Mathematical and Computational Applications

MDPI

Article

Finite Convergence for Feasible Solution Sequence of Variational Inequality Problems

Wenling Zhao *, Ruyu Wang and Hongxiang Zhang

School of Science, Shandong University of Technology, Zibo 255049, China; ruyu_wang@126.com (R.W.); 18369906817@163.com (H.Z.)
* Correspondence: zwlsdj@163.com; Tel.: +86-135-0644-8807

Academic Editor: Fazal M. Mahomed
Received: 19 April 2017; Accepted: 7 June 2017; Published: 20 June 2017

Abstract: We establish the notion of augmented weak sharpness of solution sets for the variational inequality problems which can be abbreviated to VIPs. This notion of augmented weak sharpness is an extension of the weak sharpness of the solution set of monotone variational inequality, and it overcomes the defect of the solution set not satisfying the weak sharpness in many cases. Under the condition of the solution set being augmented weak sharp, we present a necessary and sufficient condition for finite convergence for feasible solution sequence of VIP. The result is an extension of published results, and the augmented weak sharpness also provides weaker sufficient conditions for the finite convergence of many optimization algorithms.

Keywords: variational inequality problem; feasible solution sequence; augmented weak sharpness; finite convergence

1. Introduction

The variational inequality problem (VIP) is one of classical mathematical problems. Many models of problems in engineering and physics are constructed by partial differential equations with some suitable boundary conditions and primal conditions and are described by different kinds of variational inequality problems. Consider the variational inequality problem $VIP(F, S)$ which take F as the objective function and S as the domain: seek $\bar{x} \in S$ satisfying

$$\langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \forall y \in S,$$

where $S \subset \mathbb{R}^n$, $F : S \to \mathbb{R}^n$ is a vector-valued function, $\langle \cdot, \cdot \rangle$ represents the inner product, $\mathbb{R}$ means the set of real numbers. $\bar{S}$ is consist of all optimal solution $\bar{x}$, $\bar{S} \subseteq S$. We assume the solution set $\bar{S}$ of $VIP(F, S)$ is nonempty.

Finite convergence of the feasible solution sequence produced by any algorithm for VIP has been widely concerned and researched for a long time. Early in previous studies, Rockafellar [1], Polyak [2], and Ferris [3] successively put forward the weak sharp minima and strong non-degeneracy of the solution sets of mathematical programming problems, and prove that any one of them is the sufficient condition for finite convergence of the proximal point algorithm [1,4–6] and some important iterative algorithms [7–12]. It is worth noting that the finite convergence of the feasible solution sequence produced by the algorithms above, depending on not only the weak sharp minima or strong non-degeneracy of the solution set, but also the specific structural features of the algorithms. Thus, it has more universal significance to study further the conditions for the finite convergence—which does not depend on the specific algorithm—under the condition of weak sharp minima or strong non-degeneracy. Burke and Moré [8] propose the necessary and sufficient condition for the fact that the feasible solution sequence is finitely convergent at the stationary point when the sequence converges to a strongly non-degenerate point for smooth programming problem (Corollary 3.5 in [8]). In order to generalize the study into variational inequality, Marcotte and Zhu [13] give the notion of weak sharpness for its solution set, then generalize the study into the continuous pseudo-monotone^{+} variational inequality and yield the necessary and sufficient condition for finite convergence of a feasible solution sequence (Theorem 5.2 in [13]) under the solution set being weak sharpness, a feasible solution sequence satisfying two assumption conditions, and the feasible solution set being compact. Xiu and Zhang [6] improve Theorem 5.2 in [13] by removing the conditions for the pseudo-monotone^{+} and the feasible solution set being compact, and yielded the same results (Theorem 3.2 in [6]).

However, in many cases, the weak sharpness and strong non-degeneracy for the solution set of variational inequality are not true (see Example 1 in this paper). Therefore, in order to give finite convergence of a feasible solution sequence under more relaxed conditions, we establish the notion of augmented weak sharpness relative to a feasible solution sequence for a variational inequality problem. The new notion is a substantive extension of the weak sharpness of the solution set of monotone variational inequality. We give a necessary and sufficient condition for the finite convergence of a feasible solution sequence, under the feasible solution set of VIP is augmented weak sharp. The result is an extension of published results (see the corollaries in Section 3 in this paper). In addition, the notion of augmented weak sharpness which we establish also provides weaker sufficient conditions for the finite convergence of many optimization algorithms.

