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A Phase-Fitted and Amplification-Fitted Explicit Runge–Kutta–Nyström Pair for Oscillating Systems

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Abstract: An optimized embedded 5(3) pair of explicit Runge–Kutta–Nyström methods with four stages using phase-fitted and amplification-fitted techniques is developed in this paper. The new adapted pair can exactly integrate (except round-off errors) the common test: $y'' = -w^2 y$. The local truncation error of the new method is derived, and we show that the order of convergence is maintained. The stability analysis is addressed, and we demonstrate that the developed method is absolutely stable, and thus appropriate for solving stiff problems. The numerical experiments show a better performance of the new embedded pair in comparison with other existing RKN pairs of similar characteristics.

Keywords: phase-fitted and amplification-fitted schemes; RKN pair; oscillatory systems; initial-value problems

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1. Introduction

The aim of this paper is to efficiently solve special second-order initial-value systems of the form

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0, \tag{1}$$

for which it is known that their solutions are oscillatory, where $y \in \Re^d$ and $f : \Re \times \Re^d \to \Re^d$ is sufficiently differentiable. In recent and past years, the search for new numerical algorithms to efficiently solve (1) has attracted the attention of many researchers due to the great relevance of these problems in so many areas of applied sciences (as quantum chemistry, fluid mechanics, physical chemistry, astronomy and many others). To solve (1) directly, the class of Runge–Kutta–Nyström (RKN) methods has mostly been used. Regarding the efficient use of these methods, the embedded technique was firstly proposed by Fehlberg in [1] to provide an estimate of the error committed on each step. Since then there have been many researchers who have presented pairs of embedded RKN methods. Van de Vyver developed in [2] an explicit 5(3) embedded pair of RKN methods with four stages for solving (1). Franco developed a 5(3) embedded pair of explicit ARKN schemes with four stages in [3]. Simos [4], Kalogiratou and Simos [5], Van de Vyver [6] and Liu [7] derived different adapted RKN methods. Senu et al. constructed an explicit embedded pair of RKN methods in [8], Franco et al. [9] presented two embedded explicit RKN pairs for approximating the oscillatory solution of (1). Anastassi and Kosti developed a 6(4) embedded RKN optimized pair in [10]. Fang et al. developed a new pair of explicit ARKN methods in [11], for the numerical integration of general perturbed oscillators. Similarly, Fang et al. in [12] constructed an efficient energy-preserving method for general nonlinear oscillatory Hamiltonian systems. Also, Mei et al. in [13] derived an arbitrary order ERKN method based on group theory for solving oscillatory Hamiltonian systems, and Yang et al. in [14] developed an extended RKN-type method for the numerical integration of perturbed oscillators. Recently, Demba et al. [15,16] derived two new explicit RKN methods trigonometrically adapted for solving the kind of problems in (1). Most recently, Demba et al. [17] derived an exponentially-fitted explicit RKN pair for solving (1).

This work aims at the development of a new phase-fitted and amplification-fitted 5(3) embedded pair of explicit RKN methods based on the 5(3) pair presented by Van der Vyver [2] for solving the problem in (1). The derived method accurately solves the test equation $y'' = -w^2y$. The numerical experiments reveal the efficiency of the developed method when compared with other embedded RKN codes of orders 5(3) with four stages.

The remaining part of this paper is organized in this way: the description of a pair of explicit RKN methods, the definitions of phase-lag and amplification error, and the definitions regarding the stability analysis are addressed in Section 2. Section 3 is devoted to the construction of the new code, to determine the order and error analysis, and to bring some details about the linear stability of the derived pair. Some numerical examples are presented in Section 4 along with some comments on the results obtained. Finally, Section 5 gives a conclusion.

2. Fundamental Concepts

2.1. Explicit Runge-Kutta-Nyström Methods

An explicit RKN method with r stages for the problem (1) is generally expressed by the formulas:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{l=1}^{r} b_l f(x_n + c_l h, Y_l), \tag{2}$$

$$y'_{n+1} = y'_n + h \sum_{l=1}^{r} d_l f(x_n + c_l h, Y_l), \tag{3}$$

$$Y_l = y_n + c_l hy'_n + h^2 \sum_{j=1}^{l-1} a_{lj} f(x_n + c_j h, Y_j), \quad l = 1, \ldots, r, \tag{4}$$

where y_{n+1} and y'_{n+1} denote approximations for $y(x_{n+1})$ and $y'(x_{n+1})$ respectively, and the grid points on the integration interval $[x_0, x_N]$ are given by $x_j = x_0 + jh,\ j = 0, 1, \ldots, N$, with h the fixed step-size considered. The above explicit method may be formulated compactly using the Butcher array in the form

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & d^T \end{array}$$

where $A = (a_{ij})_{r\times r}$ a lower triangular matrix of coefficients, $c = (c_1, c_2, \ldots, c_r)^T$ is the vector of stages, and $b = (b_1, b_2, \ldots, b_r)^T$, $d = (d_1, d_2, \ldots, d_r)^T$ are two vectors containing the remaining coefficients of the method. For short, this can be denoted as (c, A, b, d).

