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Fractal Behavior of a Ternary 4-Point Rational Interpolation Subdivision Scheme

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Abstract: In this paper, a ternary 4-point rational interpolation subdivision scheme is presented, and the necessary and sufficient conditions of the continuity are analyzed. The generalization incorporates existing schemes as special cases: Hassan–Ivrissimtzis's scheme, Siddiqi–Rehan's scheme, and Siddiqi–Ahmad's scheme. Furthermore, the fractal behavior of the scheme is investigated and analyzed, and the range of the parameter of the fractal curve is the neighborhood of the singular point of the rational scheme. When the fractal curve and surface are reconstructed, it is convenient for the selection of parameter values.

Keywords: subdivision scheme; rational mask; subdivision matrix; C^2 continuity; fractal behavior

1. Introduction

Due to the use of techniques that are simple and easy to handle, the subdivision method is widely used for generating smooth curves from original data points, and this method plays an important role in computer-aided geometric design, computer-aided design, and image processing.

According to the relationship of the original points and the limit curves, the two categories we are going to divide them into are interpolation subdivision [1–6] and approximation subdivision [7–10]. The limit curve of the interpolation subdivision scheme is through all original points, which can protect the shape of the original control polygon. In 2002, Hassan et al. [2] presented a ternary 4-point interpolating subdivision scheme that generates a C^2 limiting curve for the tension parameter satisfied with $\frac{1}{15} < \mu < \frac{1}{9}$. In 2010 and 2012, Mustafa et al. [3,4] introduced 6-point and 5-point ternary interpolating schemes with a shape parameter in succession, and proved the limit curves are C^3 or C^2 and continue for a certain range of parameter ω. In 2012 and 2013, Siddiqi and Rehan [5,6] proposed two schemes of ternary 4-point interpolating subdivision in which the limiting curve is C^1 or C^2 continuous. The above schemes have the following common characteristics: the masks in the scheme are all simple linear combinations of parameters, and when the limit curve is C^1 or C^2 continuous, the selection range of parameters is small, so it is impossible to know what will happen to the limit curves at other infinite intervals.

Compared with interpolation subdivision, the limit curve of the approximation subdivision scheme did not continue through the original points, but this scheme had less support and the limit curve had higher smoothness. In 2004, Hassan and Dodgson [7] derived a ternary 3-point approximating subdivision scheme that generates the C^2 limiting curve. In 2007, Ko et al. [8] introduced an improved ternary 4-point approximating subdivision scheme derived from cubic polynomial interpolation, and used similar methods to generalize ternary $(2n+2)$ point approximating subdivision schemes. In 2012, Ghaffar and Mustafa [9] investigated a general formula to generate the family of an even-point ternary approximating subdivision scheme with a shape parameter. In 2015, Rehan and Siddiqi [10] proposed a ternary 4-point approximating subdivision scheme that generates the limiting curve of C^3 continuity. These experimental results show that when the continuity was higher, the limiting curve of the approximation scheme deviated further from the original points.

This motivated us to present a ternary interpolation scheme with high smoothness and more degrees of freedom for the curve design. The proposed scheme not only provides the mask of 4-point schemes, but also generalizes and unifies several well-known schemes.

Furthermore, the subdivision scheme is important not only in the geometric design of smooth curves, but also in the construction of irregular shapes. Zheng et al. [11,12] proved that the limit curves generated by binary 4-point and ternary 3-point interpolation subdivision schemes are fractals. Siddiqi et al. [13,14] described the fractal behavior of ternary 4-point interpolation subdivision schemes.

In this paper, the fractal behavior of the ternary 4-point rational interpolation subdivision scheme is investigated and analyzed. Through examples, it was found that when the parameter selection is close to the singular point, the limit curve pattern is more turbulent. As the parameter value becomes far from the singular point, the limit curve becomes smoother and finally reaches C^1 and C^2 continuity.

