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Review

Analytical Solution to Normal Forms of Hamiltonian Systems

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Abstract: The idea of the normalisation of the Hamiltonian system is to simplify the system by transforming Hamiltonian canonically to an easy system. It is under symplectic conditions that the Hamiltonian is preserved under a specific transformation—the so-called Lie transformation. In this review, we will show how to compute the normal form for the Hamiltonian, including computing the general function analytically. A clear example has been studied to illustrate the normal form theory, which can be used as a guide for arbitrary problems.

Keywords: Hamiltonian; normal forms; generating function; Lie transformation; canonical transformation

1. Introduction

A Hamiltonian system is a dynamical system that satisfies

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \tag{1}$$

ordinary differential equations (ODEs). Here, $q \in R^n$ represents the coordinates of the configuration variable (positions) and their canonically conjugate momenta $p \in R^n$. The function $H = H(q, p, t)$ is called the Hamiltonian of System (1) with n degrees of freedom (n dof).We may write the Hamiltonian system

$$\dot{\mathbf{y}} = \mathbf{J}\nabla H(\mathbf{y}, t)$$

where $\mathbf{J}$ is the $2n \times 2n$ Poisson matrix $\mathbf{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ and $\mathbf{y} = (q, p)$. Furthermore, the Hamiltonian H possesses an equilibrium y^0 (i.e., $\frac{\partial H}{\partial \mathbf{y}}(y^0) = 0$) at the origin in R^{2n}. If not, we make the shift $y = \hat{y} + y^0$ zero. For many dynamical systems, the Hamiltonian H represents the energy in the system. Furthermore, the Hamiltonian H will be in the form $H(q, p) = T + V$, where T is the kinetic energy and V is the potential energy of the system and is a function of q alone. The energy is constant, if a Hamiltonian does not depend explicitly on the time t:

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} = 0,$$

by (1). Hence, $H(q(t), p(t)) = H(q(0), p(0)) = E$. This is called conservation of energy. Where the Hamiltonian depends on time $H(q, p, t)$, the energy is not conserved [1–3].

2. Methodology

Here, we will provide a brief description of the normal form of the Hamiltonian system process using the Lie transform [4–8]. We are going to transform a given Hamiltonian $H = H(q, p)$

into another Hamiltonian $K = K(Q, P)$ which is simpler by means of a canonical transformation $Q = Q(q, p, t)$ and $P = P(q, p, t)$.

If we define the Hamiltonian function $H(q, p, t)$ and transformed Hamiltonian function $K(Q, P, t)$ such that

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \tag{2}$$

$$\dot{Q} = \frac{\partial K}{\partial P}, \quad \dot{P} = -\frac{\partial K}{\partial Q}, \tag{3}$$

then the coordinate transformation $(q, p) \to (Q, P)$, such that $Q = Q(q, p, t)$ and $P = P(q, p, t)$, is the so-called canonical transformation [9]. However, in our example, we use the exterior product to verify whether the transformations are canonical or not by using its properties [10,11]. In detail, if $A(v, w)$ denotes the area of the parallelogram determined by the pair of vectors v and w then A has the following properties:

- $A(v, v) = 0$.
- $A(v, w) = -A(w, v)$.

Apply these two properties in the transformation $(q, p) \to (Q, P)$ such that

$$\omega^2 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 + ...,$$

and

$$\omega^2 = dQ_1 \wedge dP_1 + dQ_2 \wedge dP_2 + ...,$$

if $\sum_{i=1}^{n} dq_i \wedge dp_i = \sum_{i=1}^{n} dQ_i \wedge dP_i$, then the transformations are canonical.

Lie transformation provides a symplectic change of the variable that depends on a small parameter as the general solution of the Hamiltonian system [4,5]. In detail, the general solution $\mathcal{X}(\mathbf{y}, \epsilon)$ defines a canonical transformation such that $\mathbf{x} = \mathcal{X}(\mathbf{y})$ with the inverse $\mathbf{y} = \mathcal{Y}(\mathbf{x})$, and hence $\mathcal{X}(\mathcal{Y}(\mathbf{x})) = \mathbf{x}$, where the time ϵ maps the flow of the Hamiltonian system. It is defined by the generation function W. [4,5] We will set time $\epsilon = 1$ in order to make our transformations canonical. We then have

$$\mathbf{x} = \mathcal{X}(\mathbf{y}) := \mathcal{X}(\mathbf{y}, \epsilon = 1) = \mathcal{X}|_{\epsilon=1}(\mathbf{y}).$$