The paper is organized as follows. After some elementary definitions and notations, in Section 2, we introduce the notions of weak sharpness and strong non-degeneracy of the solution set, we establish the notion of augmented weak sharpness, and prove that augmented weak sharpness is a necessary condition for the weak sharpness and strong non-degeneracy of the solution set to VIP. In Section 3, we give the necessary and sufficient conditions for the finite convergence of a feasible solution sequence, under the condition that the solution set is augmented weak sharp on feasible solution sequence.

Next, we introduce the notions and symbols used in this paper.

Suppose $N \subseteq \{1, 2, \dots\}$ is an infinite subsequence, and $C^k \subset R^n$, $k = 1, 2, \dots$. We define

$$\limsup_{k\to\infty} C^k = \{x \in R^n |\ \text{exist } N, x^k \in C^k, \text{such that } \lim_{k\in N, k\to\infty} x^k = x\},$$

$$\liminf_{k\to\infty} C^k = \{x \in R^n |\ \text{exist }\ x^k \in C^k,\ \text{such that } \lim_{k\to\infty} x^k = x\}.$$

According to the above notions, it follows that

$$\liminf_{k\to\infty} C^k \subseteq \limsup_{k\to\infty} C^k.$$

Suppose $C \subset \mathbb{R}^n$ is a subset, *intC* denotes the interior of C. $\bar{x} \in C$, $\tau \in (0, 1)$, the tangent cone of C at point $\bar{x}$ is defined as [14]

$$T_C(\bar{x}) = \left\{d \in R^n | \exists x^k \in C, x^k \to \bar{x}, \tau^k \downarrow 0 (k \to \infty), \lim_{k\to\infty} (x^k - \bar{x})/\tau^k = d\right\}.$$

The regular normal cone of C at point $\bar{x}$ is

$$\hat{N}_C(\bar{x}) = \left\{d \in R^n\ | < d, x - \bar{x} > \leq o(\|x - \bar{x}\|), x \in C\right\}.$$

In general meaning, the normal cone of C at point $\bar{x}$ is defined as

$$N_C(\bar{x}) = \limsup_{x^k \in C, x^k \to \bar{x}} \hat{N}_C(x^k).$$

The polar cone of C is

$$C^{\circ} = \left\{y \in R^n | < y, x > \leq 0, \forall x \in C\right\}.$$

According to Proposition 6.5 in [15], we have

$$T_C(x)^\circ = \hat{N}_C(x).$$

When C is convex, according to Theorem 6.9 in [15], we have

$$N_C(\bar{x}) = \hat{N}_C(\bar{x}) = \Big\{d| < d, x - \bar{x} > \leq 0, \forall x \in C\Big\}.$$

Suppose $x \in R^n$; the projection of x on a closed set C is

$$P_C(x) = argmin_{y \in C}\|y - x\|.$$

Note by the above definition that when C is a closed convex set, $P_C(\cdot)$ is a single-valued mapping. However, when C is a closed set, $P_C(\cdot)$ may be a set-valued mapping.

The distance between x and C is given by

$$dist(x, C) = \inf_{y \in C} \|y - x\|.$$

If C is a closed set, we have

$$dist(x, C) = \|P_C(x) - x\|.$$

Suppose the subdifferential $\partial\psi(\cdot)$ of function $\psi(\cdot)$ at point $x \in C$ is a non-empty set. Then, the projection subdifferential of $\psi(\cdot)$ at point x is

$$P_{T_C(x)}\Big(-\partial\psi(x)\Big) = \Big\{P_{T_C(x)}(-u)|u \in \partial\psi(x)\Big\}.$$

If $\psi(\cdot)$ is continuously differentiable in a neighborhood of point $x \in C$, according to Practice 8.8 in [15], we have $\partial\psi(x) = \{\nabla\psi(x)\}$; i.e., the projection subdifferential is projection gradient $P_{T_C(x)}(-\nabla\psi(x))$.

We say that $\{x^k\} \subset R^n$ converges finitely to C if there exists k_0 such that $x^k \in C$ for all $k \geq k_0$.