2. Preliminaries

A general ternary subdivision scheme S with the initial values $p^0 = \{p_i^0\}_{i\in Z}$ recursively defines new discrete values as follows:

$$p_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{3j-i} p_j^k, \quad i \in \mathbb{Z}, \tag{1}$$

where the set $a = \{a_i | i \in \mathbb{Z}\}$ of coefficients is called the mask of the scheme. A necessary condition for uniform convergence of the subdivision scheme (1) is that

$$\sum_{j\in\mathbb{Z}} a_{3j} = \sum_{j\in\mathbb{Z}} a_{3j+1} = \sum_{j\in\mathbb{Z}} a_{3j+2} = 1. \tag{2}$$

The Z-transform of the mask a of subdivision scheme can be given as

$$a(z) = \sum_{i\in\mathbb{Z}} a_i z^i, \tag{3}$$

which is called the symbol or Laurent polynomial.

A subdivision scheme is said to be uniformly convergent if for every initial data $p^0 = \{p_i^0\}_{i\in Z}$, there is a continuous function f such that for any closed interval $[a, b]$

$$\lim_{k\to\infty} \sup_{i\in\mathbb{Z}\cap 3^k[a,b]} |p_i^k - f(\frac{i}{3^k})| = 0.$$

As a result, f is regarded as the limit function of the subdivision scheme, and is denoted $f = S^\infty p^0$.

In 2002, Hassan et al. [2] provided a sufficient and necessary condition for a uniform convergent subdivision scheme. Firstly, they used matrix formalism to derive necessary conditions for a scheme to be C^k based on the eigenvalues of the subdivision matrix. If the limiting curve is C^2 continuity, the eigenvalues $\{\lambda_i\}$ satisfy:

$$\lambda_0 = 1,\ |\lambda_i| > |\lambda_{i+1}|, i = 1, 2, ...,\ and\ \lambda_1^2 = |\lambda_2| > |\lambda_3|. \tag{4}$$

Secondly, a subdivision scheme S is uniform convergent if and only if there is an integer $L \geq 1$, such that $\|(\frac{1}{3}S_1)^L\|_\infty < 1$. Then, the subdivision scheme is uniform convergent.

This paper is organized as follows. In Section 3, a ternary 4-point rational interpolating subdivision scheme is presented. The continuity analyses are in Section 4. In Section 5, the fractal behavior of subdivision schemes is introduced. In Section 6, examples are considered to demonstrate the role of the parameter. Conclusions are drawn in Section 7.

3. Rational Interpolation Subdivision Scheme

A 4-point rational interpolation subdivision scheme is defined as

$$\begin{cases} p_{3i}^{k+1} = p_i^k, \\ p_{3i+1}^{k+1} = a_0 p_{i-1}^k + a_1 p_i^k + a_2 p_{i+1}^k + a_3 p_{i+2}^k, \\ p_{3i+2}^{k+1} = a_3 p_{i-1}^k + a_2 p_i^k + a_1 p_{i+1}^k + a_0 p_{i+2}^k, \end{cases} \tag{5}$$

where

$$a_0 = \frac{2-243\alpha}{27(126\alpha-1)}, a_1 = \frac{873\alpha-7}{9(126\alpha-1)}, a_2 = \frac{129\alpha-1}{3(126\alpha-1)}, a_3 = \frac{1-135\alpha}{27(126\alpha-1)}, \textit{and} \ \ \alpha \neq \frac{1}{126}.$$

Specifically, for $\alpha = \frac{1}{108}$, the mask of the subdivision scheme is:

$$a = \{\cdots, 0, 0, -\frac{1}{18}, -\frac{1}{18}, 0, \frac{7}{18}, \frac{13}{18}, 1, \frac{13}{18}, \frac{7}{18}, 0, -\frac{1}{18}, -\frac{1}{18}, 0, 0, \cdots\},$$

and this is exactly the scheme which is mentioned in Hassan et al. [2].

For $\alpha = 0$, the mask of the subdivision scheme is:

$$a = \{\cdots, 0, 0, -\frac{1}{27}, -\frac{2}{18}, 0, \frac{9}{18}, \frac{21}{27}, 1, \frac{21}{27}, \frac{9}{18}, 0, -\frac{2}{18}, -\frac{1}{27}, 0, 0, \cdots\},$$

then the subdivision scheme rebuilds the scheme in Siddiqi et al. [5].

