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On Determination of Wave Velocities through the Eigenvalues of Material Objects

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Abstract: The statement of the eigenvalue problem for a tensor–block matrix of any order and of any even rank is formulated. It is known that the eigenvalues of the tensor and the tensor–block matrix are invariant quantities. Therefore, in this work, our goal is to find the expression for the velocities of wave propagation of some medias through the eigenvalues of the material objects. In particular, we consider the classical and micropolar materials with the different anisotropy symbols and for them we determine the expressions for the velocities of wave propagation through the eigenvalues of the material objects.

Keywords: eigentensor; tensor-operator; tensor–block matrix operator; tensor–block matrix; wave velocities; dispersion tensor; symbol of anisotropy; velocity tensor

MSC: 15A18; 15A72; 47A75; 74J05; 74J10

1. Introduction

The theory of wave propagation in structural elements, for example, in rods, plates, and shells, generally speaking, is much more complicated than the theory of waves in an unbounded or semi-bounded medium. Although the individual waves are the same, but being repeatedly reflected from the boundaries, they create a wave, the exact description of which is difficult to realize. The problem of the theory is to build such a model in which an acceptable compromise between the requirements of accuracy and simplicity would be achieved. The fundamentals of the theory of waves in an elastic cylindrical rod were created by Pohgammer and Cree at the end of the 19th century. They found the various forms of eigenwaves in rods. Further, the studies of the unsteady wave propagation in the elements of elastic structures were carried out, as a rule, using approximate equations, which were obtained from the corresponding statics equations.

The addition of inertial terms to these equations made it possible to construct solutions of wave propagation problems, however, some conclusions were in conflict with the results of the theory of elasticity. Thus, the velocity of propagation of perturbations in the dynamic bending of the rod, determined by the Bernoulli–Euler equation, has no upper limit, while, according to the theory of elasticity, it should be limited by the velocity of longitudinal waves in a continuous medium. The mentioned equation does not allow establishing the presence of wave fronts at all. The longitudinal wave velocity, determined by the approximate equation of the longitudinal oscillations of the rod, although limited, but does not correspond with the velocity from the theory of elasticity [1]. There was an obvious need for a stronger justification of approximate equations, the establishment of areas of their applicability, the construction of refined equations, that is, the development of the dynamic theory of plates and shells. The first results were obtained when corrections were introduced into the equations, which made it possible to more fully take into account the main factors that determine the elastic wave propagation (Rayleigh [2], Tymoshenko [3]). A significant contribution was made by Timoshenko, who proposed a refined equation of dynamic bending (and shear) of the rod. As it was later established by Uflyand [4] and others, the Timoshenko equation, in contrast to the Bernoulli–Euler equation, determines the finite wave propagation velocities and gives results close to exact results that follow from the theory of elasticity. Tymoshenko's equations and their solutions were studied in a number of papers (see [1,5,6] and others).

Note that, in Nematic Liquid Crystals (NLC), it is possible to impose electric fields and consequently the nematic orientation. A simple linearization procedure proves that in this way we can electrically vary the acoustic properties of the solid fluid mixture studied in [7]. Therefore, we expect that the methods develop in the present paper may allow a semi-analytical study of the tangent evolution operator for solid–fluid ways in NLC.

In [8], the propagation at discontinuity surfaces in second gradient 3D continua. When endowing the discontinuity surface with material properties, one can expect linear propagation condition of the kind studied in the present paper. We conjecture that in this way it would be to possible design frequency filters at solid interfaces.

In [9], it is proven that second gradient continua must be used as homogenized model for hexagonal lattices if high frequency waves have to be considered. The mathematical method presented here can be applied to get closed form expressions generalizing the results in Figure 6 of the cited paper.

In the theory of non-stationary waves, the wave velocity usually means the front velocity of the wave. Consequently, in the dynamic theory of continuous media, the current problem is the determination of the number of waves and the wave propagation velocity in media. The main types of waves in the continuous elastic medium are a longitudinal or expansion wave and a transverse wave. The existence of these waves was determined by Poisson and Stokes in 1828. Other types of waves that can propagate along the surface of an elastic half-space were discovered by Rayleigh in 1885. The theory of surface waves, very important for seismology, was further developed in the works of Love, Stoneley, and Sobolev. Smirnov and Sobolev introduced a class of functionally invariant solutions of the wave equation and a representation of the wave field using functions of a complex variable. This has achieved significant progress in the general theory of waves in an elastic half-space [10]. One of the main problems of the theory of waves in the half-space is the problem of Lemb [11], namely, the action of a local source on the border of a half-space. One of the main conclusions obtained from the analysis of non-stationary waves in structural elements is that, in the process of propagation of a non-stationary wave through an elastic layer, as a result, in the reflections from surfaces, we obtain an interference pattern, in general, correctly described by simple approximation equations. However, to discover some details of the asymptotics, as well as to make the equations suitable for studying the process, clarifications are necessary, which increase the order of the equations [6]. Note that various methods for obtaining approximate equations for the theories of thin bodies are described in [12]. It also shows various versions of the refined equations for classical and micropolar theories of thin bodies, which can be used to study the wave propagation processes in thin structures. Here, we obtain expressions for velocities of the wave propagation in some unbounded media through the eigenvalues of material objects. Note that some issues of applications of eigenvalue problems for tensor and tensor–block matrices for mathematical modeling of micropolar thin bodies are also considered in [13], while some issues of the modeling of multilayer thin bodies are given in [14].

2. Statement of Eigenvalue Problem of a Tensor–Block Matrix of Any Even Rank

Find all tensor columns $\mathbb{U}$ which satisfy equation

$$\mathbb{M} \overset{p}{\otimes} \mathbb{U} = \lambda \mathbb{U},$$

where λ is scalar, and

$$\mathbb{U} = \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{U}_m \end{pmatrix}, \quad \mathbb{M} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \cdots & \mathbf{A}_{1m} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \cdots & \mathbf{A}_{2m} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} & \cdots & \mathbf{A}_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{m1} & \mathbf{A}_{m2} & \mathbf{A}_{m3} & \cdots & \mathbf{A}_{mm} \end{pmatrix}, \tag{1}$$

$$\mathbb{M} = \mathbb{M}_{i_1 i_2 \cdots i_p}^{\;j_1 j_2 \cdots j_p} \mathbf{R}^{i_1 \cdots i_p} \mathbf{R}_{j_1 \cdots j_p} = \mathbb{M}_i^j \mathbf{R}^i \mathbf{R}_j, \;\; \mathbf{R}_{i_1 \cdots i_p} = \mathbf{r}_1 \otimes \cdots \otimes \mathbf{r}_p,$$
$$\mathbf{R}_i \overset{p}{\otimes} \mathbf{R}^j = g_i^j, \;\; \mathbf{A}_{kl} = A_{kl,\, j_1 j_2 \cdots j_p}^{\;\;\;\; i_1 i_2 \cdots i_p} \mathbf{R}_{i_1 i_2 \cdots i_p} \mathbf{R}^{j_1 j_2 \cdots j_p} = A_{kl,\;\cdot j}^{\;\;\;\;\; i\,\cdot} \mathbf{R}_i \mathbf{R}^j,$$
$$\mathbf{U}_k = \mathbf{U}_{k,\, i_1 i_2 \cdots i_p} \mathbf{R}^{i_1 \cdots i_p}, \;\; k, l = \overline{1, m}, \;\; i_1, i_2, \ldots, i_p, \;\; j_1, j_2, \ldots, j_p = \overline{1, n},$$
$$i, j = \overline{1, N}, \;\; N = n^p.$$

Note that this problem is solved for the tensor of any even rank and the tensor–block matrix (TBM) of any even rank consisting of four tensors, as well as for the tensor and the tensor–block matrix of the fourth rank, and published in [15]. Therefore, here we do not dwell on the presentation of this problem with the aim of shortening the letter, but, if necessary, we refer to the work mentioned in the previous sentence. We also note that, solving the eigenvalue problem for a TBM of any even rank consisting of four tensors, there is no difficulty in solving the analogous problem for the TBM $\mathbb{M}$ (see (1)). Thus, we assume that the eigenvalue problem for a TBM of any order and of any even rank is solved and we consider some of its applications below. In addition, we note that the eigenvalue problem for the tensor of any even rank is a special case of the eigenvalue problem for a TBM of the same rank. Thus, solving the eigenvalue problem for a TBM of any even rank, we believe that it is solved for a tensor of the same rank [15].

3. Kinematic and Dynamic Conditions on the Surface of a Strong Discontinuity in Micropolar Mechanics

Consider a moving regular surface in an unbounded space, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$, whose equation in a fixed Cartesian coordinate system is given by

$$\psi(x, t) = 0. \tag{2}$$

The regularity of Equation (2) means the existence of the unit normal vector $\mathbf{n}(x, t)$ at each point of the surface in Equation (2) at time t:

$$\mathbf{n}(x, t) = \frac{\nabla_x \psi}{|\nabla_x \psi|}, \quad \nabla_x \psi = \mathbf{k}_i \partial_i \psi, \quad i = 1, 2, 3,$$

as well as the required number of times continuously differentiable functions ψ of x. For our purpose, is sufficient the existence of the unit normal, i.e., $\psi \subset C^1$, is sufficient. Here, $\mathbf{k}_i$ is an orthonormal basis of the Cartesian coordinate system, and C^1 is the set of continuously differentiable functions. We use the usual rules of tensor calculus [15–20]. We mainly preserve the notation and conventions of the previous works, capital Latin indices assume the values 1, 2. Over repeated indices there is a summation.

Note that the kinematics of a micropolar medium is described by independent vectors of displacements $\mathbf{u}$ and rotations $\boldsymbol{\varphi}$.

Definition 1. *If the vectors $\mathbf{u}$ and $\boldsymbol{\varphi}$ are continuous when passing through the surface in Equation (2), and the first derivatives $\partial_k\mathbf{u}$, $\partial_t\mathbf{u}$, $\partial_k\boldsymbol{\varphi}$, and $\partial_t\boldsymbol{\varphi}$ undergo a discontinuity such that on each side of this surface they take different finite values, then the surface in Equation (2) is called the surface of a strong discontinuity (or the wave of stress and moment stress).*

Definition 2. *If the vectors $\mathbf{u}$, $\partial_k\mathbf{u}$, $\partial_t\mathbf{u}$, $\boldsymbol{\varphi}$, $\partial_k\boldsymbol{\varphi}$ and $\partial_t\boldsymbol{\varphi}$ are continuous when passing through the surface in Equation (2), and the second derivatives of the vectors $\mathbf{u}$ and $\boldsymbol{\varphi}$ with respect to x_k and t undergo a discontinuity such that on each side of this surface they take different finite values, then the surface in Equation (2) is called the surface of a weak discontinuity (or the acceleration wave).*

We introduce the notation: $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$. Next, the surface of a strong discontinuity is studied.

