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Article

Optimization of Setting Take-Profit Levels for Derivative Trading

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Abstract: This paper develops an optimal stopping rule by characterizing the take-profit level. The optimization problem is modeled by geometric Brownian motion with two switchable regimes and solved by stochastic calculation. A closed-form profitability function for the trading strategies is given, and based on which the optimal take-profit level is numerically achievable with small cost of computational complexity.

Keywords: Black-Scholes model; passage time; optimization

1. Introduction

Derivative trading has been reshaped by quantitative techniques in recent decades [1]. This paper aims to solve the optimization problem of setting take-profit levels to maximize the profitability. Considerable studies on closing a deal are implemented into trading practice. Eloe et al. [2] optimized the threshold levels of taking profit and stopping loss based on a regime-switching model. On the other hand, modeling with regime-switching has the advantage of flexibility in changing parameters. Since first introduced by [3], intensive research interests have been drawn to this area, for example the research by Yao et al. [4] on pricing the European option by a regime-switching model.

In this paper, we also apply switched regimes, but not driven by another independent process or external factors. To optimize the take-profit level, regime-switching in our model is triggered by the price process itself; hence, our model is less subject to parameter estimation and prediction, and performs more neutrally to unveil the variation brought by the take-profit level.

We proceed as follows. In Section 2, we formulate the optimal selling problem. Section 3 gives the probability distribution of the transaction time. In Section 4, the profitability function is explicitly expressed in closed-form and optimal take-profit level is achievable from this expression. Section 5 proceeds the numerical simulation to show that the results obtained by our method are consistent with those by crude Monte Carlo simulation [5] , but ours consume less time.

2. Problem Formulation

Suppose a pair of opposite trades are opened by the the current ask price x^+ and bid price x^- at time t, the price X_t is recognized as $\frac{x^+ + x^-}{2}$ and denoted as x. Thereby, the cost for a pair of orders are $x^+ - x^-$, assumed as a constant $\theta < X_0$. Let $x(1+\eta)$ be the closing price for long trades, and $x(1-\eta)$ for short trades, $x\eta$ is the profit gained in each single deal, where $\eta \in (0,1)$ as the take-profit rate. The price dynamic is simulated by the process $\{X_t : t \in [0,\infty)\}$ formulated by the following equation:

$$dX_t = \left[\mu - (\mu - \frac{\sigma^2}{2})\mathbf{1}_{\{\hat{X}_{0,t} \geq X_0(1+\eta) \text{ or } \check{X}_{0,t} \leq X_0(1-\eta)\}}\right] X_t dt + \sigma X_t dW_t \tag{1}$$

where $\mu \in \mathbf{R}$, $\sigma > 0$, $X_0 > 0$, and $\hat{X}_{s_1,s_2} := \sup_{s_1 \leq r \leq s_2} X_r$, $\check{X}_{s_1,s_2} := \inf_{s_1 \leq r \leq s_2} X_r$. The nature filtration $(\mathcal{F}_t)_{t \in \mathbf{R}_+}$ is generated by a Wiener process $(W_t)_{t \in \mathbf{R}_+}$. For any $T > 0$, consider the process $\{X_t : t \in [0, T]\}$; to make it under a risk-neutral setting, a new probability measure $\hat{P}$ is defined by

$$d\hat{P} = \exp\left(\int_0^T \varphi_t dW_t - \int_0^T \frac{1}{2}\varphi_t^2 dt\right) dP,$$

where the process φ_t is given by

$$\varphi_t := \mu - (\mu - \frac{\sigma^2}{2})\mathbf{1}_{\{\hat{X}_{0,t} \geq X_0(1+\eta) \text{ or } \check{X}_{0,t} \leq X_0(1-\eta)\}}, \quad t \in [0, T],$$

hence under this measure, $\hat{P}$, $W_t - \int_0^t \varphi_u du$ is a standard Brownian motion and X_t is also a martingale. Without loss of generality and for convenience of notation, we still proceed under the original measure P instead of $\hat{P}$ for the remaining part. For any $t \in \mathbf{R}_+$, in the event that $X_0(1-\eta) < X_s < X_0(1+\eta)$ for all $s \in [0, t]$, $(X_s)_{s \in [0,t]}$ follows a geometry Brownian motion as in the Black–Scholes model. The drift factor μ is set according to the predicted trend based on the previous information. Therefore, once the threshold level is achieved, we abandon the previously obtained value of the drift factor μ, and make no more prediction for the uncertainty; rather, the price thereafter is simulated by a martingale—namely, letting the drift factor vanish. According to our stopping rule, two stopping times are defined as follows,

$$T_1 := \inf\{s \geq 0 \mid X_s \geq X_0(1+\eta) \text{ or } X_s \leq X_0(1-\eta)\}, \tag{2}$$

$$T_2 := \inf\{s \geq 0 \mid \hat{X}_{0,T_1+s} \geq X_0(1+\eta) \text{ and } \check{X}_{0,T_1+s} \leq X_0(1-\eta)\}. \tag{3}$$