2.1. Generating Function

The generating function $W(\mathbf{y}, \epsilon)$ is an auxiliary non-autonomous Hamiltonian depending on the parameter ϵ, and the coordinates $\mathbf{y} = (q, p) = (q_1, .., q_n, p_1, ..., p_n)$. The Hamiltonian system associated with the generating function is given by

$$\frac{dq}{d\epsilon} = \frac{\partial W(\mathbf{y}, \epsilon)}{\partial p},$$

$$\frac{dp}{d\epsilon} = \frac{-\partial W(\mathbf{y}, \epsilon)}{\partial q}.$$

The general solution is written as $\mathcal{X} : (\mathbf{y}, \epsilon) \mapsto \mathcal{X}(\mathbf{y}, \epsilon)$. For example, the solution curve through a particular point y^* can be written as

$$\mathcal{X}_{y^*} : \epsilon \mapsto \mathcal{X}(y^*; \epsilon),$$

with initial condition $\mathbf{x} = (\mathbf{y}, 0) = \mathcal{X}(\mathbf{y}, \epsilon)|_{\epsilon=0} = \mathbf{y}$.

To make sure that the transformation between the original Hamiltonian and the new one is valid, we resort to an indirect generating function approach that can be derived from an action principle of the form

$$S_{qp} = \int_{t_0}^{t_1} [p\dot{q} - H(q,p,t)]dt.$$

Let us consider the independent variations $\delta q, \delta p$ and ask the action to be minimized with respect to these variations:

$$\begin{aligned} 0 = \delta S_{qp} &= \int_{t_0}^{t_1} [p\delta\dot{q} + \dot{q}\delta p - \tfrac{\partial H}{\partial q}\delta q - \tfrac{\partial H}{\partial p}\delta p]dt \\ &= \int_{t_0}^{t_1} [(-\dot{p} - \tfrac{\partial H}{\partial q})\delta q + (\dot{q} - \tfrac{\partial H}{\partial p})\delta p]dt + [p\delta q]|_{t_0}^{t_1} \end{aligned}$$

Similarly, we calculate the variation of the action in P and Q:

$$\delta S_{QP} = \int_{t_0}^{t_1} [(-\dot{P} - \frac{\partial K}{\partial Q})\delta Q + (\dot{Q} - \frac{\partial K}{\partial P})\delta P]dt + [P\delta Q]|_{t_0}^{t_1}.$$

We need to show that the integral term vanishes for any variations $\delta Q, \delta P$. Furthermore, the solution to the action principle is unchanged if $S_{qp} - S_{QP} = W|_{t_0}^{t_1}$, where $W = W(q,t)$ is a function of coordinates and time. If we use this condition of the action principle and keep both q and p fixed at the initial and final times, then W is a function of coordinates and momenta $W = W(q,p,t)$. To summarize, we have demonstrated a sufficient condition for the transformation $(q,p) \to (Q,P)$ to be a canonical transformation, if there is a function $W(q,Q,t)$ such that

$$p = \frac{\partial W}{\partial q} \quad P = -\frac{\partial W}{\partial Q}, \quad K = H + \frac{\partial W}{\partial t},$$

where W is called a generating function. Note that if the system does not depend explicitly on time t then the new Hamiltonian function is the same as the old Hamiltonian function. There are four types of generating functions. All have old coordinates or old momenta and new coordinates or new momenta, respectively. Moreover, the generating function can be determined with respect to the normal forms. It has a different process to calculate. In the next section, we will explain the normal form of a Hamiltonian and then provide an example to clarify the theoretical part.

2.2. Normal Form

We write the coordinate change as

$$\mathbf{x} = \mathcal{X}(\mathbf{y}), \quad \mathbf{y} = \mathcal{Y}(\mathbf{x}),$$

with $\mathcal{X}(\mathcal{Y}(\mathbf{x})) = \mathbf{x}$, where $\mathbf{x} = (x_1, ..., x_{2n}) = (q_1, ..., q_n, p_1, ..., p_n)$ and $\mathbf{y} = (y_1, ..., y_{2n}) = (Q_1, ..., Q_n, P_1, ..., P_n)$. Suppose the Hamiltonian function H depends on a parameter ϵ:

$$H = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} H_i(\mathbf{x}, \epsilon),$$

where $x_i, 1 \geq i \geq n$ refers to the coordinates and $x_i, n+1 \geq i \geq 2n$ refers to their conjugate momenta. The transformed Hamiltonian K also depends on a parameter ϵ:

$$K = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} K_i(\mathbf{y}, \epsilon) \equiv \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} H_0^{(i)}(\mathbf{y}, \epsilon),$$

where $x_i, 1 \geq i \geq n$ refers to the transformed coordinates and $x_i, n+1 \geq i \geq 2n$ refers to their conjugate momenta.