As in [1,21–23], the velocity of moving an arbitrary point of the surface in Equation (2) in the direction of the normal of this surface at this point, we denote by c:

$$c = -\frac{1}{|\nabla_x\psi|}\partial_t\psi, \quad \nabla\psi = \mathbf{k}_i\partial_i\psi. \tag{3}$$

If the wavefront (the surface of strong discontinuity) of Equation (2) moves in a medium having a velocity field $\mathbf{v}(x,t)$, then the velocity of the wavefront relative to the particles of the medium is determined by the formula

$$\theta = c - v_n, \quad v_n = \mathbf{n}\cdot\mathbf{v}. \tag{4}$$

3.1. Kinematic Conditions on the Surface of a Strong Discontinuity

The kinematic conditions on the surface of strong discontinuity can be obtained in the same way, as is done, for example, in [1,21–23] for the classical case. In this case, the difference lies in that instead of a motion vector should be considered two independent vectors $\mathbf{u}(x,t)$, $\boldsymbol{\varphi}(x,t)$, and they have the form

$$c[\partial_i\mathbf{u}] + n_i[\partial_t\mathbf{u}] = 0, \quad c[\partial_i\boldsymbol{\varphi}] + n_i[\partial_t\boldsymbol{\varphi}] = 0, \tag{5}$$

where c is determined by the first formula of Equation (3), n_i are the components of the unit normal $\mathbf{n}$ to the wave front. Writing $[\mathbf{w}] = \mathbf{w}^+ - \mathbf{w}^-$, where $\mathbf{w} = \mathbf{u}$ or $\mathbf{w} = \boldsymbol{\varphi}$, means a jump in the value of $\mathbf{w}$ relative to the wave front, $\mathbf{w}^-$ ($\mathbf{w}^+$) is the limiting value of $\mathbf{w}$ when the arbitrarily chosen point is in front (behind) the wave front to the point at the wave front. The square brackets [ ] are called the jump operator.

3.2. Laws of Conservation of Mass and the Tensor of Moments of Inertia at the Wave Front

Applying the law of conservation of mass and the law of the tensor of moments of inertia

$$\frac{d}{dt}\int\limits_V \rho dV = 0, \quad \frac{d}{dt}\int\limits_V \underset{\sim}{\mathbf{J}} dV = 0,$$

where ρ is a material density, and $\underset{\sim}{\mathbf{J}}$ is the density of the inertia tensor (a special dynamic characteristic of the medium) of the particles of the medium [24–26], to the elementary cylinder isolated in the medium. After simple transformations analogous to the classical case [1,21–23], we obtain the required laws

$$[\theta\rho] = 0, \quad [\theta\underset{\sim}{\mathbf{J}}] = 0, \tag{6}$$

where θ is defined in Equation (4).

3.3. Dynamic Conditions on the Wave Front

Dynamic conditions on the wave front can be easily obtained with the help of the law on the change of momentum and of the theorem on the change in the angular momentum of the internal rotational motions of the particles of the medium, which are defined as follows:

$$\begin{aligned}
&\frac{d}{dt}\int\limits_V \rho\mathbf{v}dV = \int\limits_V \rho\mathbf{F}dV + \int\limits_\Sigma \mathbf{P}_{(n)}d\Sigma,\\
&\frac{d}{dt}\int\limits_V \underset{\sim}{\mathbf{J}}\cdot\boldsymbol{\omega}dV = \int\limits_V (\rho\mathbf{m} + \underset{\simeq}{\mathbf{C}}\overset{2}{\otimes}\underset{\sim}{\mathbf{P}})dV + \int\limits_\Sigma \boldsymbol{\mu}_{(n)}d\Sigma,
\end{aligned} \tag{7}$$

where $\mathbf{F}$ is is the mass force, $\mathbf{m}$ is the mass moment, $\mathbf{P}_{(n)}$ and $\boldsymbol{\mu}_{(n)}$are the voltage and moment stress vectors on the area with a unit vector of the normal $\mathbf{n}$ respectively, V is the volume of the body, Σ is the boundary of the body, and $\mathbf{v} = \dot{\mathbf{u}}$, $\boldsymbol{\omega} = \dot{\boldsymbol{\varphi}}$, where the dot over the letter indicates the time derivative. As in the classical case [1,21–23], we apply Equation (7) to the above-mentioned cylinder and, taking into account Equation (6), we obtain the conditions

$$\rho\theta[\mathbf{v}] = -[\mathbf{P}_{(n)}] = -\mathbf{n}\cdot[\underset{\sim}{\mathbf{P}}], \quad \theta\underset{\sim}{\mathbf{J}}\cdot[\boldsymbol{\omega}] = -[\boldsymbol{\mu}_{(n)}] = -\mathbf{n}\cdot[\underset{\sim}{\boldsymbol{\mu}}], \tag{8}$$

where $\rho\theta = \rho^-\theta^- = \rho^+\theta^+, \quad \theta\underset{\sim}{\mathbf{J}} = \theta^-\underset{\sim}{\mathbf{J}}^- = \theta^+\underset{\sim}{\mathbf{J}}^+$.

We note that the relations in Equation (8), in which the Cauchy formulas are taken into account, $\mathbf{P}_{(n)} = \mathbf{n}\cdot\underset{\sim}{\mathbf{P}}$ and $\boldsymbol{\mu}_{(n)} = \mathbf{n}\cdot\underset{\sim}{\boldsymbol{\mu}}$, are also valid for any medium. We note that the conditions in Equation (8) and their analogs for different media are derived in [24].

4. Determination of Wave Propagation Velocities in an Infinite Micropolar Solid

Having kinetic (Equation (5)) and dynamic (Equation (8)) conditions on the front of the wave, it is easy to find an equation for determining the propagation velocities of waves in any infinite micropolar medium, including in an infinite micropolar solid. Consider a micropolar solid body, the constitutive relations of which are represented in the form (see [12,27,28])

$$\begin{aligned}
\underset{\sim}{\mathbf{P}} &= \underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\nabla\mathbf{u} + \underset{\approx}{\mathbf{B}}\overset{2}{\otimes}\nabla\boldsymbol{\varphi} - \underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\simeq}{\mathbf{C}}\cdot\boldsymbol{\varphi} - \underset{\sim}{\mathbf{b}}\vartheta,\\
\underset{\sim}{\boldsymbol{\mu}} &= \underset{\approx}{\mathbf{C}}\overset{2}{\otimes}\nabla\mathbf{u} + \underset{\approx}{\mathbf{D}}\overset{2}{\otimes}\nabla\boldsymbol{\varphi} - \underset{\approx}{\mathbf{C}}\overset{2}{\otimes}\underset{\simeq}{\mathbf{C}}\cdot\boldsymbol{\varphi} - \underset{\sim}{\boldsymbol{\beta}}\vartheta,
\end{aligned} \tag{9}$$

where $\underset{\sim}{\mathbf{b}} = \underset{\approx}{\mathbf{A}}\overset{2}{\otimes}\underset{\sim}{\mathbf{a}} + \underset{\approx}{\mathbf{B}}\overset{2}{\otimes}\underset{\sim}{\mathbf{d}}$, $\underset{\sim}{\boldsymbol{\beta}} = \underset{\approx}{\mathbf{C}}\overset{2}{\otimes}\underset{\sim}{\mathbf{a}} + \underset{\approx}{\mathbf{D}}\overset{2}{\otimes}\underset{\sim}{\mathbf{d}}$. Here, $\overset{2}{\otimes}$ is an inner 2-product, $\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{B}}^T$, $\underset{\approx}{\mathbf{A}}$, $\underset{\approx}{\mathbf{D}}$ are the material tensors (tensors of the elastic moduli) of the fourth rank, $\vartheta = T - T_0$ is the temperature drop, $\underset{\sim}{\mathbf{b}}$ and $\underset{\sim}{\boldsymbol{\beta}}$ are the tensors of thermomechanical properties, and T in the upper corner of the tensor means the sign of transposition.

It is known [24,26] that $\underset{\approx}{\mathbf{B}}$ is an asymmetric tensor, as mentioned above. In particular, in [29], it is proved that $\underset{\approx}{\mathbf{B}}$ is a symmetric tensor, and $\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{B}}$. Here, we derive these relations when $\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{B}}^T$, since it is easy to obtain from them the corresponding relations for the case $\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{B}}$.

Next, for simplicity, let us consider isothermal processes, i.e., we assume that $\vartheta = 0$. Then, assuming that the material tensors do not undergo a discontinuity when passing through the front of the wave and applying the jump operator to Equation (9), we have

$$[\underset{\sim}{\mathbf{P}}] = \underset{\approx}{\mathbf{A}}\overset{2}{\otimes}[\nabla\mathbf{u}] + \underset{\approx}{\mathbf{B}}\overset{2}{\otimes}[\nabla\boldsymbol{\varphi}], \quad [\underset{\sim}{\boldsymbol{\mu}}] = \underset{\approx}{\mathbf{C}}\overset{2}{\otimes}[\nabla\mathbf{u}] + \underset{\approx}{\mathbf{D}}\overset{2}{\otimes}[\nabla\boldsymbol{\varphi}]. \tag{10}$$

Multiplying Equation (5) by $\mathbf{k}_i$ with subsequent summation over i (hereinafter, we omit the index x of the operator ∇_x), we obtain

$$[\nabla\mathbf{u}] = -\frac{1}{c}\mathbf{n}[\mathbf{v}], \quad [\nabla\boldsymbol{\varphi}] = -\frac{1}{c}\mathbf{n}[\boldsymbol{\omega}]. \tag{11}$$

From Equation (10) taking into account Equation (11), we find

$$[\underset{\sim}{\mathbf{P}}] = -\frac{1}{c}(\underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \mathbf{n}[\mathbf{v}] + \underset{\approx}{\mathbf{B}} \overset{2}{\otimes} \mathbf{n}[\boldsymbol{\omega}]), \quad [\underset{\sim}{\boldsymbol{\mu}}] = -\frac{1}{c}(\underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \mathbf{n}[\mathbf{v}] + \underset{\approx}{\mathbf{D}} \overset{2}{\otimes} \mathbf{n}[\boldsymbol{\omega}]),$$

with the help of which, from the dynamic conditions in Equation (8), we arrive at the relations

$$\begin{aligned} &(\mathbf{n} \cdot \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \mathbf{n}\underset{\sim}{\mathbf{E}}) \cdot [\mathbf{v}] + (\mathbf{n} \cdot \underset{\approx}{\mathbf{B}} \overset{2}{\otimes} \mathbf{n}\underset{\sim}{\mathbf{E}}) \cdot [\boldsymbol{\omega}] = c\rho\theta \underset{\sim}{\mathbf{E}} \cdot [\mathbf{v}], \\ &(\mathbf{n} \cdot \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \mathbf{n}\underset{\sim}{\mathbf{E}}) \cdot [\mathbf{v}] + (\mathbf{n} \cdot \underset{\approx}{\mathbf{D}} \overset{2}{\otimes} \mathbf{n}\underset{\sim}{\mathbf{E}}) \cdot [\boldsymbol{\omega}] = c\theta \underset{\sim}{\mathbf{J}} \cdot [\boldsymbol{\omega}], \end{aligned} \tag{12}$$

where $\underset{\sim}{\mathbf{E}}$ is the unit tensor of the second rank.

We note that, in view of the laws of conservation of mass and the tensor of moments of inertia, Equation (6), the expressions $\rho\theta$ and $\theta\underset{\sim}{\mathbf{J}}$ in Equation (12) can be replaced by $\rho^+\theta^+$ and $\theta^+\underset{\sim}{\mathbf{J}}^+$, respectively, since in front of the wave front ρ^+ and $\underset{\sim}{\mathbf{J}}^+$, and also v_n^+ can be considered known, but to simplify the recording we do not do this.