To measure the efficiency of profit-taking, we define the profitability function:

$$\phi(\eta) := E\left[\frac{2\eta X_0 - \theta}{T_1 + T_2}\right] \tag{4}$$

for any take-profit rate $\eta \in (0, 1)$. With definition (2) of T_1, SDE (1) is rewritten equivalently: $dX_t = \mu X_t \mathbf{1}_{\{t < T_1\}} dt + \sigma X_t dW_t$. Next, we define two geometric Brownian motions $(Y_t)_{t \in \mathbf{R}_+}$ and $(Z_t)_{t \in \mathbf{R}_+}$ by $dY_t = \mu Y_t dt + \sigma Y_t dW_t$ and $dZ_t = \sigma Z_t dW_t$. For any positive sequence $\{t_i\}_{i \in \mathbf{N}}$ that $t_i < t_{i+1}$ for any $i \in \mathbf{N}$, in the event $t_i < T_1$ for all i, $\{Y_{t_i}\}_{i \in \mathbf{N}}$ has the same joint distribution as $\{X_{t_i}\}_{i \in \mathbf{N}}$. On the other hand, $\{Z_{t_i}\}_{i \in \mathbf{N}}$ has the same joint distribution as $\{X_{t_i}\}_{i \in \mathbf{N}}$ in the event that $t_i > s \geq T_1$ for all i. Therefore, in the following computation, we may substitute X_t by Y_t and Z_t in each case, respectively.

For convenience, we define $W_t^\lambda := W_t + \lambda t$, $\lambda := \frac{\mu}{\sigma} - \frac{\sigma}{2}$, $S_t := \sup_{0 \leq r \leq t} W_r$, and $I_t := \inf_{0 \leq r \leq t} W_r$.

3. Computation of Transaction Time

3.1. Independence between T_1 and T_2

In this subsection, we show by Lemma 1 the independence between T_1 and T_2 for further computation.

Lemma 1. *For any $t_1, t_2 > 0$, $P(T_1 \in dt_1,\ T_2 \in dt_2) = P(T_1 \in dt_1)P(T_2 \in dt_2)$, where T_1, T_2 are defined by (2) and (3).*

Proof. By definition (2) of T_1, $T_1 \neq 0$ since $X_0 > 0$, then $X_{T_1} = X_0(1+\eta)$ or $X_{T_1} = X_0(1-\eta)$, hence we have

$$
\begin{aligned}
&P(T_1 \in dt_1, T_2 \in dt_2) = E[\mathbf{1}_{\{T_1 \in dt_1\}} E[\mathbf{1}_{\{T_2 \in dt_2\}} \mid T_1 \in dt_1, X_{T_1} = X_0(1+\eta)]] P(X_{T_1} = X_0(1+\eta)) \\
&+ E[\mathbf{1}_{\{T_1 \in dt_1\}} E[\mathbf{1}_{\{T_2 \in dt_2\}} \mid T_1 \in dt_1, X_{T_1} = X_0(1-\eta)]] P(X_{T_1} = X_0(1-\eta)).
\end{aligned} \tag{5}
$$

To simplify (5), by definition (3), we note that

$$
\begin{aligned}
&E[\mathbf{1}_{\{T_2 \in dt_2\}} \mid T_1, X_{T_1} = X_0(1-\eta)] \\
=&P(\inf\{s \geq 0 \mid \hat{Z}_{T_1, T_1+s} \geq X_0(1+\eta)\} \in dt_2 \mid T_1, Z_{T_1} = X_0(1-\eta)),
\end{aligned} \tag{6}
$$