2. The Augmented Weak Sharpness of Solution Sets

In this section, we first introduce the notions of the weak sharpness and strong non-degeneracy of solution sets for variational inequality (VIP) in some existing literature. Then, we give the definition of the augmented weak sharpness of solution sets relative to feasible solution sequence for VIP, and clarify, by examples, the new notion is an extension of the weak sharpness and strong non-degeneracy of the solution set of monotone variational inequality.

Now we give the notions of weak sharpness and strong non-degeneracy of solution sets for VIP (see [16]).

Definition 1. *The solution set $\bar{S} \subset S$ is called a set of weak sharpness if*

$$-F(\bar{x}) \in int\bigcap\nolimits_{x \in \bar{S}}\Big[T_S(x) \cap N_{\bar{S}}(x)\Big]^\circ, \ \ \forall \bar{x} \in \bar{S}. \tag{1}$$

The point $\bar{x}$ satisfying Definition 1 is called weak sharp point.

In the general case, $\bar{S}$ is not necessarily a convex set. When $\bar{S}$ is non-convex, according to Proposition 6.5 in [15], $\hat{N}_{\bar{S}}(\cdot)$ is a closed convex cone, but $N_{\bar{S}}(\cdot)$ is only a closed cone, and there is

$$\hat{N}_{\bar{S}}(\cdot) \subseteq N_{\bar{S}}(\cdot).$$

Thus, there is

$$\Big[T_S(x) \cap N_{\bar{S}}(x)\Big]^\circ \subseteq \Big[T_S(x) \cap \hat{N}_{\bar{S}}(x)\Big]^\circ.$$

So, we give the following definition of weak sharpness. It is more relaxed than Definition 1.

Definition 2. *The solution set $\bar{S} \subset S$ is called a set of weak sharpness if*

$$-F(\bar{x}) \in \ int \bigcap\nolimits_{x \in \bar{S}} \Big[T_S(x) \cap \hat{N}_{\bar{S}}(x)\Big]^\circ, \ \ \forall \bar{x} \in \bar{S}.$$

Next, we introduce the notion of strong non-degeneracy.

Definition 3. *The solution set $\bar{S} \subset S$ is a set of strong non-degeneracy, if*

$$-F(\bar{x}) \in int N_S(\bar{x}), \ \ \forall \bar{x} \in \bar{S}. \tag{2}$$

The point $\bar{x}$ satisfying (2) is called a strongly non-degenerate point.

As described earlier, in many cases, the solution set is not weak sharp or strongly non-degenerate. In order to overcome this defect, we now introduce the notion of the augmented weak sharpness of a solution set relative to a feasible solution sequence for VIP.

Definition 4. *Suppose that solution set $\bar{S} \subset S$ is closed, $\{x^k\} \subset S$. We say $\bar{S}$ is a set of augmented weak sharpness relative to the feasible solution sequence $\{x^k\}$ if one of the following is true:*

(1) $K = \{k | x^k \notin \bar{S}\}$ *is a finite set;*
(2) *When $K = \{k | x^k \notin \bar{S}\}$ is an infinite sequence, there exist a set-valued mapping $H : \bar{S} \to 2^{R^n}$ such that*

   (a) *there is a constant $\alpha > 0$, satisfying*

$$\alpha B \subset H(z) + \Big[T_S(z) \bigcap \hat{N}_{\bar{S}}(z)\Big]^\circ, \ \ \forall z \in \bar{S}.$$

   (b) *for $\forall z^k \in P_{\bar{S}}(x^k)$ and $\forall v^k \in H(z^k)$, there is*

$$\limsup\nolimits_{k \in K, k \to \infty} \psi_k = \frac{1}{\|x^k - z^k\|} \Big\langle F(x^k) - v^k, x^k - z^k \Big\rangle \geq 0.$$

The following simple example shows that the solution set is not weak sharp, but it is augmented weak sharp relative to the feasible solution sequence.

