt{\frac{-\gamma(1+l^4m^4+2l^2(m^2-2))}{2\lambda}},$$
$$c_0 = \mp\frac{\tau(1+l^2(m^2-2))}{l}\sqrt{\frac{-\gamma}{2\lambda(1+l^4m^4+2l^2(m^2-2))}},$$
$$\alpha\sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_3, \Delta_3 = -\frac{((1+l^4m^4)(m^2-2)+2l^2(m^4+2m^2-2))}{2(1+l^4m^4+2l^2(m^2-2))},$$

Case 4

$$l = c_1 = c_3 = c_4 = 0, q = \pm r, \alpha\sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_4, \Delta_4 = -\frac{(1+m^2)}{2},$$
$$p^2 = r^2, c_2 = \pm\tau r\sqrt{\frac{2\gamma(m^2-1)}{\lambda}}, c_0 = \pm\tau\sqrt{\frac{\gamma(m^2-1)}{2\lambda}}$$

Case 5

$$c_1 = c_4 = 0, p = q = 0, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_5, \Delta_5 = \frac{[r^2 - m^4l^2 - 2m^2(r^2 - l^2)]}{2(m^2l^2 - r^2)}, \varepsilon = \pm 1,$$
$$c_2 = \varepsilon\tau\sqrt{\frac{\gamma(\mathrm{r}^2 - m^2l^2)(m^2-1)}{2\lambda}}, c_3 = -\tau\sqrt{\frac{\gamma(m^2l^2 - r^2)(r^2 - l^2)}{2\lambda}}, c_0 = \pm\frac{\tau r}{2}\sqrt{\frac{2\gamma(r^2 - l^2)}{\lambda(m^2 - r^2)}}$$

Case 6

$$c_0 = c_1 = c_2 = 0, r = l = 0, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_6, \Delta_6 = -\frac{1+m^2}{2},$$
$$c_3 = \pm\tau\sqrt{\frac{\gamma(m^2p^2 - q^2)}{2\lambda}}, c_4 = \varepsilon\tau\sqrt{\frac{\gamma(p^2 - q^2)}{2\lambda}}, \varepsilon = \pm 1$$

Therefore from (3), (11), (15), Cases 1–6 and $u = v^{\frac{1}{n-1}}$, we obtain the Jacobi elliptic wave-like solutions to Equation (6):

$$u_{2.1}^2 = \mp\frac{\tau(1 + l^2(1 - 2m^2))}{l}\sqrt{\frac{-\gamma(1 + l^4 + l^2(2 - 4m^2))}{2\lambda}} \pm \frac{\frac{\tau}{l}\sqrt{\frac{-\gamma(1 + l^4 + l^2(2 - 4m^2))}{2\lambda}} cn\xi_{2.1}}{\varepsilon l\sqrt{1 - m^2} sn\xi_{2.1} + cn\xi_{2.1} + ldn\xi_{2.1}}$$

$$u_{2.2}^2 = \pm\tau m\sqrt{\frac{\gamma(1 - m^2)}{2\lambda(4 - 3\mathrm{m}^2)}} \mp \frac{\tau m\sqrt{\frac{\gamma(4 - 3\mathrm{m}^2)}{2\lambda(1 - m^2)}} cn\xi_{2.2}}{\sqrt{m^2 - 1} + \varepsilon mcn\xi_{2.2} + \sqrt{m^2 - 1}dn\xi_{2.2}}$$

$$u_{2.3}^2 = \mp\frac{\tau(1 + l^2(m^2 - 2))}{l}\sqrt{\frac{-\gamma}{2\lambda(1 + l^4m^4 + 2l^2(m^2 - 2))}} \pm \frac{\frac{\tau}{l}\sqrt{\frac{-\gamma(1 + l^4m^4 + 2l^2(m^2 - 2))}{2\lambda}} sn\xi_{2.3}}{\pm l + sn\xi_{2.3} + ldn\xi_{2.3}}$$

$$u_{2.4}^2 = \pm\tau\sqrt{\frac{\gamma(m^2 - 1)}{2\lambda}} \pm \frac{\tau\sqrt{\frac{2\gamma(m^2 - 1)}{\lambda}} sn\xi_{2.4}}{\pm 1 \pm \varepsilon sn\xi_{2.4} + cn\xi_{2.4}}$$