An $m(n)$ embedded-type pair of RKN methods comprises two such methods, one given by (c, A, b, d) with order m, and another one of order n $(n < m)$ given by $(c, A, \hat{b}, \hat{d})$ which shares the coefficients in c and A. The higher order method provides for each step an approximate solution y_{n+1}, y'_{n+1}, while a second approximate solution $\hat{y}_{n+1}$, $\hat{y}'_{n+1}$ is provided by the method of lower order. The purpose of the second approximation is to provide an estimate of the local truncation error. A RKN pair of embedded methods may be expressed using the Butcher array in the form

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & d^T \\ \hline & \hat{b}^T \\ & \hat{d}^T \end{array}$$

On the basis of the local error estimation provided by the embedding procedure, a variable step-size approach can be constructed. The local error estimate at $x_{n+1} = x_n + h$ is obtained through the differences between the two approximations of the solution and of the derivative, that is, $\eta_{n+1} = \hat{y}_{n+1} - y_{n+1}$ and $\eta'_{n+1} = \hat{y}'_{n+1} - y'_{n+1}$.

Let $\text{Est}_{n+1} = \max(\|\eta_{n+1}\|_\infty, \|\eta'_{n+1}\|_\infty)$ denote the local error estimate used to decide the step-length h_n on the $n+1$ iteration. In order to advance the solution of the problem in hand we use the step-length control strategy presented in [7]:

- if $Est_{n+1} < Tol/100$, then $h_{n+1} = 2h_n$,
- if $Tol/100 \leq Est_{n+1} < Tol$, then $h_{n+1} = h_n$,
- if $Est_{n+1} \geq Tol$, then take $h_{n+1} = h_n/2$ and redo the computations of the current step.

being Tol the prescribed tolerance.

Definition 1 ([2,18])**.** *An explicit Runge-Kutta-Nyström method as given in the Equations (2)–(4) is said to have algebraic order k if it holds*

$$\begin{cases} y(x_0 + h) - y_1 = O(h^{k+1}), \\ y'(x_0 + h) - y'_1 = O(h^{k+1}). \end{cases} \tag{5}$$

2.2. Analysis of Phase-Lag, Amplification Error and Stability

Applying the RKN method in (2)–(4) to the test equation $y'' = -w^2 y$, the phase-lag, amplification error, and the linear stability are derived. In particular, letting $\tilde{h} = -\mu^2$, $\mu = wh$, the approximate solution provided by (2)–(4) verifies the recurrence equation:

$$L_{n+1} = E(\tilde{h}) L_n,$$

where

$$L_{n+1} = \begin{bmatrix} y_{n+1} \\ hy'_{n+1} \end{bmatrix}, \; L_n = \begin{bmatrix} y_n \\ hy'_n \end{bmatrix}, \; E(\tilde{h}) = \begin{bmatrix} 1 + \tilde{h} b^T N^{-1} e & wh(1 + \tilde{h} b^T N^{-1} c) \\ -whd^T N^{-1} e & 1 + \tilde{h} d^T N^{-1} c \end{bmatrix},$$

$N = I - \tilde{h}A$, $A = (a_{ij})_{4\times 4}$, b, c, d are the corresponding matrix and vectors of coefficients, I is the identity matrix of order four, and $e = [1,1,1,1]^T$.

For sufficiently small values of $\mu = wh$, it is assumed that the matrix $E(\tilde{h})$ possesses complex conjugate eigenvalues [19]. Under this assumption, an oscillatory numerical solution is obtained, whose behavior depends on the eigenvalues of the stability matrix $E(\tilde{h})$. The characteristic equation of this matrix can be expressed as:

$$\lambda^2 - tr(E(\tilde{h}))\lambda + det(E(\tilde{h})) = 0. \tag{6}$$

Theorem 1 ([10])**.** *If we apply to the common test equation $y'' = -w^2 y$ the Runge–Kutta–Nyström scheme in (2)–(4), we get the formula for calculating the phase-lag directly (or dispersion error) $\Psi(\mu)$ given by:*

$$\Psi(\mu) = \mu - \arccos\left(\frac{tr(E(\tilde{h}))}{2\sqrt{det(E(\tilde{h}))}} \right). \tag{7}$$

If $\Psi(\mu) = O(\mu^{l+1})$, then the method is said to have phase-lag order l. For an explicit RKN method, $tr(E(\tilde{h}))$ and $det(E(\tilde{h}))$ are polynomials in μ (in case of an implicit RKN method these would be rational functions).