For $\alpha = \frac{1}{90}$, it is a ternary 4-point subdivision scheme:

$$a = \{\cdots, 0, 0, -\frac{5}{108}, -\frac{7}{108}, 0, \frac{13}{36}, \frac{3}{4}, 1, \frac{3}{4}, \frac{13}{36}, 0, -\frac{7}{108}, -\frac{5}{108}, 0, 0, \cdots\},$$

then the subdivision scheme is the scheme offered by Siddiqi et al. [6].

4. Continuity Analysis

4.1. Necessary Condition

Theorem 1. *If the limiting curve of the subdivision scheme (5) is C^2 continuity, the parameter $\alpha \in (-\infty, 0) \bigcup (\frac{1}{86}, +\infty)$.*

Proof. Similar to in Hassan's paper [2], let the matrix A be the mid-point subdivision matrix and the matrix B be the vertex subdivision matrix. We have:

$$A = \begin{pmatrix} a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \end{pmatrix}, \qquad B = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 \\ 0 & a_0 & a_1 & a_2 & a_3 \end{pmatrix}.$$

- The eigenvalues for A are $1, \frac{1}{3}, \frac{1}{9}, \frac{108\alpha-1}{9(126\alpha-1)}, \frac{1-135\alpha}{27(126\alpha-1)}, \frac{1-135\alpha}{27(126\alpha-1)}$.
- The eigenvalues for B are $1, \frac{1}{3}, \frac{1}{9}, \frac{297\alpha-2}{27(126\alpha-1)}, \frac{\alpha}{126\alpha-1}$.

From the two subdivision matrices and Equation (4), we can find the bound of α which is a necessary condition of the limiting curve to be C^2 continuity.

If the eigenvalues of the mid-point subdivision matrix satisfy the necessary conditions (4), then

$$\frac{1}{9} > \left|\frac{108\alpha - 1}{9(126\alpha - 1)}\right| > \left|\frac{1 - 135\alpha}{27(126\alpha - 1)}\right|. \tag{6}$$

If the eigenvalues of vertex subdivision matrix satisfies the necessary conditions (4), then

$$\frac{1}{9} > \left|\frac{297\alpha - 2}{27(126\alpha - 1)}\right| > \left|\frac{\alpha}{126\alpha - 1}\right|. \tag{7}$$

Combining Equation (6) and Equation (7), we find that the necessary conditions for C^2 continuity are $\alpha \in (-\infty, 0) \bigcup (\frac{1}{86}, +\infty)$. □

4.2. Sufficient Condition

Theorem 2. *If $\alpha \in (-\infty, 0) \bigcup (\frac{1}{86}, +\infty)$, then the subdivision scheme of Equation (5) is C^2 continuity.*

Proof. Considering the refinement rules of the ternary 4-point rational interpolating subdivision scheme defined in Equation (5), we calculate sufficient conditions for the continuity of C^0, C^1, and C^2 by the Laurent polynomial method.

The mask a of the subdivision scheme can be written as

$$a = \frac{1}{126\alpha - 1}\{\frac{1 - 135\alpha}{27}, \frac{2 - 243\alpha}{27}, 0, \frac{873\alpha - 7}{9}, \frac{129\alpha - 1}{3}, 1, \frac{129\alpha - 1}{3}, \frac{873\alpha - 7}{9}$$
$$\frac{873\alpha - 7}{9}, 0, \frac{2 - 243\alpha}{27}, \frac{1 - 135\alpha}{27}\}. \tag{8}$$

Taking $a_m(z) = (\frac{3z^2}{1+z+z^2})^m a(z)$, we have:

$$a_1(z) = \frac{1}{9(126\alpha - 1)}\{(1 - 135\alpha)z^{-3} + (243\alpha - 2)z^{-2} + (1 - 108\alpha)z^{-1} + (1026\alpha - 8)$$
$$+(1350\alpha - 11)z + (1026\alpha - 8)z^2 + (1 - 108\alpha)z^3 + (243\alpha - 2)z^4 + (1 - 135\alpha)z^5\},$$

for the range $\alpha \in (-\infty, 0) \bigcup (\frac{1}{86}, +\infty)$,

$$\left\|\frac{1}{3}S_1\right\|_\infty = \frac{1}{3}\max\{\sum_{i\in\mathbb{Z}} a_{3i}^{(1)}, \sum_{i\in\mathbb{Z}} a_{3i+1}^{(1)}, \sum_{i\in\mathbb{Z}} a_{3i+2}^{(1)}\} = \left|\frac{15 - 1521\alpha}{27(1 - 126\alpha)}\right| < 1. \tag{9}$$