$$u_{2.5}^2 = \pm\frac{\tau r}{2}\sqrt{\frac{2\gamma\left(r^2 - l^2\right)}{\lambda(m^2 - r^2)}} + \frac{\varepsilon\tau\sqrt{\frac{\gamma(r^2 - m^2l^2)(m^2 - 1)}{2\lambda}} sn\xi_{2.5} - \tau\sqrt{\frac{\gamma(m^2l^2 - r^2)(r^2 - l^2)}{2\lambda}} cn\xi_{2.5}}{rcn\xi_{2.5} + ldn\xi_{2.5}}$$

$$u_{2.6}^2 = \frac{\pm\tau\sqrt{\frac{\gamma(m^2p^2 - q^2)}{2\lambda}} cn\xi_{2.6} + \varepsilon\tau\sqrt{\frac{\gamma(p^2 - q^2)}{2\lambda}} dn\xi_{2.6}}{p + qsn\xi_{2.6}}$$
$$\xi_{2.i} = \tau(x + \sum_{j=1}^{N-1} l_j y_j \pm t\sqrt{\alpha\sum_{j=1}^{N-1} l_j^2 + \beta - \gamma\tau^2\Delta_i}), (i = 1, \cdots, 6)$$

Remark 2: If we let $l = 1, r = \pm 1, \alpha = \beta = \gamma = 1, \sum_{j=1}^{N-1} l_j^2 = N - 1, \tau = \sqrt{-\frac{2(N - c^2)}{1+m^2}}$, $u_{2.5}$ is equivalent to the solution $u_{2.24}$ given in [5]. If we let $p = 1, q = 0, \alpha = \beta = \gamma = 1, \sum_{j=1}^{N-1} l_j^2 = N - 1, \tau = \sqrt{-\frac{2(N - c^2)}{1+m^2}}$, $u_{2.6}$ is equivalent to the solution $u_{2.13}$ given in [5]. If we select the corresponding parameter, we can get the solutions from $u_{2.1}$ to $u_{2.25}$ given in [5]. Solutions $u_{2.i}(\xi_i)(i = 1, 3, 6)$ are degenerated to soliton-like solutions when the modulus $m \to 1$, and solutions $u_{2.i}(\xi_i)(i = 1, 3, 5, 6)$ are degenerated to trigonometric functions solutions when the modulus $m \to 0$.

The structure of new doubly periodic-like solutions $u_{2.4}$ and $u_{2.6}$ is illustrated in Figures 3 and 4.

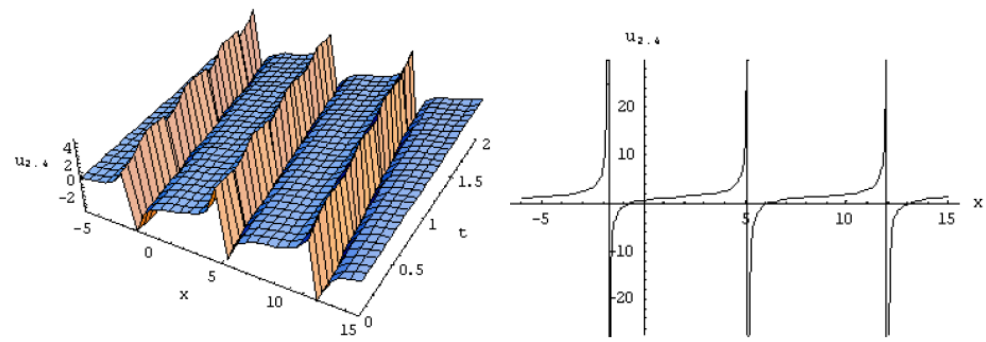


Figure 3. Solution $u_{2.4}$ when $\lambda = -1, \tau = l_j = \alpha = \beta = \gamma = N = 1, m = 0.3$ and $t = 0$.