Applying the strong Markov property of $(Z_t)_{t \in \mathbf{R}_+}$ for all $t \geq T_1$ given T_1, we have

$$
\begin{aligned}
&P(\inf\{s \geq 0 \mid \hat{Z}_{T_1, T_1+s} \geq X_0(1+\eta)\} \in dt_2 \mid T_1, Z_{T_1} = X_0(1-\eta)) \\
=&P\left(\frac{\log(1+\eta) - \log(1-\eta)}{\sigma} \in dS_{t_2}\right),
\end{aligned} \tag{7}
$$

where for the property of Wiener process, we apply Theorem 1.12 of [6]. Combining (6) and (7), we see that

$$
E[\mathbf{1}_{\{T_2 \in dt_2\}} \mid T_1, X_{T_1} = X_0(1-\eta)] = P\left(\frac{\log(1+\eta) - \log(1-\eta)}{\sigma} \in dS_{t_2}\right). \tag{8}
$$

By the symmetric property of Wiener process (see Chapter 2 of [7]),

$$
E[\mathbf{1}_{\{T_2 \in dt_2\}} \mid T_1 \in dt_1, X_{T_1} = X_0(1-\eta)] = E[\mathbf{1}_{\{T_2 \in dt_2\}} \mid T_1 \in dt_1, X_{T_1} = X_0(1+\eta)]. \tag{9}
$$

By (5), (8), and (9), we obtain

$$
P(T_1 \in dt_1, T_2 \in dt_2) = P(T_1 \in dt_1) P\left(\frac{\log(1+\eta) - \log(1-\eta)}{\sigma} \in dS_{t_2}\right). \tag{10}
$$

On the other hand, repeating the approach above, we see that

$$
P(T_2 \in dt_2) = P\left(\frac{\log(1+\eta) - \log(1-\eta)}{\sigma} \in dS_{t_2}\right), \tag{11}
$$

hence we conclude Lemma 1 by (10) and (11). □

3.2. Distribution of T_1

Define a function $G(y, a, b, t)$ for any $a > y > b > 0$ and $t \in \mathbf{R}_+$ by

$$
\begin{aligned}
G(y, a, b, t) = &\sum_{n=1}^{\infty} [\kappa(t, y - 2b + 2(n-1)(a-b)) + \kappa(t, y - 2a - 2(n-1)(a-b)) \\
&-\kappa(t, -y + 2n(a-b)) - \kappa(t, -y - 2n(a-b))],
\end{aligned} \tag{12}
$$

where the normal density function κ is defined by $\kappa(t, x) := \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$ for any $t > 0, x \in \mathbf{R}$. Then, the distribution of T_1 is given by Proposition 2 as follows.

Proposition 2. *For any $t > 0$,*

$$
\begin{aligned}
P(T_1 < t) \quad = \quad &\int_{\frac{\log(1-\eta)}{\sigma}}^{\frac{\log(1+\eta)}{\sigma}} e^{\lambda y - \frac{1}{2}\lambda^2 t} G\left(y, \frac{\log(1+\eta)}{\sigma}, \frac{\log(1-\eta)}{\sigma}, t\right) dy \\
&+1 - \Phi\left(\frac{\log(1+\eta)}{\sigma\sqrt{t}} - \sqrt{t}\lambda\right) + \Phi\left(\frac{\log(1-\eta)}{\sigma\sqrt{t}} - \sqrt{t}\lambda\right),
\end{aligned}
$$

where $\Phi(\cdot)$ denotes the distribution function of a standard normal variable.

Proof. By the definition (2) of T_1, for any $t > 0$, we have $P(T_1 < t) = P(\hat{Y}_{0,t} \geq X_0(1+\eta)$ or $\check{Y}_{0,t} \leq X_0(1-\eta))$. Applying the standard technique of Girsanov theorem, cf. Theorem 8.6.4 of B. Oksendal [8], we obtain that $P(T_1 < t) = E[\mathbf{1}_{\{S_t \geq \frac{1}{\sigma}\log(1+\eta) \text{ or } I_t \leq \frac{1}{\sigma}\log(1-\eta)\}} e^{\lambda W_t - \frac{1}{2}\lambda^2 t}]$, for which we then apply the Lemma 3 proved by some similar arguments as in Chapter 2.8 of [9]; the details of the proof are stated in the Appendix. □