$$a_2(z) = \frac{1}{3(126\alpha - 1)}\{(1 - 135\alpha)z^{-1} - 27\alpha + (351\alpha - 1)z + (648\alpha - 5)z^2 + (351\alpha - 1)z^3$$
$$-27\alpha z^4 + (1 - 135\alpha)z^5\},$$

for the range $\alpha \in (-\infty, 0) \bigcup (\frac{1}{86}, +\infty)$,

$$\left\|\frac{1}{3}S_2\right\|_\infty = \frac{1}{3}\max\{\sum_{i\in\mathbb{Z}} a_{3i}^{(2)}, \sum_{i\in\mathbb{Z}} a_{3i+1}^{(2)}, \sum_{i\in\mathbb{Z}} a_{3i+2}^{(2)}\} = \left|\frac{7 - 918\alpha}{9(1 - 126\alpha)}\right| < 1. \tag{10}$$

$$a_3(z) = \frac{1}{126\alpha - 1}\{(1 - 135\alpha)z + (162\alpha - 1)z^2 + (324\alpha - 3)z^3 + (162\alpha - 1)z^4 + (1 - 135\alpha)z^5\},$$

for the range $\alpha \in (-\infty, 0) \bigcup (\frac{1}{86}, +\infty)$,

$$\left\| \frac{1}{3} S_3 \right\|_\infty = \frac{1}{3} \max\{ \sum_{i\in\mathbb{Z}} a_{3i}^{(3)}, \sum_{i\in\mathbb{Z}} a_{3i+1}^{(3)}, \sum_{i\in\mathbb{Z}} a_{3i+2}^{(3)} \} = \left| \frac{3-324\alpha}{3(1-126\alpha)} \right| < 1. \tag{11}$$

According to (9), (10), and (11), we know that the sufficient conditions are satisfied for this scheme to be C^0, C^1, and C^2. □

5. Fractal Behavior

In this paper, the fractal behavior of a ternary 4-point interpolation subdivision scheme is developed and analyzed.

The original data points are $p^0 = \{p_i^0\}_{i=0}^n$. Let $p^k = \{p_i^k\}_{i=0}^{3^k n+1}$ be the set of control points at level k, and $p^k = \{p_i^k\}_{i=0}^{3^k n+1}$ satisfies the scheme (5). We need to analyze the effect of the parameter α on the sum of the length of all the small edges between two arbitrary fixed control points p_i^k and p_j^k after k subdivision steps. For simplicity, we only analyze the effect between two initial points p_0^0 and p_1^0 .

According to the subdivision scheme (5), it is known that $p_0^k = p_0^0$, where $k \geq 0$ and:

$$\begin{cases} p_0^{k+1} = p_0^k, \\ p_1^{k+1} = a_0 p_{-1}^k + a_1 p_0^k + a_2 p_1^k + a_3 p_2^k, \\ p_2^{k+1} = a_3 p_{-1}^k + a_2 p_0^k + a_1 p_1^k + a_0 p_2^k. \end{cases}$$

We defined the three edge vectors as:

$$\begin{cases} V_k = p_1^k - p_0^k, \\ S_k = p_2^k - p_1^k, \\ R_k = p_3^k - p_2^k. \end{cases} \tag{12}$$

Let $U_k = p_1^k - p_{-1}^k$, $W_k = p_0^k - p_{-1}^k$ and $Z_k = p_2^k - p_{-2}^k$, and we can get $U_k = V_k + W_k$. Since $U_k = p_1^k - p_{-1}^k$, U_k can be written as:

$$U_{k+1} = (a_2 - a_0) U_k + a_3 Z_k. \tag{13}$$

Similarly:

$$Z_{k+1} = (a_1 - a_3) Z_k + a_0 U_k. \tag{14}$$

Equations (13) and (14) are non-homogeneous difference equations to be solved simultaneously. Since

