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Global Modulus-Based Synchronous Multisplitting Multi-Parameters TOR Methods for Linear Complementarity Problems

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Abstract: In 2013, Bai and Zhang constructed modulus-based synchronous multisplitting methods for linear complementarity problems and analyzed the corresponding convergence. In 2014, Zhang and Li studied the weaker convergence results based on linear complementarity problems. In 2008, Zhang et al. presented global relaxed non-stationary multisplitting multi-parameter method by introducing some parameters. In this paper, we extend Bai and Zhang's algorithms and analyze global modulus-based synchronous multisplitting multi-parameters TOR (two parameters overrelaxation) methods. Moverover, the convergence of the corresponding algorithm in this paper are given when the system matrix is an H_+-matrix.

Keywords: modulus-based method; linear complementarity problems; successive relaxation method; H_+-matrix

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1. Introduction

Consider the linear complementarity problems LCP(q, A), for finding a pair of real vectors r and $z \in R^n$ such that

$$r = Az + q \geq 0, z \geq 0\ ,\ z^T(Az + q) = 0, \tag{1}$$

where $A = (a_{ij}) \in R^{n\times n}$ is the given real matrix and $q = (q_1, q_2, ..., q_n)^T \in R^n$ is the given real vector. Here, z^T and $\geq$ denote the transpose of the vector z and the componentwise defined partial ordering between two vectors, respectively. Now, H_+-matrices belong to class of P-matrices and so play an important rule in the theory of LCP.

The readers may see references [1–4] for many problems in scientific computing and engineering applications. When the matrix A is special for LCP(q, A), readers may see the references [5–14]. Lately, when LCP(q, A) is an algebra system, some scientist have studied it. Moreover, Bai and Zhang presented the modulus-based multisplitting iterative methods for LCP(q, A) and analyzed the convergence based on the corresponding methods in [10,11]. Zhang and Ren generalized the compatible H-splitting condition to an H-splitting [15]. L generalized modulus-based splitting iterative method to more general situationi [16]. Zhang et al. studied the wider convergence when system matrix is an H_+-matrix [17–19].

2. Notations and Lemmas

A matrix $A = (a_{ij})$ is called an M-*matrix* if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. The comparison matrix $\langle A\rangle = (\alpha_{ij})$ of matrix $A = (a_{ij})$ is defined by: $\alpha_{ij} = |a_{ij}|$, if $i = j$; $\alpha_{ij} = -|a_{ij}|$, if $i \neq j$. A matrix A is called an H-matrix if $\langle A \rangle$ is an M-matrix and is called an H_+-matrix if it is an H-matrix with positive diagonal entries [5,20,21]. Let $\rho(A)$ denote the spectral radius of A, a representation $A = M - N$ is called a *splitting* of A when M is nonsingular. Let A and B be M-matrices, if $A \leq B$, then $A^{-1} \geq B^{-1}$. Let A be an H-matrix, and $A = D - B, D = \text{diag}(A)$, then $\rho(|D|^{-1}|B|) < 1$. Moreover, D is nonsingular. Finally, we define by $R^n_+ = \{x | x \geq 0, x \in R^n\}$ and denote the nonnegative matrix with entries $|a_{ij}|$ by $|A|$.

Lemma 1. *Let A be an H-matrix. Then A is nonsingular, and $|A^{-1}| \leq \langle A \rangle^{-1}$ [22].*

Lemma 2. *Let $H^{(1)}, H^{(2)}, ..., H^{(l)}...$be a sequence of nonnegative matrices in $R^{n\times n}$ [23]. If there exists a real number $0 \leq \theta < 1$, and a vector $v > 0$ in R^n, such that*

$$H^{(l)}v \leq \theta v,\ l = 1, 2, ...$$

then $\rho(K_l) \leq \theta^l < 1$, where $K_l = H^{(l)}H^{(l-1)}...H^{(1)}$, and therefore $\lim\limits_{l\to\infty} K_l = 0$.

Lemma 3. *Let $A = (a_{ij}) \in Z^{n\times n}$ have all positive diagonal entries [24]. A is an M-matrix if and only if $\rho(B) < 1$, where $B = D^{-1}C,\ D = \text{diag}(A),\ A = D - C$.*

Lemma 4. *$A \in R^{n\times n}$ be an H_+-matrix. Then, the LCP(q, A) has a unique solution for any $q \in R^n$[7,9,25].*

Lemma 5. *Let $A = M - N$ be a splitting of the matrix $A \in R^{n\times n}, \Omega$ be a positive diagonal matrix, and γ a positive constant [10]. Then, for the LCP(q, A) the following statements hold true:*

(i) if (z, r) is a solution of the LCP(q, A), then $x = \frac{1}{2}\gamma(z - \Omega^{-1}r)$ satisfies the implicit fixed-point equation

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q; \tag{2}$$

(ii) if x satisfies the implicit fixed-point Equation (2), then

$$z = \gamma^{-1}(|x| + x)\ ,\ r = \gamma^{-1}\Omega(|x| - x) \tag{3}$$

is a solution of the LCP(q, A).