Example 1. *Consider $VIP(F, S)$:*

$$F(x) = (\cos x_1, e^{x_2}),$$

where

$$S = \Big\{x \in R^2 | 0 \leq x_1 \leq \tfrac{\pi}{2}, x_2 \geq 0\Big\},$$

$$\bar{S} = \Big\{(0,0), (\tfrac{\pi}{2}, 0)\Big\}.$$

When $x \in \bar{S}$, there is

$$F(x) = \begin{cases} (1,1), & if\ x = (0,0) \\ (0,1), & if\ x = (\frac{\pi}{2}, 0) \end{cases} \tag{3}$$

$$N_S(x) = \begin{cases} \Big\{\xi \in R^2 | \xi_1 \leq 0, \xi_2 \leq 0\Big\}, & if\ x = (0,0) \\ \Big\{\xi \in R^2 | \xi_1 \geq 0, \xi_2 \leq 0\Big\}, & if\ x = (\frac{\pi}{2}, 0). \end{cases} \tag{4}$$

This is a non-monotone variational inequality problem. According to (3) and (4), $\bar{S}$ is not a strongly non-degenerate set, because $x = (\frac{\pi}{2}, 0)$ is not a strongly non-generate point.

Next, we prove that $\bar{S}$ is an augmented weak sharp set relative to $\{x^k\} \subset S$, satisfying the following conditions.

(i) $\left\{x^k\right\} \subset \left\{x \in R^2 | x_1 \le \frac{\pi}{4}\right\} \cup \left\{x \in R^2 | x_1 \ge \frac{\pi}{4}, x_1 + x_2 \ge \frac{\pi}{2}\right\}$;

(ii) $\lim_{k\to\infty} dist(x^k, \bar{S}) = 0$.

Suppose $K = \{k | x^k \notin \bar{S}\}$ is an infinite sequence. Let $\lambda \in (0, \frac{1}{2})$, introducing set-valued mapping

$$H(x) = \begin{cases} (\lambda, \lambda), & x = (0,0), \\ (-\lambda, \lambda), & x = (\frac{\pi}{2}, 0). \end{cases} \tag{5}$$

By (4) and (5), it maintains (a) of Definition 4.

For $k \in K$

$$\psi_k := \frac{1}{\|x^k - P_{\bar{S}}(x^k)\|}\left\langle F(x^k) - H\left(P_{\bar{S}}(x^k)\right), x^k - P_{\bar{S}}(x^k)\right\rangle,$$

where

$$P_{\bar{S}}(x^k) = \begin{cases} (0,0), & if\ \ x_1^k \le \frac{\pi}{4}, \\ (\frac{\pi}{2}, 0), & if\ \ x_1^k \ge \frac{\pi}{4}. \end{cases} \tag{6}$$

According to (ii), the accumulation points $\bar{x}$ of the bounded sequence $\{x^k\}_{k\in K}$ can only be $(0,0)$ or $(\frac{\pi}{2}, 0)$. Without loss of generality, suppose $\bar{x} = (\frac{\pi}{2}, 0)$ is one of its accumulation points, so there is an infinite sequence $K_0 \subseteq K$ such that

$$\lim_{k\in K_0, k\to\infty} x^k = \left(\tfrac{\pi}{2}, 0\right). \tag{7}$$

By (5)–(7) , if $k \in K_0$ is large enough, we have

$$\begin{aligned} \psi_k &= \frac{1}{\|x^k - \bar{x}\|}\left[(\cos x_1^k + \lambda)(x_1^k - \tfrac{\pi}{2}) + (e^{x_2^k} - \lambda)x_2^k\right] \\ &\ge \frac{1}{\|x^k - \bar{x}\|}\left[\cos x_1^k (x_1^k - \tfrac{\pi}{2}) + (e^{x_2^k} - 2\lambda)x_2^k\right] \qquad (by\ \ (i)) \\ &\ge \frac{1}{\|x^k - \bar{x}\|}(x_1^k - \tfrac{\pi}{2})\cos x_1^k. \qquad (by\ \lambda \in (0, \tfrac{1}{2})) \end{aligned}$$

Then, according to (7), we have

$$\limsup_{k\in K, k\to\infty} \psi_k \ge \lim_{k\in K_0, k\to\infty} \frac{1}{\|x^k - \bar{x}\|}(x_1^k - \tfrac{\pi}{2})\cos x_1^k = 0.$$

So it maintains (b) of Definition 4.

Next, we will prove that the augmented weak sharpness of the solution set is an extension of the weak sharpness and strong non-degeneracy.