$$U_0 = p_1^0 - p_{-1}^0,$$
$$U_1 = p_1^1 - p_{-1}^1 = -a_3 p_{-2}^0 + (a_0 - a_2) p_{-1}^0 + (a_2 - a_0) p_1^0 + a_3 p_2^0$$

gives the special solution:

$$U_k = C_1 \lambda_1 + C_2 \lambda_2, \tag{15}$$

where $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{\alpha}{126\alpha - 1}$, and

$$C_1 = \frac{(135\alpha - 1) p_{-2}^0 - (4806\alpha - 38) p_{-1}^0 + (4806\alpha - 38) p_1^0 + (1 - 135\alpha) p_2^0}{9 - 1161\alpha},$$

$$C_2 = \frac{(1026\alpha - 8) p_{-2}^0 + (3649\alpha - 29) p_{-1}^0 + (4806\alpha - 38) p_1^0 + (1 - 135\alpha) p_2^0}{1161\alpha - 9}.$$

Since $V_k = p_1^k - p_0^k$ and $S_k = p_2^k - p_1^k$, it follows that

$$V_{k+1} = (1 - a_1)V_k + a_3 S_k - a_0 U_k, \tag{16}$$

$$S_{k+1} = (a_0 - a_3)S_k + (a_1 - a_2)V_k + (a_0 - a_3)U_k, \tag{17}$$

$$R_{k+1} = -a_0 V_k + a_2 S_k + a_3 U_k. \tag{18}$$

Using Equations (15), (16), and (17), yields:

$$V_{k+2} = (1 + a_0 - a_1 - a_3)V_{k+1} + [a_3(a_1 - a_2) - (1 - a_1)(a_0 - a_3)]V_k + U_k, \tag{19}$$

or

$$\frac{243(126\alpha - 1)}{108\alpha - 1} V_{k+2} = \frac{45(135\alpha - 1)}{108\alpha - 1} V_{k+1} + \frac{2 - 197\alpha}{108\alpha - 1} V_k + U_k, \tag{20}$$

and the corresponding characteristic equation is:

$$\frac{243(126\alpha - 1)}{108\alpha - 1} r^2 - \frac{45(1 - 135\alpha)}{108\alpha - 1} r - \frac{197\alpha - 2}{108\alpha - 1} = 0. \tag{21}$$

When $\alpha \neq \frac{1}{108}$, the roots of (21) are $r_1 = \frac{1}{9}$, $r_2 = \frac{297\alpha - 2}{27(126\alpha - 1)}$, and the solution of Equation (16) is

$$V_k = D_1 r_1^k + D_2 r_2^k + D_3 \lambda_1^k + D_4 \lambda_2^k, \tag{22}$$

where $D_i, i = 1, 2, 3, 4$ are the linear combinations of $\{p_i^0\}_{i=-2}^2$.

Similarly, the solution of Equation (17) is:

$$S_k = E_1 \rho_1^k + E_2 r_1^k + E_3 r_2^k + E_4 \lambda_1^k + E_5 \lambda_2^k, \tag{23}$$

where $\rho_1 = \frac{108\alpha - 1}{27(126\alpha - 1)}$ and $E_i, i = 1, ..., 5$ are the linear combinations of $\{p_i^0\}_{i=-2}^2$.

The solution of Equation (18) is:

$$R_k = H_1 \rho_1^k + H_2 r_1^k + H_3 r_2^k + H_4 \lambda_1^k + H_5 \lambda_2^k, \tag{24}$$

where $H_i, i = 1, ..., 5$ are the linear combinations of $\{p_i^0\}_{i=-2}^2$.

Theorem 3. *If $\alpha \in (\frac{11}{1431}, \frac{7}{837})$ and $\alpha \neq \frac{1}{126}$, then the limit curve of the subdivision scheme (5) is a fractal curve.*

Proof. From Equations (22), (23), and (24), it might be concluded by induction that 3^k small edge vectors between the two initial control points p_i^k and p_j^k after k subdivision steps can be expressed as

$$\omega_i^k = p_i^k - p_{i-1}^k = \mu_{1i}\rho_1^k + \mu_{2i} r_1^k + \mu_{3i} r_2^k + \mu_{4i}\lambda_1^k + \mu_{5i}\lambda_2^k, \quad i = 1, 2, 3. \tag{25}$$