Definition 2. *An explicit Runge–Kutta–Nyström method as given in the Equations (2)–(4) is said to be phase-fitted, if the phase-lag is zero.*

Definition 3 ([10])**.** *For the Runge–Kutta–Nyström method given in the Equations (2)–(4), the value $\beta(\mu) = 1 - \sqrt{det(E(\tilde{h}))}$ is called the amplification error (or dissipative error). If $\beta(\mu) = O(\mu^{s+1})$, then the method is said to have amplification error of order s.*

Definition 4. *An explicit Runge–Kutta–Nyström method as given in the Equations (2)–(4) is said to be amplification-fitted if the amplification-error is zero.*

Definition 5. *An interval $(0, \tilde{h}_b)$, $\tilde{h}_b \in \Re^+ \cup \{+\infty\}$ is called an absolute stability interval of the method in (2)–(4) if $\tilde{h}$ is the highest value for which $|\lambda_{1,2}| < 1$, where $\lambda_{1,2}$ are the solutions of the Equation (6).*

3. Derivation of the New Embedded Pair

In this section, we will obtain a new 5(3) pair of explicit phase-fitted and amplification-fitted embedded RKN methods based on the RKN5(3) embedded pair derived by Van de Vyver in [2]. The coefficients of the RKN pair in [2] are shown in Table 1.

Table 1. The RKN5(3) Method in [2].

0				
$\frac{1}{5}$	$\frac{1}{50}$			
$\frac{2}{3}$	$-\frac{1}{27}$	$\frac{7}{27}$		
1	$\frac{3}{10}$	$-\frac{2}{35}$	$\frac{9}{35}$	
	$\frac{1}{24}$	$\frac{25}{84}$	$\frac{9}{56}$	0
	$\frac{1}{24}$	$\frac{125}{336}$	$\frac{27}{56}$	$\frac{5}{48}$
	$-\frac{5}{24}$	$\frac{125}{168}$	$-\frac{9}{56}$	$\frac{1}{8}$
	$-\frac{1}{12}$	$\frac{25}{42}$	$\frac{9}{28}$	$\frac{1}{6}$

In order to get the new adapted pair we first consider the coefficients of the third-order scheme in the RKN5(3) pair. Equating to zero the phase-lag $\Psi(\mu)$ and the amplification error $\beta(\mu)$ we get the system:

$$\begin{cases} \Psi(\mu) = 0 \\ \beta(\mu) = 0. \end{cases} \tag{8}$$

We solve this system considering the coefficients in Table 1 except two of them, which are taken as unknowns. Specifically, we take $\hat{b}_2$ and $\hat{b}_3$ as unknowns. We obtain the following values:

$$\hat{b}_2 = \frac{P(\mu)}{840\mu^4(405,000 - 85,500\,\mu^2 + 7455\,\mu^4 - 288\,\mu^6 + 4\,\mu^8)},$$

$$\hat{b}_3 = \frac{3Q(\mu)}{280\mu^4(405,000 - 85,500\,\mu^2 + 7455\,\mu^4 - 288\,\mu^6 + 4\,\mu^8)},$$

where

$$
\begin{aligned}
P(\mu) \;=\; & \Big(186\,\mu^{14} - 9157\,\mu^{12} + 11{,}175\,\mu^{10} + 3{,}355{,}275\,\mu^{8} - 57{,}739{,}500\,\mu^{6} \\
& -2{,}232{,}000\,\mu^{6}\cos(\mu) + 349{,}515{,}000\,\mu^{4} + 4{,}860{,}000\,\mu^{4}\cos(\mu) \\
& -810{,}000{,}000\,\mu^{2} + 81{,}000{,}000\,\mu^{2}\cos(\mu) + 1{,}458{,}000{,}000 \\
& -1{,}458{,}000{,}000\,\cos(\mu)\Big),
\end{aligned}
$$

$$
\begin{aligned}
Q(\mu) \;=\; & \Big(-162{,}000{,}000 + 132{,}840{,}000\,\mu^{2} - 41{,}985{,}000\,\mu^{4} + 5{,}890{,}500\,\mu^{6} \\
& -511{,}245\,\mu^{8} + 27{,}540\,\mu^{10} - 934\,\mu^{12} + 12\,\mu^{14} - 51{,}840{,}000\,\mu^{2}\cos(\mu) \\
& +3{,}240{,}000\,\mu^{4}\cos(\mu) + 162{,}000{,}000\,\cos(\mu) - 144{,}000\,\mu^{6}\cos(\mu)\Big).
\end{aligned}
$$