Lemma 3. *For any $a > c > b > 0$, $t > 0$ and $x \in (a, b)$, we have*

$$
\begin{aligned}
&P(S_t \geq a \text{ or } I_t \leq b\,;\ W_t \leq c \mid W_0 = x) \\
&= \sum_{n=1}^{\infty} \Bigg[\Phi\left(\frac{c + x - 2n(a-b) - 2a}{\sqrt{t}}\right) + \Phi\left(\frac{c + x + 2n(a-b) - 2b}{\sqrt{t}}\right) - \Phi\left(\frac{c - x + 2n(a-b)}{\sqrt{t}}\right) \\
&\quad - \Phi\left(\frac{c - x - 2n(a-b)}{\sqrt{t}}\right) - \Phi\left(\frac{b + x - 2n(a-b) - 2a}{\sqrt{t}}\right) - \Phi\left(\frac{b + x + 2n(a-b) - 2b}{\sqrt{t}}\right) \\
&\quad + \Phi\left(\frac{b - x + 2n(a-b)}{\sqrt{t}}\right) + \Phi\left(\frac{b - x - 2n(a-b)}{\sqrt{t}}\right) \Bigg].
\end{aligned}
$$

4. Optimization of Take-Profit Levels

In this section, we express the profitability function in a closed-form and consider the maximization problem over $\eta \in (0, 1)$. By (4) and (11), and Proposition 2, we have

$$
\phi(\eta) = (2\eta X_0 - \theta) \int_{(R^+)^2} \frac{P(dT_1)P(dT_2)}{T_1 + T_2} = (2\eta X_0 - \theta) \int_{(R^+)^2} \frac{1}{t+s} \varphi_1(t)\varphi_2(s)\,dt\,ds, \tag{13}
$$

where the probability density functions are given by

$$
\begin{aligned}
\varphi_1(t) := &\int_{\frac{\log(1-\eta)}{\sigma}}^{\frac{\log(1+\eta)}{\sigma}} e^{\lambda y - \frac{1}{2}\lambda^2 t} \frac{\partial G}{\partial t}\left(y, \frac{\log(1+\eta)}{\sigma}, \frac{\log(1-\eta)}{\sigma}, t\right) dy \\
&- \frac{1}{2}\lambda^2 \int_{\frac{\log(1-\eta)}{\sigma}}^{\frac{\log(1+\eta)}{\sigma}} e^{\lambda y - \frac{1}{2}\lambda^2 t} G\left(y, \frac{\log(1+\eta)}{\sigma}, \frac{\log(1-\eta)}{\sigma}, t\right) dy \\
&- \frac{1}{\sqrt{2\pi}} e^{-\frac{(\log(1-\eta) - \sigma\lambda t)^2}{2\sigma^2 t}} \left(\frac{\log(1-\eta)}{2\sigma\sqrt{t^3}} + \frac{\lambda}{2\sqrt{t}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{(\log(1+\eta) - \sigma\lambda t)^2}{2\sigma^2 t}} \left(\frac{\log(1+\eta)}{2\sigma\sqrt{t^3}} + \frac{\lambda}{2\sqrt{t}}\right)
\end{aligned}
$$

for $t > 0$, where the function G is defined by (12), and $\varphi_2(t) = \frac{\log(1+\eta) - \log(1-\eta)}{\sigma\sqrt{2\pi t^3}} e^{-\frac{(\log(1+\eta) - \log(1-\eta))^2}{2\sigma^2 t}}$ for $t > 0$. Then, we consider the maximization problem of setting suitable η. First, to enlarge $\phi(\eta)$, we ensure it to be positive; therefore, η should satisfy the condition that $\eta > \frac{\theta}{2X_0}$. Note that $\eta \in (0, 1)$ and $\theta < X_0$ as we assumed before, η should be within $(\frac{\theta}{2X_0}, 1)$. Next, we check the convergence of the integration in (13). Actually, under the condition that $\eta \in (\frac{\theta}{2X_0}, 1)$, we see that $0 < \phi(\eta) < \frac{(2X_0 - \theta)\sigma^2}{(\log(2X_0 + \theta) - \log(2X_0 - \theta))^2}$, which also provides a uniform upper bound for the profitability function. The optimal take-profit rate $\eta^* = \max\left\{\eta \in \left(\frac{\theta}{2X_0}, 1\right) \,\middle|\, \phi(\eta) = \max_{\frac{\theta}{2X_0} \leq r \leq 1} \phi(r)\right\}$, is ready to be solved numerically.