For $\alpha \in (\frac{11}{1431}, \frac{7}{837})$ and $\alpha \neq \frac{1}{126}$, then

$$|r_2| > \frac{1}{9} = r_1, \; |r_1| > |\rho_1|, \; |r_2| > \frac{1}{3} = \lambda_1, \; |r_2| > |\lambda_2|. \tag{26}$$

Let $\left|\omega_i^k\right|$ refer to the length of a vector ω_i^k and $\left|\omega_{i0}^k\right| = \min_{i=1,2,\cdots,3^k} \left|\omega_i^k\right|$. Then, we have

$$\sum_{i=1}^{3k} \left|\omega_i^k\right| = 3^k \left|\mu_{1i}\rho_1^k + \mu_{2i} r_1^k + \mu_{3i} r_2^k + \mu_{4i}\lambda_1^k + \mu_{5i}\lambda_2^k\right|$$
$$= \left|3 \times \frac{297\alpha - 7}{27(126\alpha - 1)}\right|^k \cdot \left|\mu_{1i}\left(\frac{\rho_1}{r_2}\right)^k + \mu_{2i}\left(\frac{r_1}{r_2}\right)^k + \mu_{3i} + \mu_{4i}\left(\frac{\lambda_1}{r_2}\right)^k + \mu_{5i}\left(\frac{\lambda_2}{r_2}\right)^k\right| \to \infty (k \to \infty).$$

Hence, so far as the initial points p_0^0 and p_1^0 are concerned, the sum of the lengths of all the small edges tends to infinity as k approaches infinity. Therefore, the limit curve of ternary 4-point scheme is a fractal curve when $\alpha \in (\frac{11}{1431}, \frac{7}{837})$ and $\alpha \neq \frac{1}{126}$, and this parameter is valued near the point $\frac{1}{126}$. $\square$

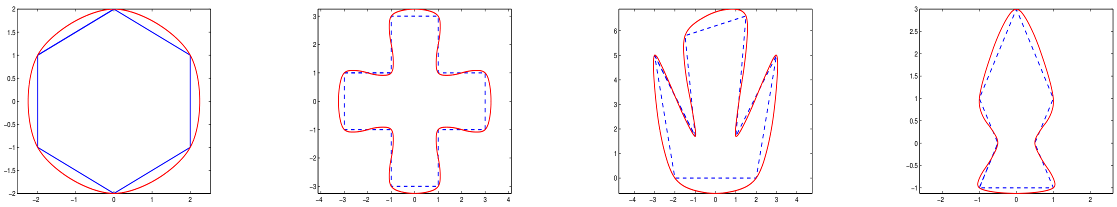
We give the comparison of the range for continuity and fractal behavior of proposed 4-point ternary scheme with other existing ternary schemes in Table 1.

Table 1. Comparison of existing ternary 4-point interpolation subdivision schemes.

Scheme	Continuity	The Range for C^k	The Range for Fractal Behavior
Hassan [2]	C^2	$(\frac{1}{15}, \frac{1}{9})$	$(-1, -\frac{1}{5}) \bigcup (\frac{3}{5}, 1)$ Siddiqi [13]
Siddiqi [6]	C^1	$(-\frac{1}{18}, \frac{1}{9})$	$(-\frac{17}{27}, -\frac{1}{3})$ Siddiqi [14]
Ours	C^2	$(-\infty, 0) \bigcup (\frac{1}{84}, +\infty)$	$(\frac{11}{1431}, \frac{7}{837})$

6. Example

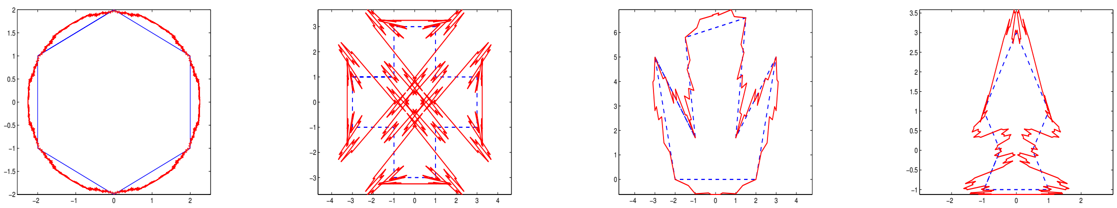
Examples of the rational interpolation subdivision schemes with C^2 continuity are given in Figure 1.