Lie transformation can be achieved by using the solution to another Hamiltonian system defined by the generating function

$$W = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} W_{i+1}(\mathbf{y}, \epsilon), \quad \text{where} \quad W_{i+1}(\mathbf{y}) = (\frac{\partial^i}{\partial \epsilon^i} W(\mathbf{y}, \epsilon)),$$

following the recursion formula

$$H_i^{(j)} = H_{i+1}^{(j-1)} + \sum_{k=0}^{i} \binom{i}{k} \{H_{i-k}^{(j-1)}, W_{k+1}\}, \tag{4}$$

with $i \geq 0, j \geq 1$, and hence, $H_{(0)}^i = H_i$. The operator $\{\cdot, \cdot\}$ is the so-called Poisson bracket of two scalar fields: given $A, B : \mathrm{R}^{2n} \to \mathrm{R}$ and is defined as the quantity

$$\{A, B\} = \sum_{i=1}^{n} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_{n+i}} - \frac{\partial A}{\partial x_{n+i}} \frac{\partial B}{\partial x_i},$$

where q, p are coordinates and momenta, respectively [12].

Note that $W(\mathbf{x}, \epsilon)$ is conserved under the transformation and can be written as $W(\mathbf{x}, \epsilon) = W(\mathbf{y}, \epsilon)$.

We express the original Hamiltonian (H) in terms of the new variable (Q, P) as $K = K(Q, P, \epsilon)$ by means of

$$K(Q, P, \epsilon) \equiv H(\underbrace{q(Q, P, \epsilon), p(Q, P, \epsilon)}_{\mathcal{X}(\mathbf{y}, \epsilon)}, \epsilon),$$

where $\mathcal{X}(\mathbf{y}, \epsilon) = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \mathcal{X}_i(\mathbf{y}, \epsilon)$. The coordinate change will be a near-identity map, which means that $\mathcal{X}_0(\mathbf{y}) = \mathbf{y}$ and thus,

$$\mathcal{X}(\mathbf{y}, \epsilon) = \mathbf{y} + \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} \mathcal{X}_i(\mathbf{y}, \epsilon).$$

A similar formula can be used to see the change of coordinates back to the old ones, which is

$$\mathcal{Y}(\mathbf{x}, \epsilon) = \mathbf{x} + \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} \mathcal{Y}_i(\mathbf{x}, \epsilon).$$

Here, we define the quantities L_j^i as the relation between the coefficients of the various series that are expressed in terms of intermediate quantities L_j^i with $0 \leq j, i \leq n$ and $j + i = n$. These quantities can compute the transformed Hamiltonian K_n from the original Hamiltonian H_n and other quantities are computed by a chain of relations:

$$H_n \equiv L_0^n \to L_1^{n-1} \to \ldots \to L_{n-1}^1 \to L_n^0 \equiv K_n.$$

The following recursion formula relates the terms K with those in H and W by quantities L_j^i

$$L_j^i = L_{j+1}^{i-1} + \sum_{k=0}^{j} \binom{j}{k} \{L_{j-k}^{i-1}, W_{k+1}\}, \tag{5}$$

with $j \geq 0$ and $i \geq 1$ [9,13,14].

The last Equation (5) has the binomial coefficient $\binom{j}{k} = \frac{j!}{k!(j-k)!}$. Note that the calculation of L_0^0 makes the first term H_0 and K_0 equal because W is a near identity transformation, and hence, the transformation is generated [12]. In addition, the first term in the expansion for W starts with W_1.