We introduce the notation

$$\begin{aligned} &\underset{\sim}{\mathbf{A}} = (1/\rho)\mathbf{n} \cdot \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \mathbf{n}\underset{\sim}{\mathbf{E}}, \ \underset{\sim}{\mathbf{B}} = (1/\rho)\mathbf{n} \cdot \underset{\approx}{\mathbf{B}} \overset{2}{\otimes} \mathbf{n}\underset{\sim}{\mathbf{E}}, \\ &\underset{\sim}{\mathbf{C}} = \underset{\sim}{\mathbf{J}}^{-1} \cdot (\mathbf{n} \cdot \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \mathbf{n}\underset{\sim}{\mathbf{E}}), \ \underset{\sim}{\mathbf{D}} = \underset{\sim}{\mathbf{J}}^{-1} \cdot (\mathbf{n} \cdot \underset{\approx}{\mathbf{D}} \overset{2}{\otimes} \mathbf{n}\underset{\sim}{\mathbf{E}}), \end{aligned} \quad \underset{\sim}{\mathbb{M}} = \begin{pmatrix} \underset{\sim}{\mathbf{A}} & \underset{\sim}{\mathbf{B}} \\ \underset{\sim}{\mathbf{C}} & \underset{\sim}{\mathbf{D}} \end{pmatrix}, \ [\mathbb{V}] = \begin{pmatrix} [\mathbf{v}] \\ [\boldsymbol{\omega}] \end{pmatrix}, \tag{13}$$

where $\underset{\sim}{\mathbb{M}}$ is TBM, $\mathbb{V}$ is a vector column of vectors of linear and angular velocities, and $[\mathbb{V}]$ is the jump of this vector column. Equation (12) can be represented in the form

$$\begin{pmatrix} \underset{\sim}{\mathbf{A}} & \underset{\sim}{\mathbf{B}} \\ \underset{\sim}{\mathbf{C}} & \underset{\sim}{\mathbf{D}} \end{pmatrix} \cdot \begin{pmatrix} [\mathbf{v}] \\ [\boldsymbol{\omega}] \end{pmatrix} = \lambda \begin{pmatrix} [\mathbf{v}] \\ [\boldsymbol{\omega}] \end{pmatrix}, \ \lambda = c\theta \tag{14}$$

or, in short,

$$\underset{\sim}{\mathbb{M}} \cdot [\mathbb{V}] = \lambda[\mathbb{V}]. \tag{15}$$

The equality in Equation (15) (see also Equation (14)) represents a homogeneous system of six algebraic equations with respect to six unknowns (two vectors $[\mathbf{v}]$ and $[\boldsymbol{\omega}]$ having a nontrivial solution). For this system to have a nontrivial solution, it is necessary and sufficient that its determinant be zero. Since the determinant is of the sixth order, and $\theta^+ = c - v_n^+$, then, from the equality to zero of this determinant, we obtain an algebraic equation of degree 6 with respect to c^2, which is the desired dispersion equation for determining the velocities of waves in an infinite anisotropic micropolar body and their number in a given direction.

Thus, we grt the eigenvalue problem in the form of Equation (15) for TBM. Obviously, $\lambda = c\theta$ is an eigenvalue, and $[\mathbb{V}]$ is its corresponding jump of the vector column. By virtue of Equation (15), the dispersion equation (the characteristic equation for $\underset{\sim}{\mathbb{M}}$) can be written in the form

$$\det(\underset{\sim}{\mathbb{M}} - \lambda\underset{\sim}{\mathbb{E}}) = 0, \tag{16}$$

where $\mathbb{E}$ is a unit TBM of the second rank. In the expanded form, Equation (16) can be written as follows:

$$\begin{aligned} &\lambda^6 - I_1(\underset{\sim}{\mathbb{M}})\lambda^5 + I_2(\underset{\sim}{\mathbb{M}})\lambda^4 - I_3(\underset{\sim}{\mathbb{M}})\lambda^3 + I_4(\underset{\sim}{\mathbb{M}})\lambda^2 - I_5(\underset{\sim}{\mathbb{M}})\lambda + I_6(\underset{\sim}{\mathbb{M}}) = 0, \\ &I_6(\underset{\sim}{\mathbb{M}}) = \det(\underset{\sim}{\mathbb{M}}). \end{aligned} \tag{17}$$

It is seen that the dispersion in Equation (17) is an algebraic equation of the sixth degree and must have six roots (eigenvalues), counting each root as many times as its multiplicity. Each multiple root determines the square of the velocity of one wave. Hence, in an arbitrary anisotropic infinite micropolar medium, in the general case, no more than six waves can arise in each direction. Note that, based on the dispersion equation of the form in Equation (17), it is easy to establish the number of waves arising in a micropolar elastic medium for different anisotropy. Thus, it is sufficient to find the invariants of the TBM that appear in Equation (17), and then solve the equation itself. Invariants are easily found through the first invariants of powers $\underset{\sim}{\mathbb{M}}$. We have ([15,17,30–32])

$$S_k = I_k(\underset{\sim}{\mathbb{M}}) = \frac{1}{k\,!}\begin{vmatrix} s_1 & 1 & \cdots & 0 & 0 \\ s_2 & s_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{k-1} & s_{k-2} & \cdots & s_1 & k \\ s_k & s_{k-1} & \cdots & s_2 & s_1 \end{vmatrix}, \quad k = \overline{1,6}, \tag{18}$$

$$S_k = I_k(\underset{\sim}{\mathbb{M}}), \quad s_k = I_1(\underset{\sim}{\mathbb{M}}^k), \quad k = \overline{1,6}, \quad \underset{\sim}{\mathbb{M}}^k = \overbrace{\underset{\sim}{\mathbb{M}} \cdot \underset{\sim}{\mathbb{M}} \cdot \ldots \cdot \underset{\sim}{\mathbb{M}}}^{k},$$

where $I_k(\underset{\sim}{\mathbb{M}})$, $k = \overline{1,6}$, denote the invariants of TBM $\underset{\sim}{\mathbb{M}}$. In this case, the inverse relations to Equation (18) are represented in the form

$$s_k = I_1(\underset{\sim}{\mathbb{M}}^k) = \begin{vmatrix} S_1 & 1 & 0 & \cdots & 0 \\ 2S_2 & S_1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ kS_k & S_{k-1} & S_{k-2} & \cdots & S_1 \end{vmatrix}, \quad k = \overline{1,6}.$$

If the material has a center of symmetry, then $\underset{\approx}{\mathbf{B}} = \underset{\approx}{\mathbf{C}}^T = 0$. Then, $\underset{\sim}{\mathbf{B}} = 0$ and $\underset{\sim}{\mathbf{C}} = 0$, and the TBM $\underset{\sim}{\mathbb{M}}$ becomes a diagonal TBM. For such a matrix, the characteristic equation and eigenvalues (wave velocities) are easily found. We have

$$\det(\underset{\sim}{\mathbb{M}} - \lambda\underset{\sim}{\mathbb{E}}) = \det\begin{pmatrix} \underset{\sim}{\mathbf{A}} - \lambda\underset{\sim}{\mathbf{E}} & \underset{\sim}{\mathbf{0}} \\ \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{D}} - \lambda\underset{\sim}{\mathbf{E}} \end{pmatrix} = \det(\underset{\sim}{\mathbf{A}} - \lambda\underset{\sim}{\mathbf{E}})\det(\underset{\sim}{\mathbf{D}} - \lambda\underset{\sim}{\mathbf{E}}) = 0, \tag{19}$$

where $\underset{\sim}{\mathbf{0}}$ is the zero tensor of rank 2. Equation (19) is equivalent to the following equations:

$$\lambda^3 - I_1(\underset{\sim}{\mathbf{A}})\lambda^2 + I_2(\underset{\sim}{\mathbf{A}})\lambda - I_3(\underset{\sim}{\mathbf{A}}) = 0, \ \lambda^3 - I_1(\underset{\sim}{\mathbf{D}})\lambda^2 + I_2(\underset{\sim}{\mathbf{D}})\lambda - I_3(\underset{\sim}{\mathbf{D}}) = 0. \tag{20}$$

Thus, if the material has a center of symmetry, then in this case to determine the wave velocities we have two cubic equations in Equation (20), which are easily solved. Based on Equation (20), we can conclude that, in an arbitrary anisotropic micropolar elastic medium with a center of symmetry, no more than six waves appear in each direction.

We note that in the case of a classical medium, we have one cubic equation analogous to the first of the equations in Equation (20) provided that $\underset{\sim}{\mathbf{A}}$ is determined using the first relation in Equation (13), where $\underset{\approx}{\mathbf{A}}$ is the elastic modulus tensor. Consequently, in an arbitrary anisotropic classical elastic medium, no more than three waves appear in each direction. Note also that, in the case of a micropolar medium, it is reasonable to call $\underset{\sim}{\mathbb{M}}$ the dispersion TBM, in the case that the classical medium $\underset{\sim}{\mathbf{A}}$ is the dispersion tensor, since their characteristic equations are the dispersion equations.

Having the dispersion equations derived above for the micropolar (classical) medium, we can find the propagation velocities of the waves in the media under consideration for any anisotropy. Of course, the required velocities can be found both in the traditional representation of material objects (TBM, tensors of elastic moduli) and with the help of their eigenvalues, solving in advance the eigenvalue problems of the corresponding object in the case under consideration. The eigenvalue problems for the tensor and TBM of any even rank are solved in [15,17,30,31], in which the notion of the anisotropy (structure) symbol of the material is introduced and the classifications of micropolar and classical media are given.

Therefore, for each material included in these classifications, on the basis of the corresponding dispersion equation obtained for the micropolar (classical) medium, one can obtain a dispersion equation for the considered micropolar (classical) medium, and then determine the number of waves and their propagation velocity.

Definition 3. *The symbol $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, where k is the number of different eigenvalues of the TBM (tensor), and α_i is the multiplicity of the eigenvalue λ_i, $i = 1, 2, \ldots, k$, is called the symbol of the anisotropy (structure) of the TBM (tensor).*

Note that the anisotropy symbol is defined for the TBM (tensor) of even rank. In this case, the symbol of the anisotropy of the material TBM (material tensor) is also called the symbol of anisotropy of the material.

Next, we consider some particular cases of materials and find the propagation velocities of the waves in them using the eigenvalues of the corresponding tensor objects.

5. Application

5.1. Classical Materials with Anisotropy Symbols $\{1, 5\}$ and $\{5, 1\}$

For materials with anisotropy symbols $\{1, 5\}$ and $\{5, 1\}$, the elastic modulus tensor is represented in the form

$$\underset{\approx}{\mathbf{A}} = (\lambda_1 - \lambda_2)\underset{\sim}{\mathbf{a}}_1\underset{\sim}{\mathbf{a}}_1 + \lambda_2\underset{\approx}{\mathbf{E}}, \quad \underset{\approx}{\mathbf{B}} = \mu_1\underset{\approx}{\mathbf{E}} - (\mu_1 - \mu_6)\underset{\sim}{\mathbf{b}}_6\underset{\sim}{\mathbf{b}}_6. \tag{21}$$

Here, $\underset{\approx}{\mathbf{E}} = \sum_{k=1}^{6} \underset{\sim}{\mathbf{a}}_k\underset{\sim}{\mathbf{a}}_k = \sum_{k=1}^{6} \underset{\sim}{\mathbf{b}}_k\underset{\sim}{\mathbf{b}}_k = (1/2)(\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)})$ is the unit tensor of rank 4, $\underset{\sim}{\mathbf{a}}_k$ and $\underset{\sim}{\mathbf{b}}_k$, $k = \overline{1,6}$ are complete orthonormal systems of proper tensors for tensors $\underset{\approx}{\mathbf{A}}$ and $\underset{\approx}{\mathbf{B}}$, respectively, $\underset{\approx}{\mathbf{C}}_{(2)}$ and $\underset{\approx}{\mathbf{C}}_{(3)}$ are isotropic tensors of rank 4, λ_1, λ_2 and μ_1, μ_6 are eigenvalues, and $\underset{\sim}{\mathbf{a}}_1 = \underset{\sim}{\mathbf{a}}_1^T$ and $\underset{\sim}{\mathbf{b}}_6 = \underset{\sim}{\mathbf{b}}_6^T$ are proper tensors corresponding to eigenvalues λ_1 and μ_6, respectively.