3. Global Modulus-Based Synchronous Multisplitting Multi-Parameters TOR Methods

Firstly, we will introduce the idea of multisplitting algorithm and the parallel iterative process. $\{M_k, N_k, E_k\}_{k=1}^{l}$ is a *multisplitting* of A if

(1) $A = M_k - N_k$ is a splitting for $k = 1, 2, ..., l$;
(2) $E_k \geq 0$ is a nonnegative diagonal matrix, called weighting matrix;
(3) $\sum\limits_{k=1}^{l} E_k = I$, where I is the identity matrix.

If Ω is a positive diagonal matrix, γ is a positive constant, form Lemma 5, we may find that if x satisfies the following implicit fixed-point systems,

$$(\Omega + M_k)x = N_k x + (\Omega - A)|x| - \gamma q, k = 1, 2, ..., l, \tag{4}$$

we have,

$$z = \gamma^{-1}(|x| + x)\ ,\ r = \gamma^{-1}\Omega(|x| - x) \tag{5}$$

which is a solution of the Equation (1).

Let

$$A = D - L_k - F_k - U_k,\ k = 1, 2, ..., l,$$

where $D = \mathrm{diag}(A), L_k$ and F_k are the strictly lower triangular, and U_k are such that $A = D - L_k - F_k - U_k$, then $(D - L_k - F_k, U_k, E_k)$ is a multisplitting of A. With the equivalent reformulations (4), (5) and TOR method of the Equation (1), we may obtain the global modulus-based synchronous multisplitting multi-parameters TOR algorithm (GMSMMTOR). Please see the following Method 1.

Method 1. *The GMSMMTOR algorithm for the Equation (1). If $(M_k, N_k, E_k)(k = 1, 2, ...l)$ are the multisplitting of matrix $A \in R^{n\times n}$. Given an initial value $x^{(0)} \in R^n$ for $m = 0, 1, ...$ until the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty} \subset R^n_+$ is convergent, compute $z^{(m+1)} \in R^n_+$ by*

$$z^{(m+1)} = \frac{1}{\gamma}(|x^{(m+1)}| + x^{(m+1)})$$

and $x^{(m,k)} \in R^n$ according to

$$x^{(m+1)} = \omega \sum_{k=1}^{l} E_k x^{(m,k)} + (1-\omega)x^{(m)},$$

where $x^{(m,k)}, k = 1, 2, ..., l$, are obtained by solving the linear systems:

$$\begin{aligned}[\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k)]x^{(m,k)} &= [(1-\alpha_k)D + (\alpha_k - \beta_k)L_k + (\alpha_k - \gamma_k)F_k + \alpha_k U_k]x^{(m)} \\ &\quad +\alpha_k[(\Omega - A)|x^{(m)}| - \gamma q], \\ &\quad k = 1, 2, ..., l,\end{aligned} \tag{6}$$

respectively.

Remark 1. *In this paper, TOR method has more splitting and parameters, so the faster convergence rate can be obtained by selecting parameters. In Method 1, when $\alpha_k = \alpha, \beta_k = \beta, \gamma_k = \gamma, \omega = 1$, the GMSMMTOR algorithm reduces to MSMTOR (Modulus-Based Synchronous Multisplitting Two Parameters Overrelaxation Method) algorithm; when $\alpha_k = \alpha, \beta_k = \beta, \gamma_k = \gamma$, GMSMMTOR algorithm reduces to GMSMTOR (Global Modulus-Based Synchronous Multisplitting Two Parameters Overrelaxation Method) algorithm; when $\gamma_k = 0, \omega = 1$, GMSMMTOR algorithm reduces to MSMMAOR (Modulus-Based Synchronous Multisplitting Multi-Parameters Accelerated Overrelaxation Method) algorithm; when $\gamma_k = 0$, GMSMMTOR algorithm reduces to GMSMMAOR (Global Modulus-Based Synchronous Multisplitting Multi-Parameters Accelerated Overrelaxation Method) algorithm; when $\alpha_k = \alpha, \beta_k = \beta, \gamma_k = 0, \omega = 1$, GMSMMTOR algorithm reduces to MSMAOR (Modulus-Based Synchronous Multisplitting Accelerated Overrelaxation Method) algorithm [26]; when $\alpha_k = \alpha, \beta_k = \beta, \gamma_k = 0$, GMSMMTOR algorithm reduces to GMSMAOR (Global Modulus-Based Synchronous Multisplitting Accelerated Overrelaxation Method) algorithm.*

Remark 2. *From Table 1, one can find that GMSMMTOR algorithm is the generalization of MSMMAOR algorithm. Moreover, when selecting proper parameters and E_k, we can get faster convergence rate.*

Table 1. The relaxed modulus-based synchronous multisplitting multi-parameter algorithm and the corresponding convergence.