Theorem 1. For $VIP(F,S)$, F is monotonous, $\bar{S} \subset S$ is a closed set. If $\bar{S}$ is a set of weak sharpness, then $\bar{S}$ is a set of augmented weak sharpness relative to any one $\{x^k\} \subset S$.

Proof of Theorem 1. As $VIP(F,S)$ is a monotonous variational inequality problem, then its solution set $\bar{S}$ is a closed convex set, so $\hat{N}_{\bar{S}}(\cdot) = N_{\bar{S}}(\cdot)$, and $P_{\bar{S}}(\cdot)$ is single valued mapping. Let $\{x^k\} \subset S$. Suppose $K = \{k | x^k \notin \bar{S}\}$ is an infinite sequence, then let

$$H(z) = F(z), \ \ \forall z \in \bar{S}.$$

According to the hypothesis, $\bar{S}$ is a set of weak sharp minima, so according to (1) , we know (a) in Definition 4 is true. Additionally, because $F(\cdot)$ is monotonous, it maintains (b) of Definition 4. The proof is complete. □

Remark 1. *Theorem 1 shows that the augmented weak sharpness of the solution sets for (VIP) are an extension of the weak sharp minima of the solution sets of monotone variational inequality.*

Theorem 2. For $VIP(F,S)$, suppose $\bar{S} \subset S$ is a closed set, $\{x^k\} \subset S$. If $\{F(x^k)\}$ is bounded and any one of its accumulation points $\bar{p}$ satisfies $-\bar{p} \in int \bigcap_{x\in\bar{S}} \left[T_S(x) \cap \hat{N}_{\bar{S}}(x)\right]^{\circ}$. Then, $\bar{S}$ is a set of augmented weak sharpness relative to $\{x^k\}$.

Proof of Theorem 2. Suppose $K = \{k|x^k \notin \bar{S}\}$ is an infinite sequence. According to the hypotheses, $\{F(x^k)\}$ is bounded, so $\{F(x^k)\}_{k\in K}$ must have an accumulation point $\bar{p}$, such that $-\bar{p} \in int \bigcap_{x\in\bar{S}}[T_S(x) \cap \hat{N}_{\bar{S}}(x)]^{\circ}$; that is, a constant $\alpha > 0$ exists, such that

$$\alpha B \subset \bar{p} + \left[T_S(z) \cap \hat{N}_{\bar{S}}(z)\right]^{\circ}, \forall z \in \bar{S}. \tag{8}$$

Let $H(z) = \bar{p}, \forall z \in \bar{S}$, by (2.8), which maintains (a) of Definition 4.

Let $K_0 \subset K$ such that

$$\lim_{k\in K_0, k\to\infty} F(x^k) = \bar{p}. \tag{9}$$

By (9), we have

$$\begin{aligned} \limsup_{k\in K, k\to\infty} \psi_k &\geq \limsup_{k\in K_0, k\to\infty} \psi_k \\ &= \lim_{k\in K_0, k\to\infty} \frac{1}{\|x^k - P_{\bar{S}}(x^k)\|} \left\langle F(x^k) - \bar{p}, x^k - P_{\bar{S}}(x^k) \right\rangle = 0. \end{aligned}$$

It maintains (b) of Definition 4. The proof is complete. □

Theorem 3. For $VIP(F,S)$, suppose $F(\cdot)$ is continuous on S. If $\{x^k\} \subset S$ is bounded and any one of its accumulation points is a strong non-degenerate point, then $\bar{S}$ is a set of augmented weak sharpness relative to $\{x^k\}$.

Proof of Theorem 3. Suppose $K = \{k|x^k \notin \bar{S}\}$ is an infinite sequence. Then, by the assumption $\{x^k\}_{k\in K}$ must have an accumulation point $\bar{x}$. Suppose $K_0 \subset K$ satisfies

$$\lim_{k\in K_0, k\to\infty} x^k = \bar{x}. \tag{10}$$

According to the hypothesis, $\bar{x}$ is a strong non-degenerated point and $F(\cdot)$ is continuous; by Proposition 5.1 in [14], $\bar{x}$ is an isolated point of $\bar{S}$, so we have $T_{\bar{S}}(\bar{x}) = \{0\}$, $\hat{N}_{\bar{S}}(\bar{x}) = T_{\bar{S}}(\bar{x})^{\circ} = R^n$.