Since $\underset{\sim}{\mathbf{a}}_1$ ($\underset{\sim}{\mathbf{b}}_6$) is a symmetric tensor satisfying the orthonormality condition $\underset{\sim}{\mathbf{a}}_1 \overset{2}{\otimes} \underset{\sim}{\mathbf{a}}_1 = 1$ ($\underset{\sim}{\mathbf{b}}_6 \overset{2}{\otimes} \underset{\sim}{\mathbf{b}}_6 = 1$), in a basis constructed using a basis of an arbitrary coordinate system, it is characterized by five components, and in the main basis for $\underset{\sim}{\mathbf{a}}_1$ ($\underset{\sim}{\mathbf{b}}_6$)—two components. Hence, the tensor $\underset{\approx}{\mathbf{A}}$ ($\underset{\approx}{\mathbf{B}}$) in a basis formed using a basis to an arbitrary coordinate system is characterized by seven parameters, namely two eigenvalues and five components of the tensor $\underset{\sim}{\mathbf{a}}_1$ ($\underset{\sim}{\mathbf{b}}_6$), and in the basis formed by means of the canonical basis for $\underset{\sim}{\mathbf{a}}_1$ ($\underset{\sim}{\mathbf{b}}_6$), four parameters, namely two eigenvalues and two components of the tensor $\underset{\sim}{\mathbf{a}}_1$ ($\underset{\sim}{\mathbf{b}}_6$).

Proposition 1. *Let $\underset{\sim}{\mathbf{a}} = \underset{\sim}{\mathbf{a}}^T$. Then, $\underset{\sim}{\mathbf{a}}\underset{\sim}{\mathbf{a}}$ is an isotropic tensor of rank 4 if and only if $\underset{\sim}{\mathbf{a}}$ is a spherical tensor. In this case, if $\underset{\sim}{\mathbf{a}} \overset{2}{\otimes} \underset{\sim}{\mathbf{a}} = 1$, then $\underset{\sim}{\mathbf{a}} = [(\pm\sqrt{3})/3]\underset{\sim}{\mathbf{E}}$, and $\underset{\sim}{\mathbf{a}}\underset{\sim}{\mathbf{a}} = (1/3)\underset{\approx}{\mathbf{C}}_{(1)}$.*

Here, $\underset{\approx}{\mathbf{C}}_{(1)} = \underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}}$ is the first of three isotropic tensors of rank 4. From this statement, it follows that the tensors $\underset{\approx}{\mathbf{A}}$ and $\underset{\approx}{\mathbf{B}}$ of Equation (21) is traditionally isotropic if and only if $\underset{\sim}{\mathbf{a}}_1\underset{\sim}{\mathbf{a}}_1 = \underset{\sim}{\mathbf{b}}_6\underset{\sim}{\mathbf{b}}_6 = (1/3)\underset{\approx}{\mathbf{C}}_{(1)}$. In this case, they can be written in the form

$$\underset{\approx}{\mathbf{A}} = (1/3)(\lambda_1 - \lambda_2)\underset{\approx}{\mathbf{C}}_{(1)} + \lambda_2\underset{\approx}{\mathbf{E}} = \lambda\underset{\approx}{\mathbf{C}}_{(1)} + 2\mu\underset{\approx}{\mathbf{E}}, \quad \underset{\approx}{\mathbf{B}} = \mu_1\underset{\approx}{\mathbf{E}} - (1/3)(\mu_1 - \mu_6)\underset{\approx}{\mathbf{C}}_{(1)}. \tag{22}$$

We note that an isotropic material whose properties are characterized by the tensor $\underset{\approx}{\mathbf{A}}$($\underset{\approx}{\mathbf{B}}$) has a positive (negative) Poisson's ratio [15,17,31]. In addition, the tensor $\underset{\approx}{\mathbf{A}}$ of Equation (22) is presented in the traditional form (by the Lamé coefficients $\lambda = 1/3(\lambda_1 - \lambda_2)$, $\mu = 1/2\lambda_2$).

It is easy to see that as tensors $\underset{\approx}{\mathbf{A}}$ and $\underset{\approx}{\mathbf{B}}$ of Equation (21), and the corresponding dispersion tensors $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ have the same structure and are represented as

$$\underset{\sim}{\mathbf{A}} = \frac{1}{\rho}\mathbf{n} \cdot \underset{\approx}{\mathbf{A}} \cdot \mathbf{n} = a(\underset{\sim}{\mathbf{E}} + \mathbf{nn}) + b\underset{\sim}{\mathbf{u}}, \quad \underset{\sim}{\mathbf{B}} = \frac{1}{\rho}\mathbf{n} \cdot \underset{\approx}{\mathbf{B}} \cdot \mathbf{n} = f(\underset{\sim}{\mathbf{E}} + \mathbf{nn}) + g\underset{\sim}{\mathbf{v}}, \tag{23}$$

$$
\begin{aligned}
&a = \lambda_2/(2\rho) = \mu/\rho > 0, \quad b = (\lambda_1 - \lambda_2)/\rho = (3\lambda)/\rho > 0, \quad \underset{\sim}{\mathbf{u}} = \mathbf{n} \cdot \underset{\sim}{\mathbf{a}}_1 \mathbf{n} \cdot \underset{\sim}{\mathbf{a}}_1, \\
&f = \mu_1/(2\rho) > 0, \quad g = (\mu_6 - \mu_1)/\rho < 0, \quad \underset{\sim}{\mathbf{v}} = \mathbf{n} \cdot \underset{\sim}{\mathbf{b}}_1 \mathbf{n} \cdot \underset{\sim}{\mathbf{b}}_1.
\end{aligned}
\tag{24}
$$

Assuming in Equation (23) $\underset{\sim}{\mathbf{a}}_1 = \underset{\sim}{\mathbf{b}}_6 = (\pm\sqrt{3}/3)\underset{\sim}{\mathbf{E}}$, we obtain the dispersion tensors $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$, corresponding to isotropic tensors in Equation (22),

$$
\underset{\sim}{\mathbf{A}} = a\underset{\sim}{\mathbf{E}} + b_1\mathbf{nn}, \quad \underset{\sim}{\mathbf{B}} = f\underset{\sim}{\mathbf{E}} + g_1\mathbf{nn},
\tag{25}
$$

$$
\begin{aligned}
&a = \frac{\lambda_2}{2\rho} = \frac{\mu}{\rho} > 0, \ \ b_1 = \frac{2\lambda_1 + \lambda_2}{6\rho} = \frac{\lambda + \mu}{\rho} > 0, \ \ \underset{\sim}{\mathbf{u}} = \mathbf{n} \cdot \underset{\sim}{\mathbf{a}}_1 \mathbf{n} \cdot \underset{\sim}{\mathbf{a}}_1, \\
&f = \mu_1/(2\rho) > 0, \quad g_1 = (\mu_1 + 2\mu_6)/(6\rho) > 0, \quad \underset{\sim}{\mathbf{v}} = \mathbf{n} \cdot \underset{\sim}{\mathbf{b}}_1 \mathbf{n} \cdot \underset{\sim}{\mathbf{b}}_1.
\end{aligned}
\tag{26}
$$

Obviously, the tensors in Equation (25), which are special cases of the tensors in Equation (23), similar to the latter have the same structure. In this regard, below we consider the first tensor in Equation (23), and for the rest of tensors in Equations (23) and (25) we obtain the corresponding relations by an appropriate renaming of the coefficients and tensors.

Thus, we find the propagation velocities of the waves in the material $\{1,5\}$ whose dispersion tensor has the form of Equation (23), and the characteristic equation is represented as the first equation of Equation (20). To do this, we first find $I_1(\underset{\sim}{\mathbf{A}}^k)$, $k = 1,2,3$, and then using Equation (18), which is true for a tensor of the corresponding rank, we find $I_k(\underset{\sim}{\mathbf{A}})$, $k = 1,2,3$. After simple calculations, we have

$$
\begin{aligned}
&I_1(\underset{\sim}{\mathbf{A}}) = 4a + bI_1(\underset{\sim}{\mathbf{u}}), \quad I_1(\underset{\sim}{\mathbf{A}}^2) = 6a^2 + 2ab[I_1(\underset{\sim}{\mathbf{u}}) + I^2] + b^2 I_1^2(\underset{\sim}{\mathbf{u}}), \\
&I_1(\underset{\sim}{\mathbf{A}}^3) = 10a^3 + 3a^2b[I_1(\underset{\sim}{\mathbf{u}}) + 3I^2] + 3ab^2 I_1(\underset{\sim}{\mathbf{u}})[I_1(\underset{\sim}{\mathbf{u}}) + I^2] + b^3 I_1^3(\underset{\sim}{\mathbf{u}}), \\
&I_2(\underset{\sim}{\mathbf{A}}) = 5a^2 + 3abI_1(\underset{\sim}{\mathbf{u}}) - abI^2, \quad I_3(\underset{\sim}{\mathbf{A}}) = 2a^3 + 2a^2bI_1(\underset{\sim}{\mathbf{u}}) - a^2bI^2, \\
&I_1(\underset{\sim}{\mathbf{u}}) = \mathbf{n} \cdot \underset{\sim}{\mathbf{a}}_1^2 \cdot \mathbf{n}, \ \ I = \underset{\sim}{\mathbf{a}}_1 \overset{2}{\otimes} \mathbf{nn}.
\end{aligned}
\tag{27}
$$

Further, by virtue of the corresponding invariants in Equation (27), constructing the characteristic equation from Equation (20) for the considered tensor $\underset{\sim}{\mathbf{A}}$ and solving it, we obtain the following expressions for the roots:

$$
\eta_1 = a, \quad \eta_{2,3} = \frac{1}{2}\left[3a + bI_1(\underset{\sim}{\mathbf{u}}) \pm \sqrt{[a - bI_1(\underset{\sim}{\mathbf{u}})]^2 + 4abI^2}\right].
\tag{28}
$$

Theorem 1. *The dispersion tensor is positive definite, and the dispersion TBM is positively determined.*

It follows from this theorem that the eigenvalues of the dispersion tensor and the dispersion tensor–block matrix are positive. It is easy to prove that the eigenvalues in Equation (28) obtained above are positive. Knowing the roots of the characteristic equation of the tensor $\underset{\sim}{\mathbf{A}}$ (see Equation (28)), it is easy to find the propagation velocities of waves in an initially resting medium. We have

$$
c_1 = \sqrt{\eta_1}, \quad c_2 = \sqrt{\eta_2}, \quad c_3 = \sqrt{\eta_3}.
\tag{29}
$$

Thus, in an initially resting medium with the structure symbol $\{1,5\}$ using Equation (29), we can find the propagation velocities of the waves in an arbitrary direction. In the general case, their number is not more than three. Note that, if the medium does not rest, then to determine the wave velocities instead of Equation (29) we have formulas

$$
c_1(c_1 - v_n^+) = \eta_1, \quad c_2(c_2 - v_n^+) = \eta_2, \quad c_3(c_3 - v_n^+) = \eta_3.
\tag{30}
$$

It is seen that each relation from Equation (30) is a square equation with respect to the wave propagation velocity. Solving them we obtain explicit expressions for the propagation velocities of waves in the initially disturbing medium with the structure symbol $\{1,5\}$. It can be assumed that the number of waves arising in such a medium in an arbitrary direction are no more than six, but not fewer than three. To conduct this study, and similar to the above for $\underset{\sim}{\mathbf{A}}$ (the first equation of Equation (23)) and for $\underset{\sim}{\mathbf{B}}$ (the second equation of Equation (23)) is not difficult. Obviously, in the latter case, it is sufficient in the above formulas for $\underset{\sim}{\mathbf{A}}$, as mentioned above, to replace a, b, $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{u}}$ by f, g, $\underset{\sim}{\mathbf{B}}$ and $\underset{\sim}{\mathbf{v}}$, respectively. In connection with this simplicity, we do not dwell on these questions.