The normal form process can be stopped at any existing order. The Lie triangle summarizes the recursion process [4,5]. Thus, the process and Lie triangle are as follows:

$$\begin{array}{lllll}
K_0: & K_0 \equiv L_0^0 & = & H_0 \\
K_1: & K_1 \equiv L_1^0 & = & L_1^0 + \{W_3, L_0^0\} \\
 & & = & H_1 + \{W_1, H_0\} \\
K_2: & K_2 \equiv L_2^0 & = & L_1^1 + \{W_1, L_1^0\} \\
 & & = & L_2^0 + \{W_1, L_1^0\} + \{W_2, L_0^0\} + \{W_1, L_1^0\} \\
 & & = & H_2 + \{W_1, H_1\} + \{W_2, H_0\} + \{H_1 + \{W_1, H_0\}, W_1\} \\
 & & = & H_2 + \{W_1, H_1\} + \{W_2, H_0\} + \{W_1, H_1\} + \{\{W_1, H_0\}, W_1\} \\
 & & = & H_2 + 2\{W_1, H_1\} + \{W_2, H_0\} + \{\{W_1, H_0\}, W_1\},
\end{array}$$

where K_i is the transformed Hamiltonian and

$$\begin{array}{ccccccccc}
L_0^0 & & & & & & & & \\
\downarrow & \searrow & & & & & & & \\
L_1^0 & \rightarrow & L_0^1 & & & & & & \\
\downarrow & & \downarrow & \searrow & & & & & \\
L_2^0 & \rightarrow & L_1^1 & \rightarrow & L_0^2 & & & & \\
\downarrow & & \downarrow & & \downarrow & \searrow & & & \\
L_3^0 & \rightarrow & L_2^1 & \rightarrow & L_1^2 & \rightarrow & L_0^3 & & \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots,
\end{array} \tag{6}$$

respectively [9]. By using Lie transformations to compute the normal form, the transformed Hamiltonian K is defined as

$$K = K_0 + K_1 + ...$$

In general, K is in normal form, where K_n is the polynomial of the degree $n+2$. In addition, for any smooth function f, then

$$\{f, K_i\} = 0, \text{with} \quad i = 0, 1, ...$$

This is the so-called normal form with respect to a given function. However, the same property can be applied on its own quadratic terms, namely K_2 such that $\{K_0, K_i\} = 0$. Thus, the normal form of quadratic terms K_0 has the form

$$K_0(\mathbf{y}) = \sum_{j=1}^{n} \lambda_j (q_j p_j),$$

where q_j and p_j are configuration space coordinates and their conjugate momenta, respectively. Additionally, the coefficients of products $q_j p_j$ are given by the vector $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{C}^n$.

To sum up, assume that we have the Hamiltonian such that

$$H = H_0 + H_1 + ...,$$

with H_n is the polynomial of degree $n+2$ and its coordinates denoted by $\mathbf{x} = (q, p)$. The aim of normalization is to find the easiest change of coordinates

$$\mathbf{x} = \mathcal{X}(\mathbf{y}), \quad \text{(canonical)}$$

with the inverse

$$\mathbf{y} = \mathcal{Y}(\mathbf{x}),$$

through the generating function W, such that the function H expressed in terms of $\mathbf{y}$ by means of

$$H(\mathcal{X}(\mathbf{y})) = K(\mathbf{y}),$$

with $K = K_0 + K_1 + ...$ results in a transformed Hamiltonian K, that is in the normal form through the degree $n + 2$.

The above method was first proposed by Deprit [5]. Here, we have followed the presentation style of [9].

2.3. Computing a Generating Function W

In more detail, we provide an example to show how to find generating functions W_3 and W_4 of the degree three and four in formal norm [15].

Consider H to be a Hamiltonian of (n dof). Let us expand H in power series such that

$$H(q,p) = H_2(q,p) + \epsilon H_3(q,p) + \dots, \tag{7}$$

where $H_n(q,p)$ is a homogeneous polynomial of degree n in the variables (q,p). The aim is to perform transformations canonically to make the expansion simple. We will perform all series manipulations formally, and set $\epsilon = 1$ afterwards.

As we have (7), then

$$\begin{aligned} K &= e^{\epsilon\{W_3,\cdot\}}H, \\ &= [I + \epsilon\{W_3,\cdot\} + \tfrac{\epsilon^2}{2}[W_3,\{W_3,\cdot\}\}]H, \\ &= H_2 + \epsilon(H_3 + \{W_3,H_2\}) + \mathcal{O}(\epsilon^2), \end{aligned}$$

where K is the transformed Hamiltonian. It is easy to see that the monomials of degree three of K can be obtained using generating function W_3 by

$$\bar{H}_3 = H_3 + \{W_3,H_2\}.$$

We choose the coefficients of W_3 such that $\bar{H}_3$ is zero.