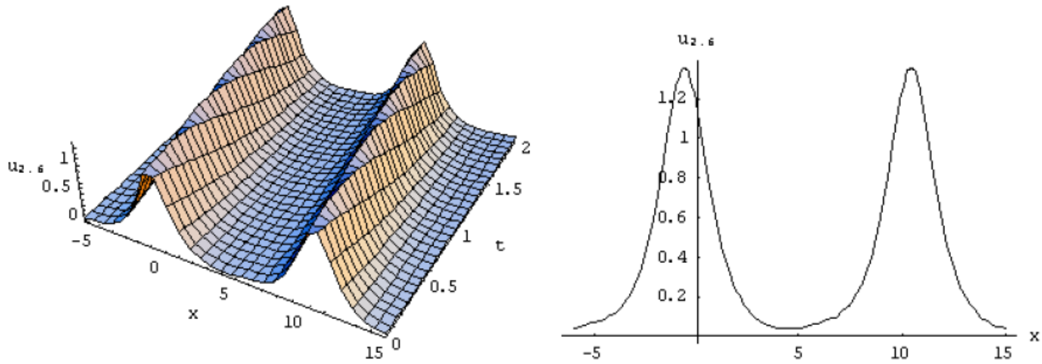


Figure 4. Solution $u_{2.6}$ when $p = 2, q = 1, \tau = \lambda = l_j = \alpha = \beta = \gamma = N = 1, m = 0.93$ and $t = 0$.

State 3 $n > 3$

Case 1

$$m = 1, l = 0, r = 1, p = \pm\sqrt{1+q^2}, \alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_1, \Delta_1 = -\frac{1}{(n-1)^2},$$
$$c_0 = c_1 = c_2 = c_4 = 0, c_3 = \frac{\gamma\tau^2(1+n)}{\lambda(n-1)^2}$$

Case 2

$$m = 0, l = 0, p = 1, r = \pm\sqrt{1-q^2}, c_0 = c_2 = c_3 = c_4 = 0, c_1 = -\frac{\gamma\tau^2(1+n)}{\lambda(n-1)^2},$$
$$\alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_2, \Delta_2 = -\frac{1}{(n-1)^2},$$

Case 3

$$m = 1, q = l = 0, p = 1, r = \pm 1, c_0 = \frac{\gamma\tau^2(1+n)}{2\lambda(n-1)^2}, d_2 = -\frac{\gamma\tau^2(1+n)}{2\lambda(n-1)^2},$$
$$\alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_3, \Delta_3 = -\frac{1}{(n-1)^2},$$

Case 4

$$m = 0, q = l = 0, p = \pm 1, r = \varepsilon, \varepsilon = \pm 1, c_0 = -\frac{\gamma\tau^2(1+n)}{2\lambda(n-1)^2}, d_2 = -\frac{\gamma\tau^2(1+n)}{2\lambda(n-1)^2},$$
$$\alpha \sum_{j=1}^{N-1} l_j^2 + \beta - c^2 = \gamma\tau^2\Delta_4, \Delta_4 = \frac{1}{(n-1)^2},$$

Therefore from (3), (11), (15), Cases 1–4 and $u = v^{\frac{1}{n-1}}$, we obtain the following soliton-like and trigonometric function solutions for Equation (6) are expressed by

$$u_{3.1}(x,t) = \{\frac{\frac{\gamma\tau^2(1+n)}{\lambda(n-1)^2}\mathrm{sech}[\xi_1]}{\pm\sqrt{1+q^2}+q\tanh[\xi_1]+\mathrm{sech}[\xi_1]}\}^{\frac{1}{n-1}}$$

$$u_{3.2}(x,t) = \{\frac{-\frac{\gamma\tau^2(1+n)}{\lambda(n-1)^2}\sec[\xi_2]}{\pm\sqrt{1-q^2}+q\tan[\xi_2]+\sec[\xi_2]}\}^{\frac{1}{n-1}}$$

$$u_{3.3}(x,t) = \{\frac{\gamma\tau^2(1+n)}{2\lambda(n-1)^2} - \frac{\gamma\tau^2(1+n)}{2\lambda(n-1)^2}[\frac{\tanh[\xi_3]}{1\pm\mathrm{sech}[\xi_3]}]^2\}^{\frac{1}{n-1}}$$

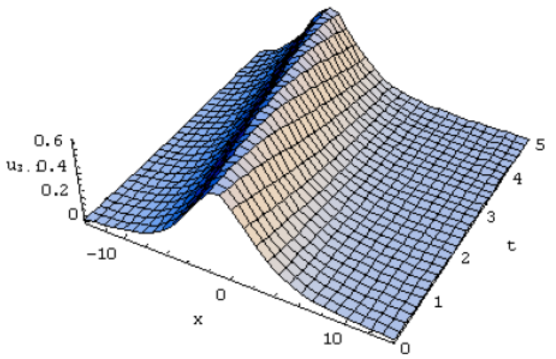
$$u_{3.4}(x,t) = \{-\frac{\gamma\tau^2(1+n)}{2\lambda(n-1)^2} - \frac{\gamma\tau^2(1+n)}{2\lambda(n-1)^2}[\frac{\tan[\xi_4]}{\varepsilon\pm\sec[\xi_4]}]^2\}^{\frac{1}{n-1}}$$

where

$$\xi_i = \tau(x+\sum_{j=1}^{N-1} l_j y_j \pm t\sqrt{\alpha\sum_{j=1}^{N-1} l_j^2+\beta-\gamma\tau^2\Delta_i}),(i=1,\cdots,4).$$