Next, consider the first dispersion tensor in Equation (25), which corresponds to the first tensor in Equation (22), characterizing the properties of a traditionally isotropic material and being a particular case of the first tensor in Equation (23). It is easy to see that in this case, analogous to Equation (27), the relations are represented in the form

$$\begin{aligned}&I_1(\underset{\sim}{\mathbf{A}}) = 3a + b_1, \quad I_1(\underset{\sim}{\mathbf{A}}^2) = 2a^2 + (a+b_1)^2, \quad I_1(\underset{\sim}{\mathbf{A}}^3) = 2a^3 + (a+b_1)^3,\\ &I_2(\underset{\sim}{\mathbf{A}}) = a(3a+2b_1), \quad I_3(\underset{\sim}{\mathbf{A}}) = a^2(a+b_1).\end{aligned} \tag{31}$$

Taking into account the invariants corresponding to Equation (31), from the first equation of Equation (20), we obtain the characteristic equation for the investigated dispersion tensor $\underset{\sim}{\mathbf{A}}$. Solving it, we get

$$\eta_1 = \eta_2 = a = \frac{\lambda_2}{2\rho} = \frac{\mu}{\rho}, \quad \eta_3 = a + b_1 = \frac{\lambda_1 + 2\lambda_2}{3\rho} = \frac{\lambda + 2\mu}{\rho}. \tag{32}$$

The velocities of the propagation of waves in an initially resting infinite isotropic elastic medium by virtue of Equation (32) are determined by formulas

$$c_1 = \sqrt{\eta_1} = \sqrt{\frac{\lambda_2}{2\rho}} = \sqrt{\frac{\mu}{\rho}}, \quad c_2 = \sqrt{\eta_3} = \sqrt{\frac{\lambda_1 + 2\lambda_2}{3\rho}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{33}$$

It can be seen that by Equation (33) the propagation velocities of the waves are expressed both in terms of the eigenvalues and the Lame's parameters. In the case under consideration, writing out the analogous to Equation (30) relations and investigating them is not difficult, so we do not dwell on this.

From the first tensor of Equation (25), we see that any vector perpendicular to $\mathbf{n}$ and located in the tangent plane to the wave surface, is an eigenvector of the dispersion tensor $\underset{\sim}{\mathbf{A}}$, and that $\mathbf{n}$ is its eigenvector. Consequently, the system of vectors $(\mathbf{s}, \mathbf{l}, \mathbf{n})$, where $\mathbf{s}$ and $\mathbf{l}$ are mutually perpendicular unit tangent vectors to the wave surface, is the complete orthonormal system of eigenvectors of the considered dispersion tensor $\underset{\sim}{\mathbf{A}}$ (the first tensor of Equation (25)). The eigenvectors of the tensor $\underset{\sim}{\mathbf{A}}$ can be found by solving the system of equations corresponding to these vectors, but, in the case under consideration, there is no such need, since they can be easily guessed. Consequently, the canonical representation of the dispersion tensor $\underset{\sim}{\mathbf{A}}$ in view of what has been said above, has the form

$$\underset{\sim}{\mathbf{A}} = \eta_1(\mathbf{ss} + \mathbf{ll}) + \eta_3\mathbf{nn} = \eta_1\underset{\sim}{\mathbf{I}} + \eta_3\mathbf{nn}, \quad \underset{\sim}{\mathbf{I}} = \mathbf{ss} + \mathbf{ll}. \tag{34}$$

It is easy to see that, by virtue of Equation (33), from Equation (34) for the initially resting medium, we obtain

$$\underset{\sim}{\mathbf{A}} = c_1^2(\mathbf{ss} + \mathbf{ll}) + c_2^2\mathbf{nn} = c_1^2\underset{\sim}{\mathbf{I}} + c_2^2\mathbf{nn} = \mathbf{v}_s\mathbf{v}_s + \mathbf{v}_l\mathbf{v}_l + \mathbf{v}_n\mathbf{v}_n, \tag{35}$$

and we also have

$$\underset{\sim}{\mathbf{V}} = \sqrt{\underset{\sim}{\mathbf{A}}} = c_1(\mathbf{ss}+\mathbf{ll}) + c_2\mathbf{nn} = c_1\underset{\sim}{\mathbf{I}} + c_2\mathbf{nn} = \mathbf{v}_s\mathbf{s} + \mathbf{v}_l\mathbf{l} + \mathbf{v}_n\mathbf{n} = \mathbf{s}\mathbf{v}_s + \mathbf{l}\mathbf{v}_l + \mathbf{n}\mathbf{v}_n, \tag{36}$$

where $\mathbf{v}_s = c_1\mathbf{s}$, $\mathbf{v}_l = c_1\mathbf{l}$ and $\mathbf{v}_n = c_2\mathbf{n}$.

From Equations (34), (35) and (36), it is seen that the tensors $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{V}}$ are represented by the sum of two or three orthogonal tensors. In addition, it is seen from Equation (33) that the eigenvalues of the tensor $\underset{\sim}{\mathbf{V}}$ are the wave propagation velocities, and the eigenvectors coincide with the eigenvectors of the tensor $\underset{\sim}{\mathbf{A}}$, which is quite natural. In this connection, the tensor $\underset{\sim}{\mathbf{V}}$ can be called the velocity tensor. Note that the vector equation for determining the eigen-tensors for $\underset{\sim}{\mathbf{A}}$ (the first equation of Equation (25)) can be written in the form

$$\{(a-\eta)\underset{\sim}{\mathbf{I}}+(b_1+a-\eta)\mathbf{nn}\}\cdot[\mathbf{v}]=0. \tag{37}$$

Consequently, for an arbitrary motion of the medium, the expression $[\mathbf{v}]$ can be represented in the form

$$[\mathbf{v}]=[\mathbf{v}_\tau]+[\mathbf{v}_n],\quad \mathbf{v}_\tau=v_\tau\boldsymbol{\tau},\quad \mathbf{v}_n=v_n\mathbf{n},\quad \boldsymbol{\tau}\perp\mathbf{n}. \tag{38}$$

Taking into account Equation (38), from Equation (37), we have

$$\{(a-\eta)\underset{\sim}{\mathbf{I}}\}\cdot[\mathbf{v}_\tau]+\{(b_1+a-\eta)\mathbf{nn}\}\cdot[\mathbf{v}_n]=0. \tag{39}$$

Note that the following theorem holds, which can be proved by the kinematic conditions (the first equation of Equation (5)).

Theorem 2. *$rot\mathbf{u}=\nabla\times\mathbf{u}=0$ if and only if $[\mathbf{v}]\parallel\mathbf{n}$, and $div\mathbf{u}=\nabla\cdot\mathbf{u}=0$ if and only if $[\mathbf{v}]\perp\mathbf{n}$.*

Note also that, if $[\mathbf{v}]\parallel\mathbf{n}$ or $[\mathbf{v}]\perp\mathbf{n}$, then the wave is called longitudinal or transverse, respectively.

If now, the motion of the medium is such that $[\mathbf{v}]\parallel\mathbf{n}$ ($[\mathbf{v}]\perp\mathbf{n}$), i.e., $\mathbf{v}=\mathbf{v}_n=v\mathbf{n}$, ($\mathbf{v}=\mathbf{v}_\tau=v\boldsymbol{\tau}$), then, from Equation (39), we get

$$\{(b_1+a-\eta)\mathbf{nn}\}\cdot[\mathbf{v}_n]=0\quad\left(\{(a-\eta)\underset{\sim}{\mathbf{I}}\}\cdot[\mathbf{v}_\tau]=0\right), \tag{40}$$

and hence it is easy to obtain the propagation velocities of longitudinal (transverse) waves. They are written above in Equation (33) (see also Equations (30) and (32)), therefore, we do not write them out again here.

Note that a study analogous to the one above can be carried out in a more general case, but for brevity we do not dwell on this.

Further, before considering the materials of other structures, we note that for a fourth rank tensor we apply the four-index and two-index representations ([15,17,30,31])

$$\begin{aligned}
&\underset{\approx}{\mathbf{A}}=A_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l=\sum_{m,n=1}^{9}A_{mn}\underset{\sim}{\mathbf{e}}_m\underset{\sim}{\mathbf{e}}_n=A_{mn}\underset{\sim}{\mathbf{e}}_m\underset{\sim}{\mathbf{e}}_n,\ i,j,k,l=1,2,3,\ m,n=\overline{1,9},\\
&\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij},\ i,j=1,2,3;\ \underset{\sim}{\mathbf{e}}_s=\mathbf{e}_s\mathbf{e}_s,\ s=1,2,3,\ \underset{\sim}{\mathbf{e}}_4=\frac{1}{\sqrt{2}}(\mathbf{e}_1\mathbf{e}_2+\mathbf{e}_1\mathbf{e}_2),\\
&\underset{\sim}{\mathbf{e}}_5=\frac{1}{\sqrt{2}}(\mathbf{e}_2\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_2),\ \underset{\sim}{\mathbf{e}}_6=\frac{1}{\sqrt{2}}(\mathbf{e}_3\mathbf{e}_1+\mathbf{e}_1\mathbf{e}_3),\ \underset{\sim}{\mathbf{e}}_7=\frac{1}{\sqrt{2}}(\mathbf{e}_1\mathbf{e}_2-\mathbf{e}_1\mathbf{e}_2),\\
&\underset{\sim}{\mathbf{e}}_8=\frac{1}{\sqrt{2}}(\mathbf{e}_2\mathbf{e}_3-\mathbf{e}_3\mathbf{e}_2),\ \underset{\sim}{\mathbf{e}}_9=\frac{1}{\sqrt{2}}(\mathbf{e}_3\mathbf{e}_1-\mathbf{e}_1\mathbf{e}_3),\ \underset{\sim}{\mathbf{e}}_m\overset{2}{\otimes}\underset{\sim}{\mathbf{e}}_m=\delta_{mn},\ m,n=\overline{1,9}.
\end{aligned} \tag{41}$$

At the same time, if the components of the tensor $\underset{\approx}{\mathbf{A}}$ have symmetries $A_{ijkl}=A_{klij}=A_{jikl}$, then we have

$$\begin{aligned}
&\underset{\approx}{\mathbf{A}}=A_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l=\sum_{m=1}^{6}\sum_{n=1}^{6}A_{mn}\underset{\sim}{\mathbf{e}}_m\underset{\sim}{\mathbf{e}}_n=A_{mn}\underset{\sim}{\mathbf{e}}_m\underset{\sim}{\mathbf{e}}_n,\\
&i,j,k,l=1,2,3,\ \ m,n=\overline{1,6}.
\end{aligned} \tag{42}$$

Further, the two-index notation of the fourth rank tensor is mainly applied (see Equations (41) and (42)).