Method	$\alpha_k, \beta_k, \omega$	Description	Ref.
MSMJ	$\alpha_k = 1, \beta_k = 0, \omega = 1$	Modulus-based synchronous multisplitting Jacobi algorithm	[27]
MSMGS	$\alpha_k = \beta_k = 1, \omega = 1$	Modulus-based synchronous multisplitting Gauss-Seidel algorithm	[27]
MSMSOR	$0 < \alpha(\alpha_k) = \beta(\beta_k) < \frac{1}{\rho(D^{-1}\|B\|)}, \omega = 1$	Modulus-based synchronous multisplitting SOR algorithm	[27]

Table 1. *Cont.*

Method	$\alpha_k, \beta_k, \omega$	Description	Ref.
MSMAOR	$0 < \beta(\beta_k) \leq \alpha(\alpha_k) < \frac{1}{\rho(D^{-1}\|B\|)}$	Modulus-based synchronous multisplitting AOR algorithm	[27]
MSMMAOR	$\omega = 1, 0 < \beta_k \leq \alpha_k \leq 1$ or $0 < \beta_k < \frac{1}{\rho(D^{-1}\|B\|)}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}\|B\|)}$	Modulus-based synchronous multisplitting multi-parameters AOR algorithm	[17]
GMSMMTOR	$0 < \beta_k, \gamma_k \leq \alpha_k \leq 1, 0 < \omega < \frac{2}{1+\rho'}$ or $0 < \beta_k, \gamma_k < \frac{1}{\rho(D^{-1}\|B\|)}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}\|B\|)}$ $0 < \omega < \frac{2}{1+\rho'}$ where $\rho' = \max\limits_{1 \leq k \leq l}\{1 - 2\alpha_k + 2\alpha_k\rho_\epsilon,$ $2\delta_k\rho_\epsilon - 1, 2\alpha_k\rho_\epsilon - 1\}, \delta_k = \max\{\beta_k, \gamma_k\}$	Global modulus-based synchronous multisplitting multi-parameter TOR algorithm	this paper

4. Convergence Analysis

In 2013, based on modulus-based synchronous multisplitting AOR method, Bai and Zhang got the following Theorem [27].

Theorem 1. *Let $A \in R^{n\times n}$ be an H_+-matrix, with $D =$diag(A) and $B = D - A$, and let (M_k, N_k, E_k) $(k = 1, 2, ..., l)$ and $(D - L_k, U_k, E_k)(k = 1, 2, ..., l)$ be a multisplitting and a triangular multisplitting of the matrix A, respectively [27]. Assume that $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$. If $A = D - L_k - U_k(k = 1, 2, ..., l)$ satisfies $\langle A\rangle = D - |L_k| - |U_k|(k = 1, 2, ..., l)$, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the MSMAOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R^n_+$, provided the relaxation parameters α and β satisfy*

$$0 < \beta \leq \alpha < \frac{1}{\rho(D^{-1}|B|)}.$$

In 2014, based on modulus-based synchronous multisplitting AOR algorithm, Zhang et al. [17] obtained Theorem 2.

Theorem 2. *Let $A \in R^{n\times n}$ be an H_+-matrix, with $D =$diag(A) and $B = D - A$, and let (M_k, N_k, E_k) $(k = 1, 2, ..., l)$ and $(D - L_k, U_k, E_k)(k = 1, 2, ..., l)$ be a multisplitting and a triangular multisplitting of the matrix A, respectively [17]. Assume that $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$. If $A = D - L_k - U_k(k = 1, 2, ..., l)$ satisfies $\langle A\rangle = D - |L_k| - |U_k|(k = 1, 2, ..., l)$, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the MSMMAOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R^n_+$, provided the relaxation parameters α_k and β_k satisfy*

$$0 < \beta_k \leq \alpha_k \leq 1 \text{ or } 0 < \beta_k < \frac{1}{\rho(D^{-1}|B|)}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}|B|)}.$$

In 2008, based on global relaxed non-stationary multisplitting multi-parameter TOR algorithm (GRNMMTOR) for the large sparse linear system [26], Zhang, Huang and Gu [28] got the corresponding theorem:

Theorem 3. *Let A be an H-matrix, and for $k = 1, 2, ..., l$, L_k and F_k be strictly lower triangular matrices [26]. Define the matrix $U_k, k = 1, 2, ..., l$, such that $A = D - L_k - F_k - U_k$ and assume that we have $\langle A\rangle = |D| - |L_k| - |F_k| - |U_k| = |D| - |B|$. If*

$$0 \leq \beta_k \leq \gamma_k, 0 \leq \alpha_k \leq \gamma_k, 0 < \gamma_k < \frac{2}{1+\rho}, 0 < \omega < \frac{2}{1+\rho_{\gamma_k}},$$

then GRNMMTOR method converges for any initial vector $x^{(0)}$, *where* $\rho = \rho(J), J = |D|^{-1}|B|$, $\rho_{\gamma_k} = \max_{1 \le k \le \alpha} \{|1-\gamma_k| + \gamma_k \rho_\epsilon\}, q(m,k) \ge 1, m = 0,1,...,k = 1,2,...,l.$

Based on global modulus-based synchronous multisplitting multi-parameter TOR algorithm, we analyze the wider results of the presented algorithms for LCPs, which is as follows:

Theorem 4. *Let* $A \in R^{n\times n}$ *be an* H_+*-matrix, with* $D =$diag(A) *and* $B = D - A$, *and let* (M_k, N_k, E_k) $(k = 1,2,...,l)$ *and* $(D - L_k - F_k, U_k, E_k)(k = 1,2,...,l)$ *be a multisplitting and a triangular multisplitting of the matrix A, respectively. Assume that* $\gamma > 0$ *and the positive diagonal matrix* Ω *satisfies* $\Omega \ge D$. *If* $A = D - L_k - F_k - U_k(k = 1,2,...,l)$ *satisfies* $\langle A \rangle = D - |L_k| - |F_k| - |U_k|(k = 1,2,...,l)$, *then the iteration sequence* $\{z^{(m)}\}_{m=0}^{\infty}$ *generated by the GMSMMTOR iteration method converges to the unique solution* z_* *of* $LCP(q, A)$ *for any initial vector* $z^{(0)} \in R^n_+$, *provided the relaxation parameters* α_k *and* β_k, ω *satisfy*

$$\begin{array}{l} 0 < \beta_k, \gamma_k \le \alpha_k \le 1, 0 < \omega < \frac{2}{1+\rho'} \text{ or} \\ 0 < \beta_k, \gamma_k < \frac{1}{\rho(J|)}, 1 < \alpha_k < \frac{1}{\rho(J)}, 0 < \omega < \frac{2}{1+\rho'}, \end{array} \tag{7}$$

where $\rho = \rho(J) < 1, J = D^{-1}|B|, \rho' = \max_{1\le k\le l}\{1 - 2\alpha_k + 2\alpha_k\rho_\epsilon, 2\delta_k\rho_\epsilon - 1, 2\alpha_k\rho_\epsilon - 1\}, \delta_k = \max\{\beta_k, \gamma_k\}$. *Moreover,* β_k, γ_k *should be greater than or less than* α_k *at once.*

Proof 1. From Lemma 3 and Equation (6), for GMSMMTOR algorithm, we have

$$\begin{aligned} (\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))x_* &= [(1-\alpha_k)D + (\alpha_k - \beta_k)L_k + (\alpha_k - \gamma_k)F_k \\ &\quad + \alpha_k U_k]x_* + \alpha_k[(\Omega - A)|x_*| - \gamma q], \\ &\quad k = 1,2,...,l, \end{aligned} \tag{8}$$

by subtracting Equation (8) from Equation (6), we obtain

$$\begin{aligned} x^{(m,k)} - x_* &= (\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))^{-1} \\ &\quad [(1-\alpha_k)D + (\alpha_k - \beta_k)L_k + (\alpha_k - \gamma_k)F_k + \alpha_k U_k](x^{(m)} - x_*) \\ &\quad + (\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))^{-1}\alpha_k(\Omega - A)(|x^{(m)}| - |x_*|), k = 1,2,...,l, \end{aligned}$$

then, the error about the GMSMMTOR algorithm is as follows:

$$\begin{aligned} x^{(m+1)} - x_* &= \omega \textstyle\sum_{k=1}^{l} E_k(\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))^{-1} \\ &\quad [(1-\alpha_k)D + (\alpha_k - \beta_k)L_k + (\alpha_k - \gamma_k)F_k + \alpha_k U_k](x^{(m)} - x_*) \\ &\quad + \omega \textstyle\sum_{k=1}^{l} E_k(\alpha_k\Omega + D - (\beta_k L_k + \gamma_k F_k))^{-1} \\ &\quad \alpha_k(\Omega - A)(|x^{(m)}| - |x_*|) + (1-\omega)(x^{(m)} - x_*), \end{aligned} \tag{9}$$