The corresponding Taylor series expansions in powers of μ are given by

$$
\begin{aligned}
\hat{b}_2 \;=\; & \frac{125}{168} - \frac{11}{1050}\mu^2 - \frac{613}{235{,}200}\mu^4 - \frac{129{,}473}{1{,}587{,}600{,}000}\mu^6 - \frac{61{,}471}{39{,}293{,}100{,}000}\mu^8 \\
& - \frac{25{,}533{,}945{,}907}{128{,}724{,}195{,}600{,}000{,}000}\mu^{10} - \frac{161{,}252{,}152{,}727}{3{,}707{,}256{,}833{,}280{,}000{,}000}\mu^{12} + \cdots \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\hat{b}_3 \;=\; & -\frac{9}{56} + \frac{17}{1400}\mu^2 - \frac{9101}{17{,}640{,}000}\mu^4 - \frac{12{,}353}{396{,}900{,}000}\mu^6 - \frac{29{,}522{,}267}{7{,}858{,}620{,}000{,}000}\mu^8 \\
& - \frac{65{,}718{,}361{,}603}{128{,}724{,}195{,}600{,}000{,}000}\mu^{10} - \frac{25{,}219{,}254{,}688{,}909}{463{,}407{,}104{,}160{,}000{,}000{,}000}\mu^{12} + \cdots \quad .
\end{aligned}
$$

As $\mu \to 0$, the newly obtained coefficients $\hat{b}_2, \hat{b}_3$ become the coefficients of the counterpart scheme in the original pair.

Similarly, if we take the coefficients of the fifth-order scheme in the RKN5(3) pair, except b_1 and b_2, which are taken as unknowns in the Equation (8), the solution of this system results in

$$
\begin{aligned}
b_1 &= -\frac{R(\mu)}{360\mu^4(1200\,\mu^2 - 60\,\mu^4 - 7200 + \mu^6)}, \\
b_2 &= \frac{S(\mu)}{252\mu^4(1200\,\mu^2 - 60\,\mu^4 - 7200 + \mu^6)},
\end{aligned}
$$

where

$$
\begin{aligned}
R(\mu) \;=\; & \Big(25{,}920{,}000 + 38{,}160\,\mu^{8} - 1161\,\mu^{10} + 16\,\mu^{12} + 5{,}810{,}400\,\mu^{4} \\
& -475{,}200\,\mu^{4}\cos(\mu) + 8{,}294{,}400\,\mu^{2}\cos(\mu) + 14{,}400\,\mu^{6}\cos(\mu) \\
& -651{,}600\,\mu^{6} - 25{,}920{,}000\,\cos(\mu) - 21{,}254{,}400\,\mu^{2}\Big), \\
S(\mu) \;=\; & \Big(5{,}508{,}000\,\mu^{4} - 18{,}144{,}000\,\mu^{2} - 716{,}400\,\mu^{6} + 50{,}310\,\mu^{8} \\
& -1815\,\mu^{10} + 28\,\mu^{12} - 756{,}000\,\mu^{4}\cos(\mu) + 18{,}144{,}000 \\
& +25{,}200\,\mu^{6}\cos(\mu) - 18{,}144{,}000\,\cos(\mu) + 9{,}072{,}000\,\mu^{2}\cos(\mu)\Big).
\end{aligned}
$$

The corresponding Taylor series expansions in powers of μ of the above coefficients are given by

$$\begin{aligned} b_1 &= \frac{1}{24} - \frac{37}{50,400}\mu^4 + \frac{17}{4,536,000}\mu^6 - \frac{1}{3,991,680}\mu^8 \\ &\quad + \frac{4393}{217,945,728,000}\mu^{10} + \frac{1949}{485,222,400,000}\mu^{12} + \cdots \quad , \\ b_2 &= \frac{25}{84} + \frac{13}{10,080}\mu^4 - \frac{1}{36,288}\mu^6 + \frac{1}{3,991,680}\mu^8 \\ &\quad - \frac{4211}{217,945,728,000}\mu^{10} - \frac{45,823}{10,461,394,944,000}\mu^{12} + \cdots \quad . \end{aligned}$$

As $\mu \to 0$, the newly obtained coefficients b_1, b_2 in the fifth-order adapted scheme become those of the counterpart scheme in the original pair.

The new adapted RKN pair will be named as PFAFRKN5(3).