5.2. Classical Material with the Anisotropy Symbol $\{1,2,3\}$ (Cubic Symmetry)

In this case ,we have the following canonical representation of the elastic modulus tensor $\underset{\approx}{\mathbf{A}}$:

$$\begin{aligned}
&\underset{\approx}{\mathbf{A}}=(\lambda_1-\lambda_4)\underset{\sim}{\mathbf{W}}_1\underset{\sim}{\mathbf{W}}_1+(\lambda_2-\lambda_4)(\underset{\sim}{\mathbf{W}}_2\underset{\sim}{\mathbf{W}}_2+\underset{\sim}{\mathbf{W}}_3\underset{\sim}{\mathbf{W}}_3)+\lambda_4\underset{\approx}{\mathbf{E}}\\
&(\underset{\approx}{\mathbf{E}}=\sum_{m=1}^{6}\underset{\sim}{\mathbf{W}}_m\underset{\sim}{\mathbf{W}}_m),
\end{aligned} \tag{43}$$

where the eigenvalues and proper tensors are represented in the form

$$\begin{aligned}
&\lambda_1 = A_{11} + 2A_{12}, \quad \lambda_2 = \lambda_3 = A_{11} - A_{12}, \quad \lambda_4 = \lambda_5 = \lambda_6 = A_{44}, \\
&\underset{\sim}{\mathbf{W}}_1 = \frac{\pm\sqrt{3}}{3}(\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_2 + \underset{\sim}{\mathbf{e}}_3) = \frac{\pm\sqrt{3}}{3}\underset{\sim}{\mathbf{E}}, \quad \underset{\sim}{\mathbf{W}}_2 = \frac{\pm\sqrt{2}}{2}(\underset{\sim}{\mathbf{e}}_1 - \underset{\sim}{\mathbf{e}}_2), \\
&\underset{\sim}{\mathbf{W}}_3 = \frac{\pm\sqrt{6}}{6}(\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_2 - 2\underset{\sim}{\mathbf{e}}_3), \quad \underset{\sim}{\mathbf{W}}_4 = \underset{\sim}{\mathbf{e}}_4, \quad \underset{\sim}{\mathbf{W}}_5 = \underset{\sim}{\mathbf{e}}_5, \quad \underset{\sim}{\mathbf{W}}_6 = \underset{\sim}{\mathbf{e}}_6.
\end{aligned} \tag{44}$$

Taking into account the expressions for the intrinsic tensors in Equation (44), from Equation (43), we get

$$\underset{\approx}{\mathbf{A}} = \frac{1}{3}(\lambda_1 - \lambda_2)\underset{\approx}{\mathbf{C}}_1 + (\lambda_2 - \lambda_4)\sum_{k=1}^{3}\underset{\sim}{\mathbf{e}}_k\underset{\sim}{\mathbf{e}}_k + \lambda_4\underset{\approx}{\mathbf{E}}, \quad \underset{\approx}{\mathbf{E}} = \frac{1}{2}(\underset{\approx}{\mathbf{C}}_2 + \underset{\approx}{\mathbf{C}}_3). \tag{45}$$

By virtue of Equation (44), for the dispersion tensor $\underset{\sim}{\mathbf{A}}$, we have the expression

$$\underset{\sim}{\mathbf{A}} = a\mathbf{nn} + b\underset{\sim}{\mathbf{E}} + d\sum_{k=1}^{3} n_k^2\underset{\sim}{\mathbf{e}}_k. \tag{46}$$

Based on Equation (46), it is easy to find $I_1(\underset{\sim}{\mathbf{A}}^k)$, $k = 1,2,3$, and then using Equation (18), which are also true for $\underset{\sim}{\mathbf{A}}$, to find $I_k(\underset{\sim}{\mathbf{A}})$, $k = 2,3$. In fact, after simple calculations, we find

$$\begin{aligned}
&I_1(\underset{\sim}{\mathbf{A}}) = a + 3b + d, \quad I_1(\underset{\sim}{\mathbf{A}}^2) = a(a+2b) + 2bd + 3b^2 + (2ad + d^2)\sum_{k=1}^{3} n_k^4, \\
&I_1(\underset{\sim}{\mathbf{A}}^3) = a(a^2+3ab+3b^2) + 3b^2(b+d) + d[3a(a+2b)+3bd]\sum_{k=1}^{3} n_k^4 + d^2(3a+d)\sum_{k=1}^{3} n_k^6, \\
&I_2(\underset{\sim}{\mathbf{A}}) = \frac{1}{2}[6b^2+4ab+2ad+4bd+d^2-d(2a+d]\sum_{k=1}^{3} n_k^4], \;\; I_3(\underset{\sim}{\mathbf{A}}) = \frac{1}{3!}[6ab^2+3ad^2+ \\
&+3bd^2 + +6abd + 6b^3 + 6b^2d + d^3 - (9ad^2 + 6abd + 3bd^2 + 3d^3)\sum_{k=1}^{3} n_k^4 + \\
&+(6ad^2 + 2d^3)\sum_{k=1}^{3} n_k^6].
\end{aligned}$$

Knowing the invariants $I_k(\underset{\sim}{\mathbf{A}})$, $k = 1,2,3$, in view of, for example, the first equation of Equation (20), we make the characteristic equation for the tensor under consideration, which is an algebraic equation of the third degree and always has three positive roots, counting each root as many times as its multiplicity. Thus, in the case under consideration, for any direction, depending on the multiplicity of the roots, one can determine the number of waves and find their propagation velocities. Consider, for example, three mutually perpendicular directions of wave propagation in the medium under consideration, which are determined by the following values of the components of the normal: $n_i = \delta_{i1}$, $n_i = \delta_{i2}$ $n_i = \delta_{i3}$, $i = 1,2,3$. Note that for each of these directions the first invariant of the tensor $\underset{\sim}{\mathbf{A}}$ does not depend on the direction $\mathbf{n}$, and the second and third invariants have the same values. In particular,

$$I_1(\underset{\sim}{\mathbf{A}}) = a + 3b + d, \quad I_2(\underset{\sim}{\mathbf{A}}) = b^2(2a + 3b + 2d), \quad I_3(\underset{\sim}{\mathbf{A}}) = b^2(a + b + d). \tag{47}$$

By Equation (47), the dispersion equation for each of these directions is the same and has roots

$$\mu_1 = \frac{\lambda_1 + 2\lambda_2}{3\rho}, \quad \mu_2 = \mu_3 = \frac{\lambda_4}{2\rho}. \tag{48}$$

Consequently, in each of these directions, in an infinite medium, in the case of cubic symmetry, two waves arise and on the basis of Equation (48) their velocities are given by formulas

$$c_1 = \sqrt{(\lambda_1 + 2\lambda_2)/(3\rho)}, \quad c_2 = \sqrt{\lambda_4/(2\rho)}. \tag{49}$$

5.3. Classical Material with the Anisotropy Symbol $\{1,1,2,2\}$ (Transversal Isotropy)

In this case, the canonical representation of the elastic modulus tensor $\underset{\approx}{\mathbf{A}}$ has the form

$$\underset{\approx}{\mathbf{A}} = \mu_1 \underset{\sim}{\mathbf{w}}_1 \underset{\sim}{\mathbf{w}}_1 + \mu_2 \underset{\sim}{\mathbf{w}}_2 \underset{\sim}{\mathbf{w}}_2 + \mu_3 (\underset{\sim}{\mathbf{w}}_3 \underset{\sim}{\mathbf{w}}_3 + \underset{\sim}{\mathbf{w}}_4 \underset{\sim}{\mathbf{w}}_4) + \mu_5 (\underset{\sim}{\mathbf{w}}_5 \underset{\sim}{\mathbf{w}}_5 + \underset{\sim}{\mathbf{w}}_6 \underset{\sim}{\mathbf{w}}_6), \tag{50}$$

where the eigenvalues are given by formulas

$$\mu_1 = \frac{1}{2}(A_{11} + A_{12} + A_{33} - \sqrt{(A_{11} + A_{12} - A_{33})^2 + 8A_{13}^2}), \quad \mu_3 = \mu_4 = A_{11} - A_{12},$$
$$\mu_2 = \frac{1}{2}(A_{11} + A_{12} + A_{33} + \sqrt{(A_{11} + A_{12} - A_{33})^2 + 8A_{13}^2}), \quad \mu_5 = \mu_6 = A_{55},$$

and the proper tensors are represented in the form

$$\begin{aligned} \underset{\sim}{\mathbf{w}}_1 &= -\frac{\sqrt{2}}{2} \sin\alpha (\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_2) + \cos\alpha \underset{\sim}{\mathbf{e}}_3 = -\frac{\sqrt{2}}{2} \sin\alpha \underset{\sim}{\mathbf{I}} + \cos\alpha \underset{\sim}{\mathbf{e}}_3, \\ \underset{\sim}{\mathbf{w}}_2 &= \frac{\sqrt{2}}{2} \cos\alpha (\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_2) + \sin\alpha \underset{\sim}{\mathbf{e}}_3 = \frac{\sqrt{2}}{2} \cos\alpha \underset{\sim}{\mathbf{I}} + \sin\alpha \underset{\sim}{\mathbf{e}}_3, \\ \underset{\sim}{\mathbf{w}}_3 &= \frac{\sqrt{2}}{2}(\underset{\sim}{\mathbf{e}}_1 - \underset{\sim}{\mathbf{e}}_2), \quad \underset{\sim}{\mathbf{w}}_i = \underset{\sim}{\mathbf{e}}_i, \ i = 4,5,6, \ \mathrm{tg}2\alpha = \frac{2\sqrt{2}A_{13}}{A_{11} + A_{12} - A_{33}}. \end{aligned} \tag{51}$$

Note that transversely isotropic materials according to the classification adopted in [15,17,31], can be of the following types: $\{1,1,2,2\}, \{1,2,1,2\}, \{1,2,2,1\}, \{2,1,1,2\}, \{2,1,2,1\}, \{2,2,1,1\}$.