Equation (9) is the base for discussing the convergence results of GMSMMTOR algorithm. If we take the absolute values on both sides of Equation (9) and compute $||x^{(m)}| - |x_*|| \le |x^{(m)} - x_*|$, defining $\epsilon^{(m)} = x^{(m)} - x_*$ and assembling homothetic terms together, we have

$$|\epsilon^{(m)}| = |x^{(m+1)} - x_*| \le \mathcal{H}_{GMSMMTOR}|x^{(m)} - x_*|, \tag{10}$$

where

$$\begin{aligned} \mathcal{H}_{GMSMMTOR} &= \omega \textstyle\sum_{k=1}^{l} E_k(\alpha_k\Omega + D - (\beta_k|L_k| + \gamma_k|F_k|))^{-1} \\ &\quad [|1-\alpha_k|D + |\alpha_k - \beta_k||L_k| + |\alpha_k - \gamma_k||F_k| \\ &\quad + \alpha_k|U_k| + \alpha_k|\Omega - A|] + |1-\omega|I. \end{aligned} \tag{11}$$

□

Problem 1. *If* $0 < \beta_k, \gamma_k \leq \alpha_k \leq 1, 0 < \omega < \frac{2}{1+\rho'}$. *We define*

$$\begin{array}{rcl} M_k & = & \alpha_k\Omega + D - (\beta_k|L_k| + \gamma_k|F_k|), \\ N_k^1 & = & (1-\alpha_k)D + (\alpha_k - \beta_k)|L_k| + (\alpha_k - \gamma_k)|F_k| + \alpha_k|U_k| + \alpha_k|\Omega - A|. \end{array} \tag{12}$$

By Equation (12), $\Omega \geq D$ *and* $A = D - B$, *so the diagonal part of* $|\Omega - A|$ *is* $\Omega - D$ *and off-diagonal part is B. So* $|\Omega - A| = (\Omega - D) + |B|, |B| = |L_k| + |F_k| + |U_k|, k = 1, 2, ..., l$, *we have* $N_k^1 = M_k - 2\alpha_k D + 2\alpha_k|B|$. *So*

$$H_{GMSMMTOR} = M_k^{-1}N_k^1 = M_k^{-1}(M_k - 2\alpha_k D + 2\alpha_k|B|) = I - 2\alpha_k M_k^{-1}(D - |B|),$$

and

$$\begin{array}{rcl} |H_{GMSMMTOR}| & \leq & M_k^{-1}[M_k - 2\alpha_k(D - |B|)] \\ & \leq & I - 2\alpha_k M_k^{-1}D(I - D^{-1}|B|). \end{array}$$

Let e denote vector $e = (1, 1, ..., 1)^T \in R^n$. *Since J is a nonnegative matrix, this matrix* $J + \epsilon ee^T$ *has only positive entries and is irreducible for any* $\epsilon > 0$. *By Perron-Frobenius theorem for any* $\epsilon > 0$, *there is a vector* $x_\epsilon > 0$ *such that*

$$(J + \epsilon ee^T)x_\epsilon = \rho_\epsilon x_\epsilon,$$

where $\rho_\epsilon = \rho(J + \epsilon ee^T) = \rho(J_\epsilon)$. *Moreover, if* $\epsilon > 0$ *is small enough, we obtain* $\rho_\epsilon < 1$ *by continuity of spectral radius. Since* $0 < \alpha_k \leq 1$, *we also obtain* $1 - 2\alpha_k + 2\alpha_k\rho < 1$, *and* $1 - 2\alpha_k + 2\alpha_k\rho_\epsilon < 1$. *So*

$$\begin{array}{rcl} |H_{GMSMMTOR}| & \leq & I - 2\alpha_k M_k^{-1}D[I - (D^{-1}|B| + \epsilon ee^T)] \\ & = & I - 2\alpha_k M_k^{-1}D[I - J_\epsilon]. \end{array}$$

Multiplying x_ϵ *in both sides of the equation, and* $M_k^{-1} \geq D^{-1}$, *we have*

$$\begin{array}{rcl} |H_{GMSMMTOR}|x_\epsilon & \leq & x_\epsilon - 2\alpha_k M_k^{-1}D[1 - \rho(J_\epsilon)]x_\epsilon \\ & \leq & x_\epsilon - 2\alpha_k D^{-1}D[1 - \rho(J_\epsilon)]x_\epsilon \\ & = & (1 - 2\alpha_k + 2\alpha_k\rho(J_\epsilon))x_\epsilon \end{array}$$

By Equation (11), we have

$$\begin{array}{rcl} |\mathcal{H}_{GMSMMTOR}|x_\epsilon & \leq & \omega\Sigma_{k=1}^l E_k(1 - 2\alpha_k + 2\alpha_k\rho(J_\epsilon))x_\epsilon + |1-\omega|x_\epsilon \\ & \leq & \omega\Sigma_{k=1}^l E_k(1 - 2\alpha_k + 2\alpha_k\rho_\epsilon)x_\epsilon + |1-\omega|x_\epsilon \\ & = & (\omega\rho_1 + |1-\omega|)x_\epsilon \\ & = & \theta_1 x_\epsilon(\epsilon \to 0), \end{array}$$

where $\theta_1 = \omega\rho_1 + |1-\omega| < 1, \rho_1 = \Sigma_{k=1}^l E_k(1 - 2\alpha_k + 2\alpha_k\rho_\epsilon)$.