Remark 3: All the solutions obtained in this paper for Equation (6) have been checked by Mathematica software.

The properties of the new soliton-like wave solutions $u_{3.1}$ and periodic-like solutions $u_{3.2}$ is shown in Figures 5 and 6. **Remark 4:** To our knowledge, the explicit solutions except ($u_{1.9}, u_{2.5}, u_{2.6}$) we obtained here to Equation (6) are not shown in the previous literature. They are new exact solutions of Equation (6). Our method contains all the results mentioned by the G′/G method [19], the improved sub-ODE method [20] and auxiliary equation technique [21], *etc.*, which were discussed in [22].



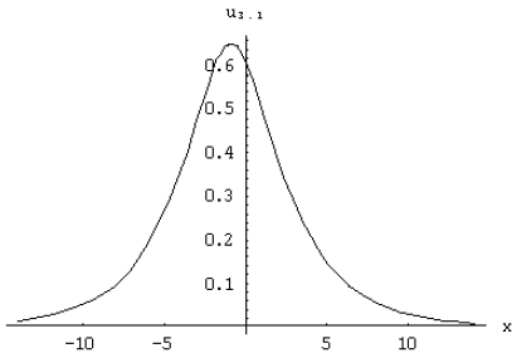


Figure 5. Solution $u_{3.1}$ when $\tau = \lambda = l_j = \alpha = \beta = \gamma = N = 1, q = 1, n =$ 4 and $t = 0$.

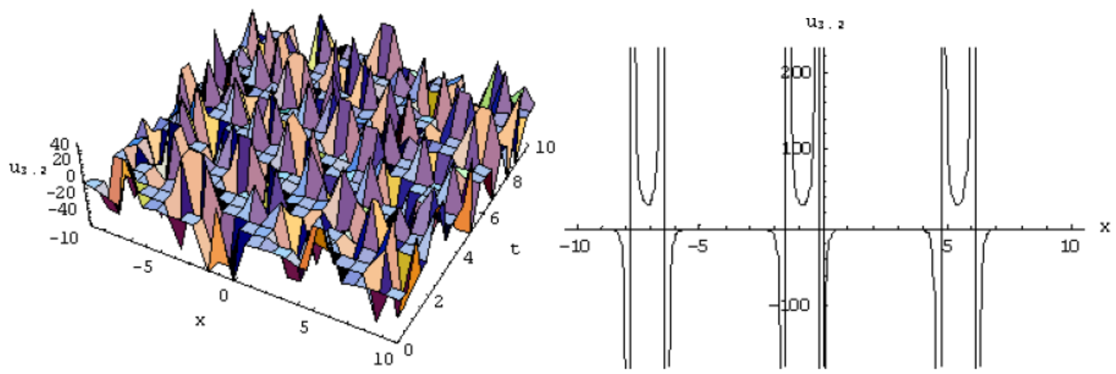


Figure 6. Solution $u_{3.2}$ when $\tau = \lambda = l_j = \alpha = \beta = \gamma = N = 1, q = \frac{1}{2}, n = \frac{4}{3}$ and $t = 0$.

4. Conclusions

In this paper, we have found abundant new types of exact solutions for the $(N + 1)$-dimensional generalized Boussinesq equation by using the generalized Jacobi elliptic functions expansion method and computerized symbolic computation. More importantly, our method is very simple and powerful at finding new solutions to various kinds of nonlinear evolution equations, such as Schrödinger equation, Boussinesq equation, *etc.* We believe that this method should play an important role for finding exact solutions in mathematical physics.

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Author Contributions: Baojian Hong obtained the data, drawn the figures and wrote the paper. Dianchen Lu conceived theoretical background and analyzed the data. All authors discussed the results and commented on the manuscript.

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