Given Equation (51), the tensor in Equation (50) can be written in the form

$$\begin{aligned} &\underset{\approx}{\mathbf{A}} = a_2 \underset{\approx}{\mathbf{C}}_{(1)} + (a_1 - a_2)\underset{\approx}{\mathbf{E}} + a_3(\underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{e}}_3 + \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{I}}) + a_4 \underset{\sim}{\mathbf{e}}_3 \underset{\sim}{\mathbf{e}}_3 + \\ &+ \frac{1}{2} a_5 (\mathbf{e}_3 \underset{\sim}{\mathbf{I}} \mathbf{e}_3 + \mathbf{e}_I \underset{\sim}{\mathbf{e}}_3 \mathbf{e}_I + \mathbf{e}_I \mathbf{e}_3 \mathbf{e}_I \mathbf{e}_3 + \mathbf{e}_3 \mathbf{e}_I \mathbf{e}_3 \mathbf{e}_I), \end{aligned} \tag{52}$$

$$\begin{aligned} &\underset{\approx}{\mathbf{C}}_{(1)} = \underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}}, \quad \underset{\approx}{\mathbf{E}} = \frac{1}{2}(\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)}) = \underset{\sim}{\mathbf{e}}_1 \underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_2 \underset{\sim}{\mathbf{e}}_2 + \underset{\sim}{\mathbf{e}}_4 \underset{\sim}{\mathbf{e}}_4, \quad \underset{\sim}{\mathbf{I}} = \underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_2, \\ &a_1 = \frac{1}{2}(\mu_1 \sin^2\alpha + \mu_2 \cos^2\alpha + \mu_3), \quad a_2 = \frac{1}{2}(\mu_1 \sin^2\alpha + \mu_2 \cos^2\alpha - \mu_3), \\ &a_3 = \frac{\sqrt{2}}{2}(\mu_2 - \mu_1)\sin\alpha\cos\alpha, \quad a_4 = \mu_1 \cos^2\alpha + \mu_2 \sin^2, \quad a_5 = \mu_5. \end{aligned} \tag{53}$$

The dispersion tensor for Equation (52) after simple calculations is written as follows:

$$\underset{\sim}{\mathbf{A}} = A_1 \mathbf{m}\mathbf{m} + A_2 \underset{\sim}{\mathbf{I}} + A_3(\mathbf{m}\mathbf{e}_3 + \mathbf{e}_3\mathbf{m}) + A_4 \underset{\sim}{\mathbf{e}}_3, \tag{54}$$

$$\begin{aligned} &\mathbf{m} = n_I \mathbf{e}_I, \quad A_1 = \frac{1}{\rho}(a_1 + \frac{1}{2}a_2), \quad A_2 = \frac{1}{\rho}(\frac{1}{2}a_2 m^2 + a_5 n_3^2), \\ &A_3 = \frac{1}{\rho}(a_3 + a_5) n_3, \quad A_4 = \frac{1}{\rho}(a_4 n_3^2 + a_5 m^2). \end{aligned} \tag{55}$$

Then, it's easy to find $\underset{\sim}{\mathbf{A}}^2$ and $\underset{\sim}{\mathbf{A}}^3$. By virtue of Equation (54), we have

$$\begin{aligned} &\underset{\sim}{\mathbf{A}}^2 = B_1 \mathbf{m}\mathbf{m} + B_2 \underset{\sim}{\mathbf{I}} + B_3(\mathbf{m}\mathbf{e}_3 + \mathbf{e}_3\mathbf{m}) + B_4 \underset{\sim}{\mathbf{e}}_3, \\ &\underset{\sim}{\mathbf{A}}^3 = C_1 \mathbf{m}\mathbf{m} + C_2 \underset{\sim}{\mathbf{I}} + C_3(\mathbf{m}\mathbf{e}_3 + \mathbf{e}_3\mathbf{m}) + C_4 \underset{\sim}{\mathbf{e}}_3, \end{aligned} \tag{56}$$

$$\begin{aligned} &B_1 = A_1^2 m^2 + 2A_1 A_2 + A_3^2, \quad B_2 = A_2^2, \quad B_3 = A_1 A_3 m^2 + A_2 A_3 + A_3 A_4, \\ &B_4 = A_3^2 m^2 + A_4^2, \ C_1 = A_1^3 m^4 + (3A_1^2 A_2 + 2A_1 A_3^2) m^2 + 3A_1 A_2^2 + 2A_2 A_3^2 + A_3^2 A_4, \\ &C_3 = A_1^2 A_3 m^4 + (2A_1 A_2 A_3 + A_1 A_3 A_4 + A_3^3) m^2 + A_2^2 A_3 + A_2 A_3 A_4 + A_3 A_4^2, \\ &C_2 = A_2^3, \quad C_4 = A_1 A_3^2 m^4 + (2A_3^2 A_4 + A_2 A_3^2) m^2 + A_4^3. \end{aligned} \tag{57}$$

From Equations (54) and (56), we find $I_1(\underset{\sim}{\mathbf{A}}^k)$, $k = 1,2,3$, and, by virtue of Equation (18), we get $I_m(\underset{\sim}{\mathbf{A}})$, $m = 2,3$. We have

$$\begin{aligned}
&I_1(\underset{\sim}{\mathbf{A}}) = A_1m^2+2A_2+A_4,\ I_1(\underset{\sim}{\mathbf{A}}^2) = B_1m^2+2B_2+B_4,\ I_1(\underset{\sim}{\mathbf{A}}^3) = C_1m^2+2C_2+C_4,\\
&I_2(\underset{\sim}{\mathbf{A}}) = \frac{1}{2}\{A_1^2m^4+[2A_1(2A_2+A_4)-B_1]m^2+(2A_2+A_4)^2-2B_2-B_4\},\\
&I_3(\underset{\sim}{\mathbf{A}}) = \frac{1}{3!}\{A_1^3m^6+3[A_1^2(2A_2+A_4)-A_1B_1]m^4+[3A_1(2A_2+A_4)^2-3A_1(2B_2+B_4)-\\
&-3B_1(2A_2+A_4)+2C_1]m^2+(2A_2+A_4)^3-3(2A_2+A_4)(2B_2+B_4)+2(2C_2+C_4)\}.
\end{aligned} \tag{58}$$

Knowing $I_m(\underset{\sim}{\mathbf{A}})$, $m = 1,2,3$ (see Equation (58)), from the dispersion equation for any direction $\mathbf{n}$, we find the eigenvalues $\underset{\sim}{\mathbf{A}}$, and then the wave velocities.

Let us find, for example, the wave velocities in the following directions: $n_i = \delta_{i1}$, $n_i = \delta_{i2}$ and $n_i = \delta_{i3}$, $i = 1,2,3$. Note that from the expression for $I_m(\underset{\sim}{\mathbf{A}})$, $m = 1,2,3$ (see Equation (58)) follows that for directions $n_i = \delta_{i1}$ and $n_i = \delta_{i2}$, $i = 1,2,3$, they will accept the same values. In fact, by virtue of the corresponding Equations (55) and (58) for these directions, we have

$$\begin{aligned}
&I_1(\underset{\sim}{\mathbf{A}}) = \frac{a_1+a_2}{\rho}+\frac{a_2}{2\rho}+\frac{a_5}{\rho},\quad I_2(\underset{\sim}{\mathbf{A}}) = \frac{a_1+a_2}{\rho}\frac{a_2}{2\rho}+\frac{a_1+a_2}{\rho}\frac{a_5}{\rho}+\frac{a_2}{2\rho}\frac{a_5}{\rho},\\
&I_3(\underset{\sim}{\mathbf{A}}) = \frac{a_1+a_2}{\rho}\frac{a_2}{2\rho}\frac{a_5}{\rho}.
\end{aligned} \tag{59}$$

From Equation (58), we see that the characteristic equation of the tensor $\underset{\sim}{\mathbf{A}}$ has roots

$$\mu_1 = \frac{a_1+a_2}{\rho},\quad \mu_2 = \frac{a_2}{2\rho},\quad \mu_3 = \frac{a_5}{\rho}. \tag{60}$$

Hence, for the wave velocities in the initially resting medium, we have the following values:

$$c_1 = \sqrt{\frac{a_1+a_2}{\rho}},\quad c_2 = \sqrt{\frac{a_2}{2\rho}},\quad c_3 = \sqrt{\frac{a_5}{\rho}}. \tag{61}$$

For the direction $n_i = \delta_{i3}$, $i = 1,2,3$, similar to Equations (59)–(61), the relations have the form

$$\begin{aligned}
&I_1(\underset{\sim}{\mathbf{A}}) = \frac{a_4}{\rho}+2\frac{a_5}{\rho},\quad I_2(\underset{\sim}{\mathbf{A}}) = 2\frac{a_4}{\rho}\frac{a_5}{\rho}+\left(\frac{a_5}{\rho}\right)^2,\quad I_3(\underset{\sim}{\mathbf{A}}) = \frac{a_4}{\rho}\left(\frac{a_5}{\rho}\right)^2,\\
&\mu_1 = \frac{a_4}{\rho},\quad \mu_2 = \mu_3 = \frac{a_5}{\rho},\quad c_1 = \sqrt{\frac{a_4}{\rho}},\quad c_2 = \sqrt{\frac{a_5}{\rho}}.
\end{aligned}$$

Note that the consideration of the material with the symbol of the structure $\{1,1,2,2\}$ of the trigonal syngonies (six essential components) reduces to the previous case.

5.4. Micropolar Material with a Center of Symmetry and the Anisotropy Symbol $\{1,5,3\}$

In this case, the properties of the medium are characterized by two tensors of the fourth rank $\underset{\approx}{\mathbf{A}}$ and $\underset{\approx}{\mathbf{D}}$, which have the same structure. Therefore, it suffices to consider one of these tensors, since all the relations obtained for one tensor are obtained in a completely analogous way for the other (see Equation (20)). Consider, for example, the tensor $\underset{\approx}{\mathbf{A}}$. In this case, its canonical representation has the form

$$\underset{\approx}{\mathbf{A}} = (\lambda_1-\lambda_2)\underset{\sim}{\mathbf{u}}_1\underset{\sim}{\mathbf{u}}_1+\lambda_2\underset{\approx}{\mathbf{E}}+(\lambda_7-\lambda_2)\sum_{m=7}^{9}\underset{\sim}{\mathbf{u}}_m\underset{\sim}{\mathbf{u}}_m,\quad \underset{\approx}{\mathbf{E}} = \underset{\approx}{\mathbf{C}}_{(2)} = \sum_{m=1}^{9}\underset{\sim}{\mathbf{u}}_m\underset{\sim}{\mathbf{u}}_m. \tag{62}$$

By virtue of Equation (62), it is easy to find $I_1(\underset{\approx}{\mathbf{A}}^k)$, $k = 1,2,3$, and then by virtue of Equation (18) get $I_m(\underset{\sim}{\mathbf{A}})$, $m = 2,3$. Knowing $I_m(\underset{\sim}{\mathbf{A}})$, $m = 1,2,3$, we can write the characteristic equation $\underset{\sim}{\mathbf{A}}$ and find its roots, and then determine the desired velocity of the waves. Formulas are cumbersome, but the main thing is that they can be obtained in an explicit form. Note that the case of the medium $\{5,1,3\}$ is a similar consideration as $\{1,5,3\}$. Therefore, in the general case, we do not stop. Next, we consider the case of a micropolar isotropic elastic medium with a center of symmetry and the symbol of structure $\{1,5,3\}$. In this case, the tensors $\underset{\approx}{\mathbf{A}}$ and $\underset{\approx}{\mathbf{D}}$ have representations

$$\underset{\approx}{\mathbf{A}} = a_1 \underset{\approx}{\mathbf{C}}_{(1)} + a_2 \underset{\approx}{\mathbf{C}}_{(2)} + a_1 \underset{\approx}{\mathbf{C}}_{(3)}, \quad \underset{\approx}{\mathbf{D}} = d_1 \underset{\approx}{\mathbf{C}}_{(1)} + d_2 \underset{\approx}{\mathbf{C}}_{(2)} + d_1 \underset{\approx}{\mathbf{C}}_{(3)}, \tag{63}$$

where the coefficients $a_i, d_i,\ i = 1,2,$ are determined in terms of the eigenvalues of these tensors by the formulas:

$$a_1 = \frac{1}{3}(\lambda_1 - \lambda_2), \quad a_2 = \frac{1}{2}(\lambda_2 + \lambda_7), \quad a_3 = \frac{1}{2}(\lambda_2 - \lambda_7),$$
$$d_1 = \frac{1}{3}(\mu_1 - \mu_2), \quad d_2 = \frac{1}{2}(\mu_2 + \mu_7), \quad d_3 = \frac{1}{2}(\mu_2 - \mu_7).$$