Problem 2. *If* $0 < \beta_k, \gamma_k < \frac{1}{\rho(D^{-1}|B|)}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}|B|)}, 0 < \omega < \frac{2}{1+\rho'}$.
Subproblem 2.1.: $\alpha_k \geq \beta_k$ *and* $\alpha_k \geq \gamma_k$. *We define:*

$$\begin{array}{rcl} N_k^2 & = & (\alpha_k - 1)D + (\alpha_k - \beta_k)|L_k| + (\alpha_k - \gamma_k)|F_k| + \alpha_k|U_k| \\ & & +\alpha_k|\Omega - A| = M_k - 2D + 2\alpha_k|B|. \end{array} \tag{13}$$

So

$$\begin{array}{rcl} |H_{GMSMMTOR}| & \leq & M_k^{-1}[M_k - 2(D - \alpha_k|B|)] \\ & \leq & I - 2M_k^{-1}D(I - \alpha_k D^{-1}|B|). \end{array}$$

Similar to the Problem 1, let e denote vector $e = (1, 1, ..., 1)^T \in R^n$*, and* $x_\epsilon > 0$ *such that* $J_\epsilon x_\epsilon = (J + \epsilon ee^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon$*. Moreover, if* $\epsilon > 0$ *is small enough, we can obtain* $\rho_\epsilon < 1$ *by continuity of spectral radius. Since* $1 < \alpha_k < \frac{1}{\rho(D^{-1}|B|)}$*, we may obtain*

$$2\alpha_k\rho - 1 < 1 \text{ and } 2\alpha_k\rho_\epsilon - 1 < 1,$$

so

$$\begin{aligned} |H_{GMSMMTOR}| &\leq I - 2M_k^{-1}D[I - \alpha_k(D^{-1}|B| + \epsilon ee^T)] \\ &= I - 2M_k^{-1}D[I - \alpha_k J_\epsilon]. \end{aligned}$$

Multiplying x_ϵ *in both sides of the above equation, and* $M_k^{-1} \geq D^{-1}$*, we have*

$$\begin{aligned} |H_{RMSMMAOR}|x_\epsilon &\leq x_\epsilon - 2M_k^{-1}D[1 - \alpha_k\rho(J_\epsilon)]x_\epsilon \\ &\leq x_\epsilon - 2(1 - \alpha_k\rho(J_\epsilon))]x_\epsilon \\ &= (2\alpha_k\rho(J_\epsilon) - 1)x_\epsilon. \end{aligned}$$

By Equation (11), we have

$$\begin{aligned} |\mathcal{H}_{GMSMMTOR}|x_\epsilon &\leq \omega\Sigma_{k=1}^{l}E_k(2\alpha_k\rho(J_\epsilon) - 1)x_\epsilon + |1-\omega|x_\epsilon \\ &\leq \omega\Sigma_{k=1}^{l}E_k(2\alpha_k\rho_\epsilon - 1)x_\epsilon + |1-\omega|x_\epsilon \\ &= (\omega\rho_2 + |1-\omega|)x_\epsilon \\ &= \theta_2 x_\epsilon(\epsilon \to 0), \end{aligned}$$

where $\theta_2 = \omega\rho_2 + |1-\omega| < 1, \rho_2 = \Sigma_{k=1}^{l}E_k(2\alpha_k\rho_\epsilon - 1)$.

Subproblem 2.2.: $\alpha_k \leq \beta_k$ *and* $\alpha_k \leq \gamma_k$*. We define*

$$\begin{aligned} N_k^3 &= (\alpha_k - 1)D + (\beta_k - \alpha_k)|L_k| + (\gamma_k - \alpha_k)|F_k| + \alpha_k|U_k| + \alpha_k|\Omega - A| \\ &= M_k - 2D + 2\beta_k|L_k| + 2\gamma_k|F_k| + 2\alpha_k|U_k| \\ &\leq M_k - 2D + 2\delta_k|B|. \end{aligned} \tag{14}$$

where $\delta_k = \max\{\beta_k, \gamma_k\}$*, so*

$$\begin{aligned} |H_{GMSMMTOR}| &\leq M_k^{-1}[M_k - 2(D - \delta_k|B|)] \\ &\leq I - 2M_k^{-1}D(I - \delta_k D^{-1}|B|). \end{aligned}$$