Therefore,

$$\left[T_S(\bar{x}) \cap \hat{N}_{\bar{S}}(\bar{x})\right]^{\circ} = T_S(\bar{x})^{\circ} = N_S(\bar{x}). \tag{11}$$

Thus, by (2) and (11), there exists a constant $\alpha > 0$, such that

$$\alpha B \subset F(\bar{x}) + N_S(\bar{x}) = F(\bar{x}) + \left[T_S(\bar{x}) \cap \hat{N}_{\bar{S}}(\bar{x})\right]^{\circ}. \tag{12}$$

Now we define a set-valued mapping $H : \bar{S} \to 2^{R^n}$ as follows

$$H(z) = \begin{cases} F(\bar{x}), \ z = \bar{x}, \\ R^n, \ z \in \bar{S} \setminus \{\bar{x}\}. \end{cases} \tag{13}$$

By (12) and (13), we have

$$\alpha B \subset H(z) + [T_S(z) \cap \hat{N}_{\bar{S}}(z)]^\circ, \forall z \in \bar{S}.$$

It maintains (a) of Definition 4.

Then, because $\bar{x}$ is an isolated point of $\bar{S}$, by (10) for all large enough $k \in K_0$, there is $P_{\bar{S}}(x^k) = \bar{x}$. So, by (10) and (13) and continuity of $F(\cdot)$, we have

$$\begin{aligned} \limsup_{k \in K, k \to \infty} \psi_k &\geq \limsup_{k \in K_0, k \to \infty} \psi_k \\ &= \lim_{k \in K_0, k \to \infty} \frac{1}{\|x^k - \bar{x}\|} \left\langle F(x^k) - F(\bar{x}), x^k - \bar{x} \right\rangle = 0. \end{aligned}$$

It maintains (b) of Definition 4. The proof is complete. □

The theorems above show that the augmented weak sharpness of the solution sets for VIP is an extension of the weak sharp and the weak sharpness.

3. Finite Convergence

In this section, when a feasible solution set of the variational inequality problem is augmented weak sharp, a necessary and sufficient condition of the finite convergence is given, and some corollaries of the above conclusion include the corresponding results in the existing literatures, under the condition of weak sharpness or strong non-degeneracy.

Theorem 4. *For VIP, suppose $\bar{S} \subset S$ is a closed set, $\bar{S}$ is a set of augmented weak sharpness relative to $\{x^k\} \subset S$. Then, $\{x^k\}$ converges finitely to $\bar{S}$ if and only if*

$$\lim_{k \to \infty} P_{T_S(x^k)}\Big(-F(x^k)\Big) = 0. \tag{14}$$

Proof of Theorem 4. (Necessity) If $x^k \in \bar{S}$, according to the hypothesis $\bar{S} \subset S$, we know that exists $u^k \in F(x^k)$, such that $-u^k \in N_S(x^k)$. So, by the convexity and projection decomposition of S, we have $P_{T_S(x^k)}(-u^k) = 0$; i.e., (14) is true.

(Sufficiency) If (14) is true. The following to prove $\{x^k\}$ converges finitely to $\bar{S}$. Otherwise, $K = \{k | x^k \notin \bar{S}\}$ is infinite, we will get a contradiction. According to the augmented weak sharpness of $\bar{S}$ about $\{x^k\} \subset S$, there exists a mapping $H : \bar{S} \to 2^{R^n}$ and a constant $\alpha > 0$ such that

$$\alpha B \subset H(z) + \Big[T_S(z) \cap \hat{N}_{\bar{S}}(z)\Big]^\circ, \forall z \in \bar{S}, \tag{15}$$

and for $\forall u^k \in F(x^k)$, $\forall z^k \in P_{\bar{S}}(x^k)$, and $v^k \in H(z^k)$, there is

$$\limsup_{k \in K, k \to \infty} \Psi_k = \frac{1}{\|x^k - z^k\|} \langle u^k - v^k, x^k - z^k \rangle \geq 0. \tag{16}$$

Since $z^k \in P_{\bar{S}}(x^k)$, so for $\forall z \in \bar{S}$, we have $\|z^k - x^k\|^2 \leq \|z - x^k\|^2$.