It is easy to see that in the present case the dispersion tensors $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{D}}$, corresponding to tensors $\underset{\approx}{\mathbf{A}}$ and $\underset{\approx}{\mathbf{D}}$ of Equation (63), respectively, have the form

$$\underset{\sim}{\mathbf{A}} = \frac{1}{\rho}[(a_1 + a_3)\mathbf{nn} + a_2 \underset{\sim}{\mathbf{E}}], \quad \underset{\sim}{\mathbf{D}} = \frac{1}{J}[(d_1 + d_3)\mathbf{nn} + d_2 \underset{\sim}{\mathbf{E}}]. \tag{64}$$

From Equation (64), we see that $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{D}}$ are similar to the tensors considered above from Equation (25). Introducing the notation

$$a = \frac{a_2}{\rho}, \quad b = \frac{a_1 + a_3}{\rho}, \quad f = \frac{d_2}{J}, \quad g = \frac{d_1 + d_3}{J},$$

the tensors in Equation (64) can be written as follows:

$$\underset{\sim}{\mathbf{A}} = a \underset{\sim}{\mathbf{E}} + b\mathbf{nn}, \quad \underset{\sim}{\mathbf{D}} = f \underset{\sim}{\mathbf{E}} + g\mathbf{nn}. \tag{65}$$

Then, similar to Equations (32) and (33), the roots of the dispersion equations of the tensors $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{D}}$ and the wave velocities in the initially at rest media are determined by the formulas

$$\begin{aligned} &\eta_1 = \eta_2 = a = \frac{\lambda_2 + \lambda_7}{2\rho}, \quad \eta_3 = a + b = \frac{\lambda_1 + 2\lambda_2}{3\rho} = \frac{\lambda + 2\mu}{\rho}, \\ &\eta_4 = \eta_5 = f = \frac{\mu_2 + \mu_7}{2\rho}, \quad \eta_6 = f + g = \frac{\mu_1 + 2\mu_2}{3\rho}, \end{aligned} \tag{66}$$

$$c_1 = \sqrt{\frac{\lambda_2 + \lambda_7}{2\rho}}, \quad c_2 = \sqrt{\frac{\lambda_1 + 2\lambda_2}{3\rho}}, \quad c_3 = \sqrt{\frac{\mu_2 + \mu_7}{2\rho}}, \quad c_4 = \sqrt{\frac{\mu_1 + 2\mu_2}{3\rho}}. \tag{67}$$

Thus, in a micropolar isotropic, infinite, initially resting medium, four waves appear in each direction, the velocities of which are calculated from Equation (67).

Note that, to determine the direction of propagation of these waves, it is necessary to find a complete system of eigenvectors and give a canonical representation of the dispersion tensors, and then conduct a study just as it was done above in the case of classical isotropic material. Of course, such a study can always be carried out for any anisotropic medium, as well as for a more general medium.

Note that, in the traditional form, the wave velocities in classical and micropolar media are researched in [33].

6. Conclusions

The statement of the eigenvalue problem for TBM of any order and any even rank is formulated. We consider materials with the anisotropy symbols $\{1.5\}$ and $\{5.1\}$, as well as isotropic materials, and for them we determine the expressions for the velocities of wave propagation. In addition, we obtain expressions for the velocities of wave propagation for materials of cubic syngony with the anisotropy symbol $\{1,2,3\}$, hexagonal system (transversal isotropy) with anisotropy symbol $\{1,1,2,2\}$, trigonal system with anisotropy symbol $\{1,1,2,2\}$, and tetragonal system with anisotropy symbol $\{1,1,1,2,1\}$. We also obtain the expressions for the velocities of wave propagation for a micropolar medium with the anisotropy symbols $\{1.5.3\}$ and $\{5.1.3\}$, and for an isotropic micropolar material.

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References

1. Poruchikov, V.B. *Methods of the Dynamic Theory of Elasticity*; Nauka: Moscow, Russia, 1986. (In Russian)
2. Strett, D.V. *Teoriya Zvuka*; Book on Demand Ltd.: Moscow, Russia, 1955; pp. 1–2. (In Russian)
3. Timoshenko, S.P. *Kolebaniya v Inzhenernom Dele*; Fizmatgiz: Moscow, Russia, 1959. (In Russian)
4. Uflyand, Y.S. Rasprostraneniye voln pri poperechnykh kolebaniyakh sterzhney i plastin. *Prikl. Matem. Mekhan.* **1948**, *12*, 3. (In Russian)
5. Novozhilov, V.V.; Slepyan, L.I. On Saint-Venant's principle in the dynamics of beams. *J. Appl. Math. Mech.* **1965**, *29*, 293–315. [CrossRef]
6. Slepyan, L.I. *Non-Stationary Elastic Waves*; Sudostroenie: Saint-Petersburg, Russia, 1972. (In Russian)
7. Rosi, G.; Placidi, L.; dell'Isola, F. "First" and "slow" pressure waves electrically induced by nonlinear coupling in Biot-type porous medium saturated by a nematic liquid crystal. *Zeitschrift für Angewandte Mathematik und Physik* **2017**, *68*, 51. [CrossRef]
8. dell'Isola, F.; Madeo, A.; Placidi, L. Linear plane wave propagation and normal transmission and reflection at discontinuity surfaces in second gradient 3D continua. *Zeitschrift für Angewandte Mathematik und Mechanik* **2012**, *92*, 52–71. [CrossRef]
9. Rosi, G.; Auffrey, N. Anisotropic and dispersive wave propagation within strain-gradient framework. *Wave Motion* **2016**, *63*, 120–134. [CrossRef]
10. Sobolev, S.L. Nekotoryye voprosy teorii rasprostraneniya kolebaniy. In *Differentsialnyye i Integralnyye Uravneniya Matematicheskoy Fiziki*; Frank, F., Mises, R., Eds.; ONTI: Leningrad, Moscow, 1937. (In Russian)
11. Lamb, H. On the propagation of tremors over the surface on an elastic solid. *Proc. R. Soc. Lond. A* **1903**, *72*, 128–130. [CrossRef]
12. Nikabadze, M.U. *Development of the Method of Orthogonal Polynomials in the Classical and Micropolar Mechanics of Elastic Thin Bodies*; Moscow University Press: Moscow, Russia, 2014. (In Russian)
13. Nikabadze, M.U.; Ulukhanyan, A.R. Some Applications of Eigenvalue Problems for Tensor and Tensor Block Matrices for Mathematical Modeling of Micropolar Thin Bodies. *Math. Comput. Appl.* **2019**, *24*, 33. [CrossRef]
14. Nikabadze, M.U.; Ulukhanyan, A.R. Modeling of multilayer thin bodies. *Contin. Mech. Thermodyn.* **2019**, *31*, 1–26. [CrossRef]
15. Nikabadze, M.U. Topics on tensor calculus with applications to mechanics. *J. Math. Sci.* **2017**, *225*, 1–194. [CrossRef]
16. Luré, A.I. *Nonlinear Elasticity*; Nauka: Moscow, Russia, 1980. (In Russian)
17. Nikabadze, M.U. On Some Issues of Tensor Calculus with Applications to Mechanics. *Contemp. Math. Fundam. Direct.* **2015**, *55*, 3–194.
18. Pobedrya, B.E. *Numerical Methods in the Theory of Elasticity and Plasticity*, 2nd ed.; Moscow State University: Moskva, Russia, 1995; 366p. (In Russian)

19. Vekua, I.N. *Fundamentals of Tensor Analysis and Covariant Theory*; Nauka: Moscow, Russia, 1978. (In Russian)
20. Pobedrya, B.E. *Lectures in Tensor Analysis*; Moscow State University: Moskva, Russia, 1986. (In Russian)
21. Petrashen, G.I. Basics of the mathematical theory of the propagation of elastic waves. In *Issues of Dynamic Propagation of Seismic Waves*; Nauka: Moscow, Russia, 1978; 248p. (In Russian)
22. Petrashen, G.I. *Propagation of Waves in Anisotropic Elastic Media*; Nauka: Leningrad, Russia, 1980; 280p. (In Russian)
23. Sagomonyan, A.Y. *Stress Waves in Continuous Media*; Moscow University Press: Moscow, Russia, 1985; 416p. (In Russian)
24. Eringen, A.C. *Microcontinuum Field Theories I. Foundations and Solids*; Springer: New York, NY, USA, 1999; 340p.
25. Kupradze, V.D.; Gegelia, T.G.; Basheleishvili, M.O.; Burchuladze, T.V. *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*; Nauka: Moscow, Russia, 1976; 664p. (In Russian)
26. Nowacki, W. *Theory of Elasticity*; Mir: Moscow, Russia, 1975; 872p. (In Russian)
27. Nikabadze, M.U. Some issues concerning a version of the theory of thin solids based on expansions in a system of Chebyshev polynomials of the second kind. *Mech. Solids* **2007**, *42*, 391–421. [CrossRef]
28. Nikabadze, M.U.; Ulukhanyan, A.R. Analytical Solutions in the Theory of Thin Bodies. In *Generalized Continua as Models for Classical and Advanced Materials, Advanced Structured Materials*; Altenbach, H., Forest, S., Eds.; Springer: Cham, Switzerland, 2016; Volume 42, pp. 319–361. [CrossRef]
29. Baskakov, V.A.; Bestuzheva, N.P.; Konchakova, N.A. *Linear Dynamic Theory of Thermoelastic Medium with Microstructure*; Voronezh University Press: Voronezh, Russia, 2001; 162 p.
30. Nikabadze, M.U. Construction of Eigentensor Columns in the Linear Micropolar Theory of Elasticity. *Mosc. Univ. Mech. Bull.* **2014**, *69*, 1–9. [CrossRef]
31. Nikabadze, M.U. Eigenvalue Problems of a Tensor and a Tensor-Block Matrix (TMB) of Any Even Rank with Some Applications in Mechanics. In *Generalized Continua as Models for Classical and Advanced Materials, Advanced Structured Materials*; Altenbach, H., Forest, S., Eds.; Springer: Cham, Switzerland, 2016; pp. 279–317.
32. Nikabadze, M.U. To the problem of decomposition of the initial boundary value problems in mechanics. *J. Phys. Conf. Ser.* **2017**, *739*, 012056. [CrossRef]
33. Ulukhanyan, A.R. The Dispersive Equations and Velocities of Propagation of Waves in the Couple Theory. Available online: https://www.researchgate.net/publication/318300021_Dispersionnye_uravnenia_i_skorosti_rasprostranenia_voln_v_momentnoj_teorii_The_Dispersive_Equations_and_Velocities_of_Propagation_of_Waves_in_the_Couple_Theory (accessed on 10 April 2019). (In Russian)
34. Matevossian, H.; Nikabadze, M.; Ulukhanian, A. Determination of velocities of wave propagation in some media through the eigenvalues of the material tensors. *J. Phys. Conf. Ser.* **2018**, *1141*, 012154. [CrossRef]