Similar to the Problem 1, let e denote vector $e = (1, 1, ..., 1)^T \in R^n$*, and* $x_\epsilon > 0$ *such that* $J_\epsilon x_\epsilon = (J + \epsilon ee^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon$*. Furthermore, if* $\epsilon > 0$ *is small enough, we obtain* $\rho_\epsilon < 1$ *by continuity of spectral radius. Since* $0 < \beta_k, \gamma_k < \frac{1}{\rho(D^{-1}|B|)}$*, we can obtain*

$$2\delta_k\rho - 1 < 1 \text{ and } 2\delta_k\rho_\epsilon - 1 < 1,$$

so

$$\begin{aligned} |H_{GMSMMTOR}| &\leq I - 2M_k^{-1}D[I - \delta_k(D^{-1}|B| + \epsilon ee^T)] \\ &= I - 2M_k^{-1}D[I - \delta_k J_\epsilon]. \end{aligned}$$

Multiplying x_ϵ *in both sides of the equation, and* $M_k^{-1} \geq D^{-1}$*, we have*

$$\begin{aligned} |H_{GMSMMTOR}|x_\epsilon &\leq x_\epsilon - 2(1 - \delta_k\rho(J_\epsilon))]x_\epsilon \\ &= (2\delta_k\rho(J_\epsilon) - 1)x_\epsilon \end{aligned}$$

By Equation (11), we have

$$\begin{array}{rcl} |\mathcal{H}_{GMSMMTOR}|x_\epsilon & \leq & \omega\Sigma_{k=1}^{l}E_k(2\delta_k\rho(J_\epsilon)-1)x_\epsilon+|1-\omega|x_\epsilon \\ & \leq & \omega\Sigma_{k=1}^{l}E_k(2\delta_k\rho_\epsilon-1)x_\epsilon+|1-\omega|x_\epsilon \\ & = & (\omega\rho_3+|1-\omega|)x_\epsilon \\ & = & \theta_3x_\epsilon(\epsilon\to 0), \end{array}$$

where $\theta_3=\omega\rho_3+|1-\omega|<1, \rho_3=\Sigma_{k=1}^{l}E_k(2\delta_k\rho_\epsilon-1)$.

Remark 3. *Obviously, one can find that the conditions of Theorem 4 in this paper are wider than those of Theorem 2.3 in [28]. Furthermore, we have more choices for the splitting $A=B-C$ which makes multisplitting iterative methods converge. So, convergence results are generalized in applications.*

Remark 4. *In this paper, GMSMMTOR algorithm is also the generalization of MSMAOR method in [27] and MSMMAOR algorithm in [17].*

5. Numerical Experiments

In this section, numerical examples are used to illustrate the feasibility and effectiveness of the relaxed modulus-based synchronous multisplitting multi-parameter AOR methods (GMSMMAOR) ($F=U$) in terms of iteration count (denoted by IT) and computing time (denoted by CPU), and norm of absolute residual vectors (denoted by RES). Here, RES is defined as

$$\mathrm{RES}(z^{(k)})=\|\min(Az^{(k)}+q,z^{(k)})\|_2$$

where $z^{(k)}$ is the kth approximate solution to the LCP(q,A) and the minimum is taken componentwise in [10].

In our numerical computations, to compare the GMSMMAOR method with the modulus-based synchronous multisplitting multi-parameter methods (MSMAOR), all initial vectors are chosen to be

$$x^{(0)}=(1,0,1,0,\cdots,1,0,\cdots)^T\in R^n$$

all runs are performed in MATLAB 7.0 (MathWorks, Natick, MA, USA) with double machine precision, and all iterations are terminated with $\mathrm{RES}(z^{(k)})\leq 10^{-5}$. In the table, α,β denote the iteration parameters in the GMSMMAOR methods and the MSMAOR. In addition, we take $\Omega=\frac{1}{2\alpha}D$ in [10] for GMSMMAOR and MSMAOR methods. In particular, when we choose the parameter pair (α_k,β_k) to be (α_k,α_k) $(1,1)$ and $(1,0)$ respectively, the GMSMMAOR method gives the so-called GMSMMSOR (Global Modulus-Based Synchronous Multisplitting Multi-Parameters Successive Over Relaxation Method), GMSMGS (Global Modulus-Based Synchronous Multisplitting Multi-Parameters Successive Gauss-Seidel Method), and GMSMJ (Global Modulus-Based Synchronous Multisplitting Multi-Parameters Successive Jacobi Method) methods, correspondingly. For convenience, let $\alpha_k=\alpha,\beta_k=\beta,\gamma=2,\omega=1,k=1$.