(a) (b) (c) (d)

Figure 1. Examples of the rational interpolation subdivision schemes with C^2 continuity. (**a**) Behavior of the schemes for $\alpha = -1$; (**b**) Behavior of the schemes for $\alpha = 0$; (**c**) Behavior of the schemes for $\alpha = 1$; (**d**) Behavior of the schemes for $\alpha = 3$.

Examples of the subdivision schemes with fractal behavior are given in Figure 2.



(a) (b) (c) (d)

Figure 2. Example of the subdivision schemes with fractal behavior. (**a**) Behavior of the schemes for $\alpha = 0.00813$; (**b**) Behavior of the schemes for $\alpha = 0.00818$; (**c**) Behavior of the schemes for $\alpha = 0.0078$; (**d**) Behavior of the schemes for $\alpha = 0.00775$.

The change of the limit curves of the subdivision scheme is given in Figure 3.

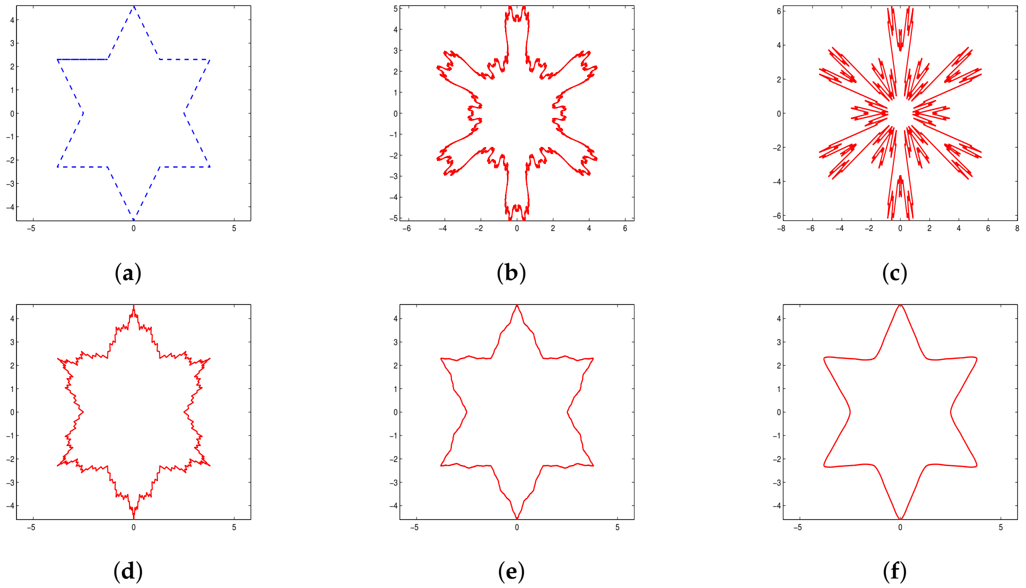


Figure 3. Example of the subdivision schemes with fractal behavior. (**a**) Control polygon; (**b**) Behavior of the schemes for $\alpha = 0.00792$; (**c**) Behavior of the schemes for $\alpha = 0.00812$; (**d**) Behavior of the schemes for $\alpha = 0.00823$; (**e**) The limit curve with C^0 of the schemes for $\alpha = 0.01$; (**f**) The limit curve with C^2 of the schemes for $\alpha = 0.007$.

The application of the fractal curve is given in Figure 4.