5. Numerical Simulation

In Table 1 below, we compare the testing errors and running time for both approaches. Note that the average testing errors are measured by comparison with the result obtained by programming with much more samples and smaller time discretization steps, which consume several times the running time. From the average running time listed, we conclude that our approach is much more efficient than that of crude Monte Carlo simulation. The reason is also mathematically obvious , as it is well known that $E[T_2] = \infty$ while $E\left[\frac{1}{T_2}\right]$ is finite, it is quite time-consuming to sample the stopping time T_2 (so is T_1). From the data of difference pairs of (μ, σ) obtained by both approaches, we find that the optimal take-profit level is more sensitive to the changes of σ than that of μ. Besides, larger volatility σ yields large optimal take-profit level, which reinforces the widely-held financial wisdom that the larger the volatility, the larger the take-profit level we can set.

Table 1. For 25 pairs of parameters (μ, σ), we report the value of η obtained by Monte Carlo simulations and our approach of maximization of the closed-form.

$\mathbf{X_0 = 10,\ \theta = 0.001}$		$\boldsymbol{\sigma = 0.10}$	$\boldsymbol{\sigma = 0.15}$	$\boldsymbol{\sigma = 0.20}$	$\boldsymbol{\sigma = 0.25}$	$\boldsymbol{\sigma = 0.30}$
Monte Carlo	¯ = 0.05	0.029	0.042	0.046	0.058	0.064
	¯ = 0.10	0.031	0.047	0.049	0.056	0.072
	¯ = 0.15	0.035	0.049	0.053	0.064	0.077
	¯ = 0.20	0.038	0.050	0.058	0.080	0.095
	¯ = 0.25	0.039	0.058	0.063	0.084	0.103
	Average Testing Errors			0.028		
	Average Running Time			54.60 min		
Our method	¯ = 0.05	0.028	0.040	0.049	0.056	0.068
	¯ = 0.10	0.032	0.045	0.051	0.061	0.075
	¯ = 0.15	0.035	0.051	0.054	0.066	0.080
	¯ = 0.20	0.040	0.052	0.059	0.077	0.091
	¯ = 0.25	0.041	0.061	0.065	0.083	0.096
	Average Testing Errors			0.011		
	Average Running Time			30.11 min		

6. Conclusion and Future Work

This paper gives an optimal stopping rule by characterizing the take-profit level. Compared to others' effects on this, ours has less computational complexity and is applicable to improving the trading strategy for the issue of closing position. Our work can be extended to other more difficult models with regime switching, such as that with Markov chains; however, it could be challenging to get a closed form, since our work benefits from the advantages of Brownian motion.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A.

In this section, we proceed to prove Lemma 3 by applying the technique in Chapter 2.8 of [9]. For any $a > c > b > 0$, $x \in (a, b)$, $t > 0$, we have

$$
\begin{aligned}
& P^x(S_t \geq a \text{ or } I_t \leq b\ ;\ W_t \leq c) \\
= & \sum_{n=1}^{\infty} \left[\Phi\left(\frac{c + x - 2n(a-b) - 2a}{\sqrt{t}} \right) + \Phi\left(\frac{c + x + 2n(a-b) - 2b}{\sqrt{t}} \right) \right. \\
& \left. - \Phi\left(\frac{c - x + 2n(a-b)}{\sqrt{t}} \right) - \Phi\left(\frac{c - x - 2n(a-b)}{\sqrt{t}} \right) \right.
\end{aligned}
$$

$$-\Phi\left(\frac{b+x-2n(a-b)-2a}{\sqrt{t}}\right)-\Phi\left(\frac{b+x+2n(a-b)-2b}{\sqrt{t}}\right)$$
$$+\Phi\left(\frac{b-x+2n(a-b)}{\sqrt{t}}\right)+\Phi\left(\frac{b-x-2n(a-b)}{\sqrt{t}}\right)\Bigg].$$

Proof. First, several sequences of stopping time are defined as follows:

$$\sigma_0=0,\ \ \tau_0=\inf\{t\geq 0 \mid I_t\leq b\};$$

$$\pi_0=0,\ \ \rho_0=\inf\{t\geq 0 \mid S_t\geq a\};$$
$$\sigma_n=\inf\{t\geq \pi_{n-1} \mid W_t=a\};$$
$$\tau_n=\inf\{t\geq \sigma_{n-1} \mid W_t=b\};$$
$$\pi_n=\inf\{t\geq \rho_{n-1} \mid W_t=b\};$$
$$\rho_n=\inf\{t\geq \pi_{n-1} \mid W_t=a\}.$$