Note that if we assume $\mathbf{x} = (x_1,x_2,...,x_n)$ and $\mathbf{k} = (k_1,...,k_n) \in N^n$,and we define

$$\mathbf{x}^{\mathbf{k}} = x_1^{k_1}...x_n^{k_n} \quad \text{and} \quad |k| = k_1 + ... + k_n,$$

hence H_3 and W_3 can be written as

$$H_3(q,p) = \sum_{|k_q|+|k_p|=3} h_{k_qk_p} q^{k_q} p^{k_p},$$

$$W_3(q,p) = \sum_{|k_q|+|k_p|=3} w_{k_qk_p} q^{k_q} p^{k_p}.$$

We determine the coefficients such that $\{W_3,H_2\} = -H_3$. Note that $\{H_2,\cdot\}$ is a linear operator and takes the diagonal form, due to

$$\{H_2, q^{k_q}p^{k_p}\} = i\langle k_p - k_q, \omega\rangle q^{k_q}p^{k_p}.$$

Hence, it is easy to find W_3 such that

$$W_3(q,p) = \sum_{|k_q|+|k_p|=3} = \frac{-h_{k_qk_p}}{i\langle k_p - k_q,\omega\rangle} q^{k_q}p^{k_p}.$$

However, $\langle k,\boldsymbol{\omega}\rangle$ do not vanish for any $k \in Z - \{0\}$. If the components of frequency vector $\boldsymbol{\omega} = (\omega_1,\omega_2,...,\omega_l)$ are linearly independent and $|k_q| + |k_p| = 3$, then this condition is satisfied. Once W_3 has been calculated, we can compute the new coordinates as a function of the old ones and vice versa [9,12,13].

We rewrite the transformed Hamiltonian as function of H such that

$$H(q,p) = H_2(q,p) + H_4(q,p) + H_5(q,p) + \dots .$$

The following step is calculating the generating function W_4 to get rid of the monomials of degree four from H. In general, this cannot be applied because $\{H_2, \cdot\}$ has some zero eigenvalues:

$$\{H_2, q^k p^k\} = 0.$$

Thus, we can only solve the equation $\{W_4, H_2\} = -H_4$, if the form of H_4 is $q^{k_q} p^{k_p}$, with $k_q \neq k_p$:

$$W_4(q,p) = \sum_{|k_q|+|k_p|=4} = \frac{-h_{k_q k_p}}{i\langle k_p - k_q, \omega\rangle} q^{k_q} p^{k_p}.$$

The presented method is formal without looking at the convergence of the variables. There are many applications presented in the series divergence. The important part of the method process is the first orders of the transformed system, which provide interesting information due to the linear approximation around the equilibrium. The process can be studied up to any existing order (ϵ) for a good approximation [16,17]. In other words, the first order terms consist of useful information to reduce the transformed system without being affected by the divergent character, where the general perturbation theorem takes place.

To summarize, once the generating function W is calculated, we can derive the new coordinates as functions of the old ones and vice versa. Additionally, the generating function W and the calculations of Poisson brackets can be used to see the coordinates changing back to the old ones without any additional calculations. There are some interesting examples in physics and engineering for the idea of Hamiltonians normal forms and generating functions which can be found in the book by Sanders and Verhulst [18]. An example follows the theoretical part for clarification.

3. Example

Let us consider the Hamiltonian function that is defined as

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{2}\omega_b^2 x^2 + \frac{1}{2}\omega_y^2 y^2 + ax^3 + bxy^2. \tag{8}$$

Firstly, we rescale p_x, p_y, x and y such that

$$q_1 = \frac{x}{\omega_b}, \quad q_2 = \frac{y}{\omega_y}, \quad p_1 = \omega_b p_x, \quad p_2 = \omega_y p_y.$$

This rescaling is canonical and easy to check. It is also easy to see that

$$dq_1 = \frac{1}{\omega_b} dx, \quad dp_1 = \omega_b dp_x.$$

According to the properties of the exterior product, the following equation holds:

$$dq_1 \wedge dp_1 = dx \wedge dp_x.$$

The same can be done for all other rescalings. As a result, the Hamiltonian function becomes

$$H = \frac{1}{2}\omega_b(p_1^2 - q_1^2) + \frac{1}{2}\omega_y(p_1^2 + q_1^2) + a\omega_b^3 q_1^3 + b\omega_b\omega_y^2 q_1 q_2^2. \tag{9}$$

Expression (9) can be written in a much simpler form by introducing the coordinates

$$Q_1 = \frac{1}{\sqrt{2}}(p_1 + q_1), \quad P_1 = \frac{1}{\sqrt{2}}(p_1 - q_1),$$

$$Q_2 = \frac{1}{\sqrt{2}}(q_2 - ip_2), \quad P_2 = \frac{1}{\sqrt{2}}(p_2 - iq_2).$$