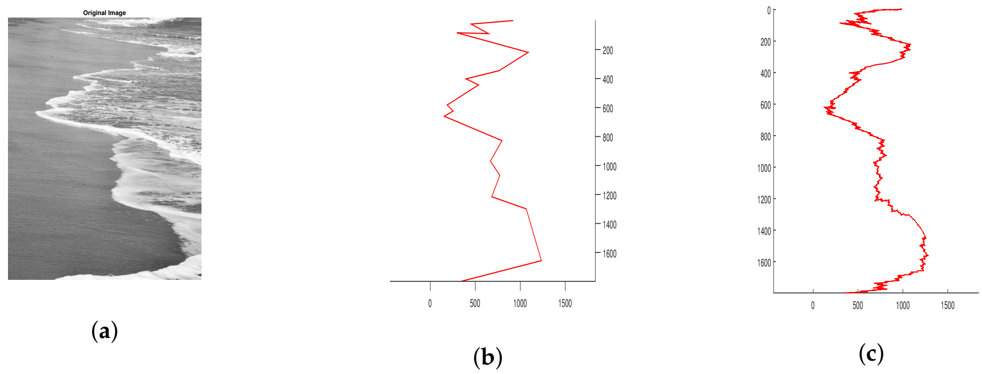


Figure 4. The application of the fractal curve. (**a**) The coastline; (**b**) Corresponds to the data points taken from (a); (**c**) The coastline is constructed by applying the subdivision scheme (5) to (b) for $\alpha = 0.00814$.

The continuous surface and the fractal surface is given in Figure 5.

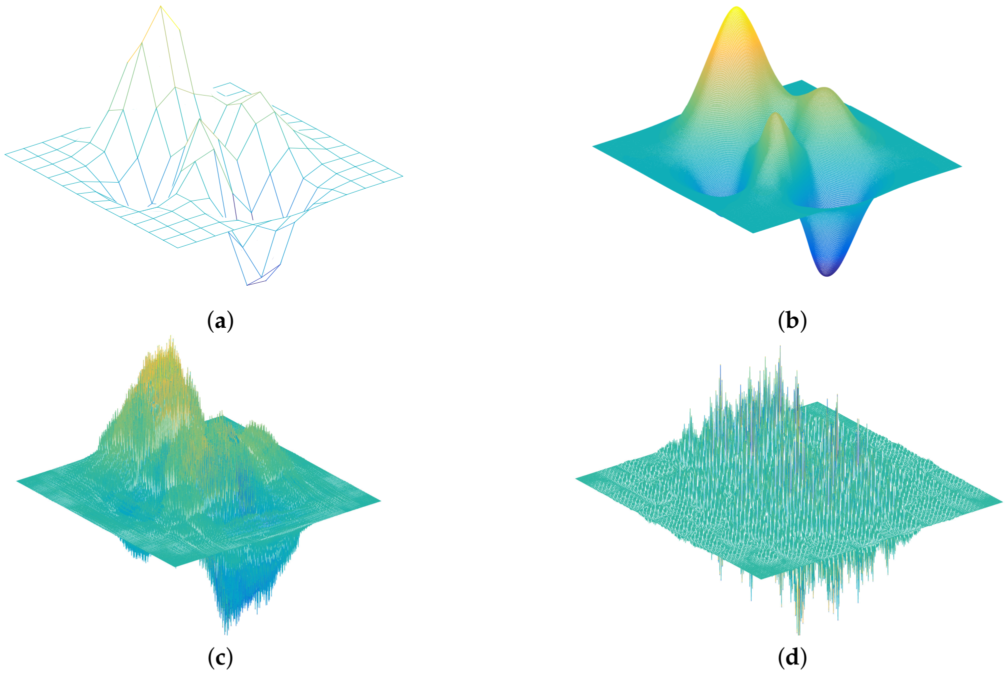


Figure 5. The continuous surface and the fractal surface. The initial control mesh in (**a**) and (**b**) shows the obtained result with $\alpha = 3$, and the surface is C^1 continuous; (**c**) shows the obtained result with $\alpha = 0.008$, it is a fractal surface; (**d**) shows the obtained result with $\alpha = 0.00795$, it is a fractal surface.

7. Conclusions

In this paper, a ternary 4-point interpolating subdivision scheme is introduced which generates a family of C^2 limiting curves for the range of parameter $\alpha \in (-\infty, 0) \bigcup \left(\frac{1}{84}, +\infty\right)$. Compared with Hassan et al. [2] and Siddiqi et al. [6], the advantage of the proposed subdivision scheme is that it generates smooth limiting curves of C^2 continuity for a wider range of the parameter. The Laurent polynomial method and matrix method are used to prove the smoothness of the proposed subdivision scheme. Furthermore, the fractal behavior of the subdivision scheme was analyzed, along with the range of parameter $\alpha \in (\frac{11}{1431}, \frac{7}{837})$.

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