3.1. Order of Convergence

This section is devoted to presenting the local truncation errors of the proposed methods and to get the algebraic orders of convergence. This is accomplished by using the usual tool of Taylor expansions. The local truncation errors (LTE) at the point x_{n+1} of the solution and the first derivative are given respectively by:

$$\begin{aligned} LTE &= y(x_0 + h) - y_1, \\ LTE_{der} &= y'(x_0 + h) - y'_1. \end{aligned} \tag{9}$$

Proposition 1. *For the lower order method, the corresponding LTEs are:*

$$\begin{aligned} LTE &= -\tfrac{h^4}{600}(w^2 y'') + O(h^5), \\ LTE_{der} &= \tfrac{h^4}{24}(f_{xxx} + 3y' f_{yxx} + 3y'' f_{xy} + 3(y')^2 f_{xyy} + 3y' f_{yy} y'' \\ &\quad + (y')^3 f_{yyy} + f_y f_x + (f_y)^2 y') + O(h^5), \end{aligned} \tag{10}$$

where the functions in the right hand sides are evaluated at x_0.

Proposition 2. *For the higher order method, we have:*

$$\begin{aligned} LTE &= \frac{h^6}{21600}(4(y')^3 f_{xyyy} + 3(y'')^2 f_{yy} + 6y'' f_{yxx} + 6(y')^2 f_{xxyy} + (y')^4 f_{yyyy} \\ &\quad + 4y' f_{xxxy} + 12 f_y f_{xx} + 12(f_y)^2 y'' + 6(y')^2 f_{yyy} y'' + 12(y')^2 f_{yy} f_y + 12y' f_{xyy} y'' \\ &\quad + 24 f_y y' f_{xy} + f_{xxxx} - 12w^4 y'') + O(h^7), \\ LTE_{der} &= \frac{h^6}{720}(f_{xxxxx} + 18y' f_{yy} f_y y'' + 15(y'')^2 f_{xyy} + 10y'' f_{xxxy} + 10y'(f_{xy})^2 + (f_y)^2 f_x \\ &\quad + 5 f_{xx} f_{xy} + f_y f_{xxx} + 5(y')^4 f_{xyyyy} + 5y' f_{xxxxy} + 10 f_{yxx} f_x + 10(y')^2 f_{xxxyy} \\ &\quad + 10(y')^3 f_{xxyyy} + 5(y')^3 (f_{yy})^2 + (y')^5 f_{yyyyy} + 15y' f_{yyy}(y'')^2 + 11(y')^3 f_{yyy} f_y \\ &\quad + 30y' f_{xxyy} y'' + 30(y')^2 f_{xyyy} y'' + 8 f_y y'' f_{xy} + 10y'' f_{yy} f_x + 10(y')^3 f_{yyyy} y'' \\ &\quad + 10(y')^2 f_{yyy} f_x + 23(y')^2 f_y f_{xyy} + 15(y')^2 f_{yy} f_{xy} + 20y' f_{xyy} f_x + 13 f_y y' f_{yxx} \\ &\quad + (f_y)^3 y' + 5y' f_{yy} f_{xx}) + O(h^7), \end{aligned}$$

where the functions in the right hand sides are evaluated at x_0.

To effectively determine the order of the proposed method, we have checked the order conditions as given in [2]. We obtained that the lower order method has algebraic order three and the higher order method has algebraic order five, thus resulting in a 5(3) RKN pair.

3.2. Absolute Stability Intervals of the New Adapted Pair

Proposition 3. *The third-order method of the PFAFRKN5(3) pair has* $(0, 23.83)$ *as interval of absolute stability and the fifth-order scheme has the absolute stability interval* $(0, 20.65)$.

Using the Maple package, from the definition in (5), the above results can be readily obtained.

4. Numerical Examples

To demonstrate the performance of the new pair, we have considered other 5(3) RKN pairs that appear in the literature to be used for the numerical comparisons:

- PFAFRKN5(3): The adapted RKN embedded pair developed in this paper,
- RKN5(3): An explicit 5(3) RKN pair presented by Van de Vyver in [2],
- ARKN5(3): An explicit 5(3) adapted RKN pair given by Franco in [3],
- EFRKN5(3): An embedded exponentially-fitted explicit RKN pair presented by Van de Vyver in [6],
- EEERKN5(3): An embedded exponentially-fitted explicit RKN method derived by Demba et al. in [17].

We will consider different oscillatory problems appeared in the literature to test the performance of the above methods:

Problem 1. *(Almost Periodic Problem in [20])*

$$\begin{aligned} y_1'' &= -y_1 + \epsilon \cos(\Psi x), y_1(0) = 1, y_1'(0) = 0, \\ y_2'' &= -y_2 + \epsilon \sin(\Psi x), y_2(0) = 0, y_2'(0) = 1,\ x \in [0, 100], \end{aligned}$$

whose exact solution is

$$y_1(x) = \frac{(1-\epsilon-\Psi^2)}{(1-\Psi^2)} \cos(x) + \frac{\epsilon}{(1-\Psi^2)} \cos(\Psi x),$$

$$y_2(x) = \frac{(1-\epsilon\Psi-\Psi^2)}{(1-\Psi^2)} \sin(x) + \frac{\epsilon}{(1-\Psi^2)} \sin(\Psi x).$$

For the numerical computations we have taken $\epsilon = 0.001$ and $\Psi = 0.1$.
To use the adapted methods we have taken the parameter value $w = 1$.