So, there is

$$\langle x^k - z^k, z - z^k \rangle \leq \frac{1}{2} \|z - z^k\|^2 = \circ(\|z - z^k\|).$$

Therefore, by the definition of $\hat{N}_{\bar{S}}(\cdot)$, we have

$$x^k - z^k \in \hat{N}_{\bar{S}}(z^k). \tag{17}$$

Then, by the convexity of S,
$$x^k - z^k \in T_S(z^k), \quad z^k - x^k \in T_S(x^k). \tag{18}$$

By (17) and (18), there is
$$x^k - z^k \in T_S(z^k) \cap \hat{N}_{\bar{S}}(z^k). \tag{19}$$

Let $g_k = \frac{x^k - z^k}{\|x^k - z^k\|}(k \in K)$. By (15), we know that there exists $\bar{v}^k \in H(z^k)$, $\bar{\xi}^k \in \left[T_S(z^k) \cap \hat{N}_{\bar{S}}(z^k)\right]^\circ$, such that
$$\alpha g_k = \bar{v}^k + \bar{\xi}^k. \tag{20}$$

According to (19) and (20), for $\forall k \in K$, we have
$$\begin{array}{rcl} \alpha & = & \langle \bar{v}^k, g_k\rangle + \langle \bar{\xi}^k, g_k\rangle \\ & \leq & \langle \bar{v}^k, g_k\rangle. \end{array} \tag{21}$$

On the other hand, by (14), we get
$$0 \in \liminf_{k\in K, k\to\infty} P_{T_S(x^k)}(-F(x^k)),$$

so, there is $\bar{u}^k \in F(x^k)$ such that
$$\lim_{k\in K, k\to\infty} P_{T_S(x^k)}(-\bar{u}^k) = 0. \tag{22}$$

Thus, by (18) and (22) and the properties of projection gradient ([8, Lemma 3.1]), for $\forall k \in K$, there is
$$\begin{array}{rcl} \alpha & \leq & \langle \bar{v}^k, g_k\rangle \\ & = & \langle -\bar{u}^k, -g_k\rangle - \langle \bar{u}^k - \bar{v}^k, g_k\rangle \\ & \leq & \max\left\{\langle -\bar{u}^k, d\rangle | d \in T_S(x^k), \|d\| \leq 1\right\} - \langle \bar{u}^k - \bar{v}^k, g_k\rangle \\ & = & \|P_{T_S(x^k)}(-\bar{u}^k)\| - \langle \bar{u}^k - \bar{v}^k, g_k\rangle. \end{array}$$

According to (16) and (22), and the above formula,
$$\begin{array}{rcl} \alpha & \leq & \liminf\limits_{k\in K, k\to\infty} \left\{\|P_{T_S(x^k)}(-\bar{u}^k)\| - \langle \bar{u}^k - \bar{v}^k, g_k\rangle\right\} \\ & = & -\limsup\limits_{k\in K, k\to\infty} \langle \bar{u}^k - \bar{v}^k, g_k\rangle \leq 0. \end{array}$$

So, a contradiction is obtained. The proof is complete. □

By Theorems 2 and 4 , we have the following conclusions.

Corollary 1. For $VIP(F, S)$, suppose $\bar{S} \subset S$ is a closed set, $\{x^k\} \subset S$. If $\{F(x^k)\}$ is bounded and any one of its accumulation points $\bar{p}$ satisfies $-\bar{p} \in int \bigcap_{x\in\bar{S}}[T_S(x) \cap \hat{N}_{\bar{S}}(x)]^\circ$, then $\{x^k\}$ converges finitely to $\bar{S}$, if and only if (14) is true.

Remark 2. *Corollary 1 is an extension and improvement of Theorem 3.3 in [17], because we replace the cone $[T_S(x) \cap N_{\bar{S}}(x)]^\circ$ in Theorem 3.3 in [17] by the cone $[T_S(x) \cap \hat{N}_{\bar{S}}(x)]^\circ$ containing the former cone; meanwhile, we weaken the continuity of $F(\cdot)$ with the condition that $\bar{S}$ is a closed set. Note that Theorem 3.3 in [17] have improved Theorem 3.1 in [6].*

By Theorems 3 and 4, we can obtain the following corollary.