Let m be a prescribed positive integer and $n=m^2$. Consider the LCP(q,A), in which $A\in R^{n\times n}$ is given by $A=\hat{A}+\mu I$ and $q\in R^n$ is given by $q=-Mz^*$ where

$$\hat{A}=\mathrm{tridiag}(-rI,S,-tI)=\begin{pmatrix} S & -tI & 0 & \cdots & 0 & 0 \\ -rI & S & -tI & \cdots & 0 & 0 \\ 0 & -rI & S & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & S & -tI \\ 0 & 0 & \cdots & \cdots & -rI & S \end{pmatrix}\in R^{n\times n} \tag{15}$$

is a block-tridiagonal matrix,

$$S = \text{tridiag}(-1, 4, -1) = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 4 & -1 \\ 0 & 0 & \cdots & \cdots & -1 & 4 \end{pmatrix} \in R^{n \times n} \tag{16}$$

is a tridiagonal matrix, and

$$z^* = (1, 2, 1, 2, \cdots, 1, 2, \cdots)^T \in R^n$$

is the unique solution of the LCP(q, A), one can see [10] for more details.

For symmetric case, we take $r = t = 1$, which is considered in [10]. In this case, the system matrix $A \in R^{n \times n}$ is symmetric positive and definite for $\mu \geq 0$. So, the LCP(q, A) has a unique solution.

In Table 2, the iteration steps, the CPU times, and the residual norms of GMSMMAOR and MSMAOR methods for the symmetric case are listed for different parameters and different problem sizes of m. When both GMSMMAOR and MSMAOR methods are applied to solve the LCP(q, A), the iteration parameters α, β about MSMAOR method satisfy Theorem 4.1 in [27] and Theorem 2 in this paper, but the iteration parameters α, β about GMSMMAOR method only satisfy Theorem 2 in this paper and don't satisfy Theorem 4.1 in [27].

From Table 2, for GMSMMAOR and MSMAOR methods with $\alpha = 1, \beta = 1.2$ and $\alpha = 1, \beta = 0.7$, fixing the value of μ, it is easy to see that the iteration steps do not change with the increasing of the problem size m. However, CPU times increase as the problem size m increases. Moreover, for GMSMMAOR and MSMAOR methods, fixing the value of m, it is also easy to see that the iteration steps and CPU times decrease as the increasing of the problem size μ. In our numerical experiments, we find that the iteration steps and CPU times of GMSMMAOR are less than that of MSMAOR under certain conditions.

Table 2. IT, CPU and Error for GMSMMAOR and MSMAOR with different parameters in symmetric case.

		m	20	30	40	50	60
$\mu = 0.5$	GMSMMAOR	IT	22	22	22	22	22
	$\alpha = 1$	CPU	0.1560	0.7800	2.4336	5.9280	12.4957
	$\beta = 1.2$	Error	7.2225×10^{-6}	7.2598×10^{-6}	7.2970×10^{-6}	7.3390×10^{-6}	7.3707×10^{-6}
$\mu = 0.5$	MSMAOR	IT	30	30	30	31	31
	$\alpha = 1$	CPU	0.2184	1.0764	3.2916	8.3773	17.6905
	$\beta = 0.7$	Error	9.7188×10^{-6}	9.8399×10^{-6}	9.9531×10^{-6}	7.3792×10^{-6}	7.4496×10^{-6}
$\mu = 1.5$	GMSMMAOR	IT	19	19	19	19	19
	$\alpha = 1$	CPU	0.1716	0.6552	2.0748	5.0856	10.7797
	$\beta = 1.2$	Error	6.6884×10^{-6}	6.8943×10^{-6}	7.0943×10^{-6}	7.2888×10^{-6}	7.4782×10^{-6}
$\mu = 1.5$	MSMAOR	IT	23	24	24	24	24
	$\alpha = 1$	CPU	0.1716	0.8424	2.6520	6.4584	13.6657
	$\beta = 0.7$	Error	9.5945×10^{-6}	6.6969×10^{-6}	7.0677×10^{-6}	7.4200×10^{-6}	7.7563×10^{-6}
$\mu = 2.5$	GMSMMAOR	IT	17	17	17	17	17
	$\alpha = 1$	CPU	0.1404	0.6084	1.8720	4.5552	9.6565
	$\beta = 1.2$	Error	7.6793×10^{-6}	8.2513×10^{-6}	8.7861×10^{-6}	9.2902×10^{-6}	9.7683×10^{-6}
$\mu = 2.5$	MSMAOR	IT	20	20	20	21	21
	$\alpha = 1$	CPU	0.1404	0.7020	2.2932	5.6472	11.9341
	$\beta = 0.7$	Error	8.3861×10^{-6}	9.4078×10^{-6}	6.1592×10^{-6}	6.6458×10^{-6}	7.0992×10^{-6}

IT: iteration count; CPU: computing time, Error: norm of residual vectors, m: problem size, $\mu : \mu \geq 0$ is a parameter to get a different matrix A.

6. Conclusions

In this paper, global modulus-based synchronous multisplitting multi-parameters TOR methods has been established and its convergence properties are discussed in detail when the system matrix is either a positive-definite matrix or an H_+-matrix. Numerical experiments show that the GMSMMTOR methods are feasible under certain conditions.

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