Problem 2. *(Two Body Problem in [21])*

$$\begin{aligned} y_1{}'' &= -\frac{y_1}{(y_1{}^2 + y_2{}^2)^{\frac{3}{2}}}, y_1(0) = 1, y_1{}'(0) = 0, \\ y_2{}'' &= -\frac{y_2}{(y_1{}^2 + y_2{}^2)^{\frac{3}{2}}}, y_2(0) = 0, y_2{}'(0) = 1,\ x \in [0, 100], \end{aligned}$$

whose exact solution is

$$y_1(x) = \cos x,$$

$$y_2(x) = \sin x.$$

To apply the adapted methods we have taken the value $w = 1$.

Problem 3. *(Nonlinear Problem in [22])*

$$y'' + y + y^3 = B\cos(\Omega x),\ y(0) = 0.20042672806900,\ y'(0) = 0,\ x \in [0, 100].$$

with $B = 0.002$ *and* $\Omega = 1.01$.

The reference solution is

$$\begin{aligned} y(x) &= 0.200179477536\cos(\Omega x) + 0.246946143 \times 10^{-3}\cos(3\Omega x) \\ &\quad +0.304016 \times 10^{-6}\cos(5\Omega x) + 0.374 \times 10^{-9}\cos(7\Omega x), \end{aligned}$$

which is an accurate approximate solution given in [22].

Now we take $w = \Omega$ *to apply our method and the ones in [3,6,17].*

Problem 4. *(Non-homogeneous System) in [8]*

$$\begin{aligned} {y_1}'' &= -v^2 y_1(x) + v^2 f(x) + f''(x),\ y_1(0) = a + f(0),\ {y_1}'(0) = f'(0), \\ {y_2}'' &= -v^2 y_2(x) + v^2 f(x) + f''(x),\ y_2(0) = f(0),\ {y_2}'(0) = av + f'(0), \end{aligned}$$

whose exact solution is

$$\begin{aligned} y_1(x) &= a\cos(vx) + f(x), \\ y_2(x) &= a\sin(vx) + f(x). \end{aligned}$$

In the numerical computations we have taken the values $v = 4$, $a = 0.1$, *and* $f(x) = e^{-10x}$.

We have solved this problem on the interval $[0, 100]$ *taking the value of the fitting parameter* $w = 4$.

Problem 5. *(Linear Problem in [23])*

$$y'' + y = 2\Omega\cos(x),\ y(0) = 1,\ y'(0) = 0,\ x \in [0, 100].$$

with $\Omega = 10^{-6}$.

The exact solution is

$$y(x) = \cos(x) + \Omega x\sin(x).$$

Now we take $w = 1$ *to apply our method and the ones in [3,6,17].*

Discussion

The numerical data are given in Tables 2–6, considering different tolerances. The tables contain the number of steps, NSTEP; the number of function evaluations, NFE; the number of rejected steps, RSTEP; the maximum absolute errors, MAXER, and the computational time in seconds.

Table 2. Numerical data corresponding to Problem 1.

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
	PFAFRKN5(3)	122	488	0	1.841691(−5)	0.036
10^{-2}	RKN5(3)	122	488	0	1.078825(−2)	0.054
	ARKN5(3)	242	968	0	9.806283(−1)	0.066
	EFRKN5(3)	242	968	0	2.139264(−3)	0.051
	EEERKN5(3)	122	488	0	2.591319(−3)	0.040
	PFAFRKN5(3)	522	2088	0	2.989273(−9)	0.037
10^{-4}	RKN5(3)	522	2088	0	7.172465(−6)	0.046
	ARKN5(3)	1044	4179	1	6.403390(−2)	0.049
	EFRKN5(3)	522	2088	0	3.930696(−4)	0.047
	EEERKN5(3)	522	2088	0	4.299671(−7)	0.047
	PFAFRKN5(3)	1123	4492	0	2.983191(−11)	0.044
10^{-6}	RKN5(3)	1123	4492	0	1.542823(−7)	0.051
	ARKN5(3)	4491	17,970	2	3.458063(−3)	0.078
	EFRKN5(3)	2246	8987	1	2.054474(−5)	0.062
	EEERKN5(3)	1123	4492	0	4.355510(−9)	0.047

Table 2. *Cont.*

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
	PFAFRKN5(3)	2420	9680	0	3.727366(−12)	0.054
10^{-8}	RKN5(3)	2420	9680	0	3.324929(−9)	0.062
	ARKN5(3)	19,347	77,397	3	1.862032(−4)	0.115
	EFRKN5(3)	19,348	77,404	4	2.760600(−7)	0.189
	EEERKN5(3)	2420	9680	0	4.516099(−11)	0.062
	PFAFRKN5(3)	10,421	41,687	1	1.644109(−11)	0.084
10^{-10}	RKN5(3)	10,421	41,687	1	1.658362(−11)	0.091
	ARKN5(3)	83,362	333,460	4	1.002430(−5)	0.334
	EFRKN5(3)	83,364	333,477	7	1.501274(−8)	0.618
	EEERKN5(3)	10,421	41,694	2	1.643935(−11)	0.129

Table 3. Numerical data corresponding to Problem 2.