Corollary 2. For $VIP(F, S)$, suppose $F(\cdot)$ is continuous on S. If $\{x^k\} \subset S$ is bounded and any one of its accumulation points is strong non-degenerate, then $\{x^k\}$ converges finitely to $\bar{S}$ if and only if (14) is true.

4. Conclusions

This paper presents the notion of augmented weak sharpness of solution sets for VIP. The notion of augmented weak sharpness is the extension of the weak sharpness and strong non-degeneracy of a solution set. Under the condition of the solution set being augmented weak sharp, we present the necessary and sufficient condition for finite convergence of the feasible solution sequence of VIP.

Finally, it is worth mentioning that the weak sharpness or strong non-degeneracy of a solution set is the sufficient condition of many optimization algorithms having finite convergence (see [6,12]); this is because the feasible solution sequence produced by these algorithms satisfies (14). So, from the theorems and these propositions in Section 2, we know the augmented weak sharpness of the solution set provides weaker sufficient conditions than the weak sharpness and strong non-degeneracy for the finite convergence of these algorithms.

Acknowledgments: This research was supported by National Natural Science Foundation of China (11271233) and Natural Science Foundation of Shandong Province (ZR2012AM016, ZR2016AM07, ZR2015AL011).

Author Contributions: W.Z. conceived and designed the paper's content; R.W. and H.Z. performed the experiments in the paper; W.Z. and R.W. wrote the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Rockafellar, R.T. Monotone operators and the proximal point algorithm. *SIAM J. Control. Optim.* **1976**, *14*, 877–898.
2. Polyak, B.T. *Introduction to Optimization*; Optimization Software: New York, NY, USA, 1987.
3. Ferris, M.C. Weak Sharp Minima and Penalty Functions in Mathematical Programming. Ph.D. Thesis, University of Cambridge, Cambridge, UK, 1988.
4. Ferris, M.C. Finite termination of the proximal point algorithm. *Math. Program.* **1991**, *50*, 359–366.
5. Matsushita, S.Y.; Xu, L. Finite termination of the proximal point algorithm in Banach spaces. *J. Math. Anal. Appl.* **2012**, *387*, 765–769.
6. Xiu, N.H.; Zhang, J.Z. On finite convergence of proximal point algorithms for variational inequalities. *J. Math. Anal. Appl.* **2005**, *312*, 148–158.
7. Al-Khayyal, F.; Kyparisis, J. Finite convergence of algorithms for nonlinear programs and variational inequalities. *J. Optim. Theory Appl.* **1991**, *70*, 319–332.
8. Burke, J.V.; Moré, J.J. On the identification of active constraints. *SIAM J. Numer. Anal.* **1988**, *25*, 1197–1211.
9. Fischer, A.; Kanzow, C. On finite termination of an iterative method for linear complementarity problem. *Math. Program.* **1996**, *74*, 279–292.
10. Solodov, M.V.; Tseng, P. Some methods based on the D-gap function for solving monotone variational inequalities. *Comput. Optim. Appl.* **2000**, *17*, 255–271.
11. Wang, C.Y.; Zhao, W.L.; Zhou, J.H.; Lian, S.J. Global convergence and finite termination of a class of smooth penalty function algorithms. *Optim. Methods Softw.* **2013**, *28*, 1–25.
12. Xiu, N.H.; Zhang, J.Z. Some recent advances in projection-type methods for variational inequalities. *J. Comput. Appl. Math.* **2003**, *152*, 559–585.
13. Marcotte, P.; Zhu, D. Weak sharp solutions of variational inequalities. *SIAM J. Optim.* **1998**, *9*, 179–189.
14. Khan, A.A.; Tammer, C.; Zalinescu, C. *Set-Valued Optimization*; Springer: New York, NY, USA, 2014; pp. 110–112.
15. Rockafellar, R.T.; Wets, R.J. *Variational Analysis*; Springer: New York, NY, USA, 1998.
16. Wang, C.Y.; Zhang, J.Z.; Zhao, W.L. Two error bounds for constrained optimization problem and their application. *Appl. Math. Optim.* **2008**, *57*, 307–328.
17. Zhou, J.C.; Wang, C.Y. New characterizations of weak sharp minima. *Optim. Lett.* **2012**, *6*, 1773–1785.