With the reflection property of Brownian motion (refer to [9]), for $y\in(b,a)$,

$$P^x\left(W_t\geq y \,\middle|\, \mathcal{F}_{\tau_n}\right)=P^x\left(W_t\leq 2b-y \,\middle|\, \mathcal{F}_{\tau_n}\right)\qquad \text{on } \{\tau_n\leq t\};$$

$$P^x\left(W_t\geq y \,\middle|\, \mathcal{F}_{\tau_n}\right)=P^x\left(W_t\leq 2b-y \,\middle|\, \mathcal{F}_{\pi_n}\right)\qquad \text{on } \{\pi_n\leq t\};$$

$$P^x\left(W_t\leq y \,\middle|\, \mathcal{F}_{\tau_n}\right)=P^x\left(W_t\geq 2a-y \,\middle|\, \mathcal{F}_{\sigma_n}\right)\qquad \text{on } \{\sigma_n\leq t\};$$

$$P^x\left(W_t\leq y \,\middle|\, \mathcal{F}_{\tau_n}\right)=P^x\left(W_t\geq 2a-y \,\middle|\, \mathcal{F}_{\rho_n}\right)\qquad \text{on } \{\rho_n\leq t\}.$$

Note that $2b-y<b$ and $2a-y>a$; thereby, for any $n\geq 1$,

$$P^x(W_t\geq y,\ \tau_n\leq t)=P^x(W_t\leq 2b-y,\ \tau_n\leq t)=P^x(W_t\leq 2b-y,\ \sigma_n\leq t);$$

$$P^x(W_t\leq y,\ \sigma_n\leq t)=P^x(W_t\geq 2a-y,\ \sigma_n\leq t)=P^x(W_t\geq 2a-y,\ \tau_{n-1}\leq t);$$
$$P^x(W_t\leq y,\ \rho_n\leq t)=P^x(W_t\geq 2a-y,\ \rho_n\leq t)=P^x(W_t\geq 2a-y,\ \pi_n\leq t);$$
$$P^x(W_t\geq y,\ \pi_n\leq t)=P^x(W_t\leq 2b-y,\ \pi_n\leq t)=P^x(W_t\leq 2b-y,\ \rho_{n-1}\leq t).$$

The above formulas are alternately and recursively applied to gain the following expressions. Therefore we have,

$$\begin{aligned}
P^x(W_t\geq y,\ \tau_n\leq t) &= P^x(W_t\leq 2b-y,\sigma_n\leq t)\\
&= P^x(W_t\geq 2a-(2b-y),\tau_{n-1}\leq t)\\
&= P^x(W_t\leq 2b-y-2(a-b),\sigma_{n-1}\leq t)\\
&= P^x(W_t\geq y+2n(a-b),\tau_{n-n}\leq t)\\
&= P^x(W_t\leq 2b-y-2n(a-b),\sigma_0\leq t),
\end{aligned}$$

and

$$\begin{aligned}
P^x(W_t\leq y,\ \sigma_n\leq t) &= P^x(W_t\geq 2a-y,\tau_{n-1}\leq t)\\
&= P^x(W_t\leq y-2(a-b),\sigma_{n-1}\leq t)\\
&= P^x(W_t\leq y-2n(a-b),\sigma_0\leq t).
\end{aligned}$$

Similarly, another two formulas are:

$$P^x(W_t \le y,\ \rho_n \le t) = P^x(W_t \ge 2a - y + 2n(a-b)).$$

$$P^x(W_t \ge y,\ \pi_n \le t) = P^x(W_t \ge y + 2n(a-b)).$$

Taking the derivative regarding y in the above four formulas, another four expressions are gained.

$$P^x(W_t \in dy,\ \tau_n \le t) = \kappa(t, x + y - 2b + 2n(a-b))dy,$$

$$P^x(W_t \in dy,\ \sigma_n \le t) = \kappa(t, x - y + 2n(a-b))dy,$$

$$P^x(W_t \in dy,\ \rho_n \le t) = \kappa(t, x + y - 2a - 2n(a-b))dy,$$

$$P^x(W_t \in dy,\ \pi_n \le t) = \kappa(t, x - y - 2n(a-b))dy,$$

where $\kappa(t,x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$ as defined before, for any $t > 0, x \in R$. Note that $\tau_{n-1} \vee \rho_{n-1} = \sigma_n \wedge \pi_n$ and $\sigma_n \vee \pi_n = \pi_n \wedge \rho_n$ for any $n \ge 1$, then for any integer $k \ge 1$,