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
	PFAFRKN5(3)	122	488	0	5.205025(−2)	0.038
10^{-2}	RKN5(3)	122	488	0	8.478978(−1)	0.071
	ARKN5(3)	270	1083	1	1.804551(+0)	0.042
	EFRKN5(3)	242	968	0	5.268747(−2)	0.059
	EEERKN5(3)	122	488	0	1.227156(−1)	0.047
	PFAFRKN5(3)	522	2088	0	3.035691(−6)	0.040
10^{-4}	RKN5(3)	522	2088	0	6.990118(−4)	0.041
	ARKN5(3)	1044	4179	1	1.480069(−1)	0.041
	EFRKN5(3)	522	2088	0	4.558690(−3)	0.046
	EEERKN5(3)	522	2088	0	3.621045(−5)	0.041
	PFAFRKN5(3)	1123	4492	0	3.943697(−8)	0.050
10^{-6}	RKN5(3)	1123	4492	0	1.520229(−5)	0.055
	ARKN5(3)	4491	17,970	2	5.473843(−3)	0.059
	EFRKN5(3)	2246	8987	1	8.448674(−5)	0.064
	EEERKN5(3)	1123	4492	0	3.722093(−7)	0.053
	PFAFRKN5(3)	2420	9680	0	5.728311(−10)	0.051
10^{-8}	RKN5(3)	2420	9680	0	3.282692(−7)	0.056
	ARKN5(3)	19,347	77,397	3	4.588825(−4)	0.109
	EFRKN5(3)	19,348	77,404	4	6.160820(−7)	0.160
	EEERKN5(3)	2420	9680	0	3.718493(−9)	0.067
	PFAFRKN5(3)	10,421	41,687	1	1.672787(−11)	0.075
10^{-10}	RKN5(3)	10,421	41,687	1	2.058225(−10)	0.075
	ARKN5(3)	83,362	333,460	4	2.680717(−5)	0.265
	EFRKN5(3)	83,364	333,477	7	3.050323(−8)	0.532
	EEERKN5(3)	10,421	41,694	2	1.717850(−11)	0.103

Table 4. Numerical data corresponding to Problem 3.

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
	PFAFRKN5(3)	122	488	0	4.384567(−5)	0.039
10^{-2}	RKN5(3)	122	536	16	2.777502(−3)	0.080
	ARKN5(3)	123	504	4	2.849535(−1)	0.043
	EFRKN5(3)	122	512	8	3.964776(−3)	0.050
	EEERKN5(3)	122	515	9	1.170545(−3)	0.048
	PFAFRKN5(3)	262	1048	0	9.677039(−7)	0.040
10^{-4}	RKN5(3)	262	1075	9	7.208088(−5)	0.041
	ARKN5(3)	510	2076	12	4.321049(−2)	0.058
	EFRKN5(3)	267	1077	3	2.048732(−4)	0.049
	EEERKN5(3)	262	1072	8	1.356514(−5)	0.073

Table 4. *Cont.*

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
	PFAFRKN5(3)	1123	4492	0	1.011154(−9)	0.047
10^{-6}	RKN5(3)	562	2260	4	1.659456(−6)	0.039
	ARKN5(3)	2085	8439	33	1.815951(−3)	0.058
	EFRKN5(3)	995	4076	32	1.381514(−5)	0.050
	EEERKN5(3)	573	2337	15	9.010356(−8)	0.045
	PFAFRKN5(3)	2420	9680	0	2.150155(−11)	0.046
10^{-8}	RKN5(3)	2324	9392	32	7.596427(−9)	0.046
	ARKN5(3)	9091	36,487	41	1.005629(−4)	0.080
	EFRKN5(3)	4485	18,048	36	5.225435(−7)	0.066
	EEERKN5(3)	1959	7932	32	9.751267(−104)	0.049
	PFAFRKN5(3)	5211	20,844	0	3.124598(−12)	0.061
10^{-10}	RKN5(3)	5183	20,816	28	4.524522(−11)	0.062
	ARKN5(3)	39,556	158,425	67	5.609382(−6)	0.180
	EFRKN5(3)	20,185	80,896	52	3.249923(−8)	0.180
	EEERKN5(3)	5213	21,140	96	4.741679(−12)	0.079

Table 5. Numerical data corresponding to Problem 4.