The same as before, these transformations are canonical and easy to verify

$$dP_1 = \frac{-1}{\sqrt{2}}dq_1 + \frac{1}{\sqrt{2}}dp_1, \quad dQ_1 = \frac{1}{\sqrt{2}}dp_1 + \frac{1}{\sqrt{2}}dq_1,$$

$$dP_1 \wedge dQ_1 = \frac{-1}{2}dq_1 \wedge dp_1 + \frac{-1}{2}dq_1 \wedge dq_1 + \frac{1}{2}dp_1 \wedge dp_1 + \frac{1}{2}dp_1 \wedge dq_1 = dp_1 \wedge dq_1.$$

According to the normal form process (more details can be found in [9,12,13]), the Hamiltonian function is defined as follows

$$H(Q,P) = H_2(Q,P) + \epsilon H_3(Q,P) + \dots,$$

thus

$$H_2(Q,P) = \omega_b Q_1 P_1 + i\omega_y Q_2 P_2.$$

Next, we calculate H_3 terms in the new coordinates Q_1, Q_2, P_1, P_2.
From the canonical transformation, we have

$$q_1 = \frac{1}{\sqrt{2}}(Q_1 - P_1), \quad q_2 = \frac{1}{\sqrt{2}}(Q_2 + iP_2).$$

Hence

$$\begin{aligned} H_3 \quad = \quad & \tfrac{a\omega_b^3}{2\sqrt{2}}(Q_1^3 - 3Q_1^2P_1 + 3Q_1P_1^2 - P_1^3) \\ & + \tfrac{b\omega_b\omega_y^2}{2\sqrt{2}}(Q_1Q_2^2 + 2iQ_1Q_2P_2 - Q_1P_2^2 - P_1Q_2^2 - 2iP_1Q_2P_2 + P_1P_2^2)\,. \end{aligned}$$

Now, the generating function W_3 of order three associated with H_3 is as follows:

$$\begin{aligned} W_3 \quad = \quad & a_1Q_1^3 + a_2Q_1Q_2^2 + a_3P_1^3 + a_4P_1P_2^2 + a_5Q_1^2P_1 + a_6Q_1P_1^2 \\ & + a_7Q_1P_2^2 + a_8P_1Q_2^2 + a_9Q_1Q_2P_2 + a_{10}P_1Q_2P_2\,. \end{aligned}$$

The coefficients $(a_1, \dots, a_{10})$ can be determined by setting

$$H_3 + \{W_3, H_2\} = 0.$$

The coefficients have been calculated, and hence, we get

$$a_1 = \frac{-a\omega_b^2}{6\sqrt{2}}, \quad a_2 = \frac{-b\omega_b\omega_y^2}{2\sqrt{2}(\omega_b + 2i\omega_y)}, \quad a_3 = \frac{a\omega_b^2}{6\sqrt{2}},$$

$$a_4 = \frac{b\omega_b\omega_y^2}{2\sqrt{2}(\omega_b + 2i\omega_y)}, \quad a_5 = \frac{3b\omega_y^2}{2\sqrt{2}}, \quad a_6 = \frac{3b\omega_b}{2\sqrt{2}},$$

$$a_7 = \frac{b\omega_b\omega_y^2}{2\sqrt{2}(\omega_b - 2i\omega_y)}, \quad a_8 = \frac{b\omega_b\omega_y^2}{2\sqrt{2}(2i\omega_y - \omega_b)},$$

$$a_9 = \frac{-ib\omega_y^2}{\sqrt{2}}, \quad a_{10} = \frac{ib\omega_y^2}{\sqrt{2}}.$$

Thus, the normal form in our example is H_2 with respect to the generating function W_3, whose coefficients are presented above.

4. Conclusions

In this paper, we presented a general calculation of normal forms of Hamiltonian systems that is quadratic polynomial in the positions only. Previously, in Section 2, we provided the Hamiltonian normal forms theory that led to a reduced Hamiltonian number of degrees of freedom. More precisely, the generalization of Hamiltonian normal forms theory contains a very important step—the so-called generating function W. This function plays an important role in transforming the old Hamiltonian H into an equivalent Hamiltonian K up to an existing order of approximation and taking into account that the Poisson bracket of each term of the transformed Hamiltonian system K and the generating function W will vanish. This procedure guarantees that the transformed Hamiltonian system, especially in the first orders of the transformed system, provides interesting information due to the linear approximation around the equilibrium. An example was provided to illustrate the theory part.

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