$$\begin{aligned} P^x(W_t \in dy, \tau_k \wedge \rho_k \le t) &= P^x(W_t \in dy, \tau_k \le t) + P^x(W_t \in dy, \rho_k \le t) \\ &\quad - P^x(W_t \in dy, \sigma_k \wedge \pi_k \le t) \\ &= P^x(W_t \in dy, \tau_k \le t) + P^x(W_t \in dy, \rho_k \le t) \\ &\quad - [P^x(W_t \in dy, \tau_{k-1} \le t) + P^x(W_t \in dy, \rho_{k-1} \le t) \\ &\quad - P^x(W_t \in dy, \tau_{k-1} \wedge \rho_{k-1} \le t)]. \end{aligned}$$

Repeatedly apply this recursive expression for k times, and have:

$$\begin{aligned} P^x(W_t \in dy, \tau_k \wedge \rho_k \le t) &= \sum_{n=1}^{k} [P^x(W_t \in dy, \sigma_n \le t) + P^x(W_t \in dy, \pi_n \le t) \\ &\quad - P^x(W_t \in dy, \tau_{n-1} \le t) - P^x(W_t \in dy, \rho_{n-1} \le t)] \\ &\quad + P^x(W_t \in dy, \tau_0 \wedge \rho_0 \le t). \end{aligned}$$

Consider the convergence of the above summation when k goes to infinity. Since the summation equals $P^x(W_t \in dy, \tau_k \wedge \rho_k \le t) - P^x(W_t \in dy, \tau_0 \wedge \rho_0 \le t)$, the above summation should decrease on k, and constrained within $[-1, 1]$, the limit exists and is bounded. Note that $S_t \ge a$ or $I_t \le b$ is the same event as $\tau_0 \wedge \rho \le t$, then pass k to infinity and gain that

$$\begin{aligned} P^x(W_t \in dy,\ S_t \ge a \text{ or } I_t \le b) &= \sum_{n=1}^{\infty} [P^x(W_t \in dy, \tau_{n-1} \le t) + P^x(W_t \in dy, \rho_{n-1} \le t) \\ &\quad - P^x(W_t \in dy, \sigma_n \le t) - P^x(W_t \in dy, \pi_n \le t)] \\ &= \sum_{n=1}^{\infty} [\kappa(t, x + y - 2b + 2(n-1)(a-b)) \\ &\quad + \kappa(t, x + y - 2a - 2(n-1)(a-b)) \\ &\quad - \kappa(t, x - y + 2n(a-b)) - \kappa(t, x - y - 2n(a-b))]dy. \end{aligned}$$

Finally, take integration regarding y from b to c, and hence complete the proof. □

References

1. Vidyamurthy, G. Pairs Trading: Quantitative Methods and Analysis. *Pearson Schweiz Ag.* **2004**, 35–47.
2. Eloe, P.; Liu, R.H.; Yatsuki, M.; Yin, G.; Zhang, Q. Optimal selling rules in a regime-switching exponential Gaussian diffusion model. *SIAM J. Appl. Math.* **2008**, *69*, 810–829.

3. Hamilton, J.D. A new approach to the economic analysis of non-stationary time series. *Econometrica* **1989**, *57*, 357–384.
4. Yao, D.D.; Zhang, Q.; Zhou, X.Y. A regime-switching model for European options. *Int. Ser. Oper. Res. Manag. Sci.* **2006**, *94*, 281–300.
5. Hammersley, J.M.; Handscomb, D.C. Monte Carlo Methods. *Springer Neth.* **1964**, *30*, 347–385.
6. Mörters, P.; Peres, Y. *Brownian Motion;* Cambridge University Press: Cambridge, UK, 2010.
7. Schilling, R.L.; Böttcher, B.; Partzsch, L. *Brownian Motion: An Introduction to Stochastic Processes*; Walter de Gruyter GmbH & Co KG: Berlin, Germany, 2012.
8. Oksendal, B. *Stochastic Differential Equations*, 6th ed.; Springer: New York, NY, USA, 2003.
9. Karatzas, I.; Shreve, S.E. *Brownian Motion and Stochastic Calculus*; Springer: New York, NY, USA; Berlin, Germany, 1991.