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
	PFAFRKN5(3)	491	1973	3	7.410682(−4)	0.045
10^{-2}	RKN5(3)	491	1973	3	4.286625(−3)	0.070
	ARKN5(3)	966	3891	9	1.563857(−1)	0.132
	EFRKN5(3)	493	1987	5	7.363927(−3)	0.059
	EEERKN5(3)	491	1973	3	7.410682(−4)	0.048
	PFAFRKN5(3)	1062	4257	3	9.654792(−6)	0.045
10^{-4}	RKN5(3)	1062	4257	3	9.105814(−5)	0.061
	ARKN5(3)	4202	16,838	10	2.511836(−2)	0.072
	EFRKN5(3)	2100	8424	8	1.548196(−4)	0.077
	EEERKN5(3)	1076	4325	7	1.094766(−5)	0.059
	PFAFRKN5(3)	4551	18,216	4	2.691087(−8)	0.067
10^{-6}	RKN5(3)	4551	18,216	4	6.186462(−8)	0.069
	ARKN5(3)	18,104	72,449	11	1.370302(−3)	0.173
	EFRKN5(3)	9036	36,177	11	8.133685(−6)	0.133
	EEERKN5(3)	4568	18,302	10	1.985082(−9)	0.088
	PFAFRKN5(3)	9947	39,803	5	7.650380(−11)	0.127
10^{-8}	RKN5(3)	9947	39,803	5	1.325328(−9)	0.174
	ARKN5(3)	78,087	312,384	12	7.383390(−5)	0.542
	EFRKN5(3)	38,992	156,013	15	4.394977(−7)	0.419
	EEERKN5(3)	19,535	78,176	12	1.616837(−11)	0.273
	PFAFRKN5(3)	42,417	169,686	6	1.017235(−11)	0.340
10^{-10}	RKN5(3)	42,271	169,099	5	1.003746(−11)	0.247
	ARKN5(3)	338,631	1,354,563	13	3.970905(−6)	2.409
	EFRKN5(3)	168,206	672,878	18	2.383359(−8)	1.731
	EEERKN5(3)	42,443	169,814	14	1.000001(−11)	0.500

Table 6. Numerical data corresponding to Problem 5.

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
	PFAFRKN5(3)	122	488	0	1.489108(−5)	0.082
10^{-2}	RKN5(3)	122	506	6	1.054508(−2)	0.099
	ARKN5(3)	233	974	14	1.390402(+0)	0.102
	EFRKN5(3)	125	500	0	1.824478(−2)	0.104
	EEERKN5(3)	122	488	0	2.593892(−3)	0.094
	PFAFRKN5(3)	522	2088	0	2.390414(−10)	0.047
10^{-4}	RKN5(3)	263	1061	3	2.255454(−4)	0.094
	ARKN5(3)	863	3548	32	1.248772(−1)	0.169
	EFRKN5(3)	477	1974	22	7.272351(−4)	0.131
	EEERKN5(3)	265	1066	2	2.292836(−5)	0.050
	PFAFRKN5(3)	1123	4492	0	2.7644136(−12)	0.126
10^{-6}	RKN5(3)	1031	4217	31	1.686848(−6)	0.131
	ARKN5(3)	3900	15,735	45	4.686084(−3)	0.578
	EFRKN5(3)	1833	7428	32	3.976141(−5)	0.426
	EEERKN5(3)	1059	4332	10	5.131505(−8)	0.254
	PFAFRKN5(3)	2420	9680	0	3.734436(−12)	0.336
10^{-8}	RKN5(3)	2389	9649	31	8.609137(−9)	0.232
	ARKN5(3)	17,271	69,285	67	2.177597(−4)	2.561
	EFRKN5(3)	8586	34,479	45	1.453551(−6)	2.477
	EEERKN5(3)	2449	9994	66	9.596512(−11)	0.617
	PFAFRKN5(3)	10,421	41,687	1	1.645062(−11)	1.482
10^{-10}	RKN5(3)	9577	38,467	53	2.599720(−11)	0.628
	ARKN5(3)	76,283	305,345	71	1.081975(−5)	12.256
	EFRKN5(3)	37,547	150,413	75	6.953047(−8)	11.711
	EEERKN5(3)	9862	39,670	74	1.510088(−11)	2.442

To better show the efficiency of the developed PFAFRKN5(3) pair, we present in Figures 1–5 the efficiency curves for the considered problems. It can be observed that the good behavior of the new pair for tolerances Tol $= 1/10^{2k}$, with $k = 1, 2, 3, 4$.

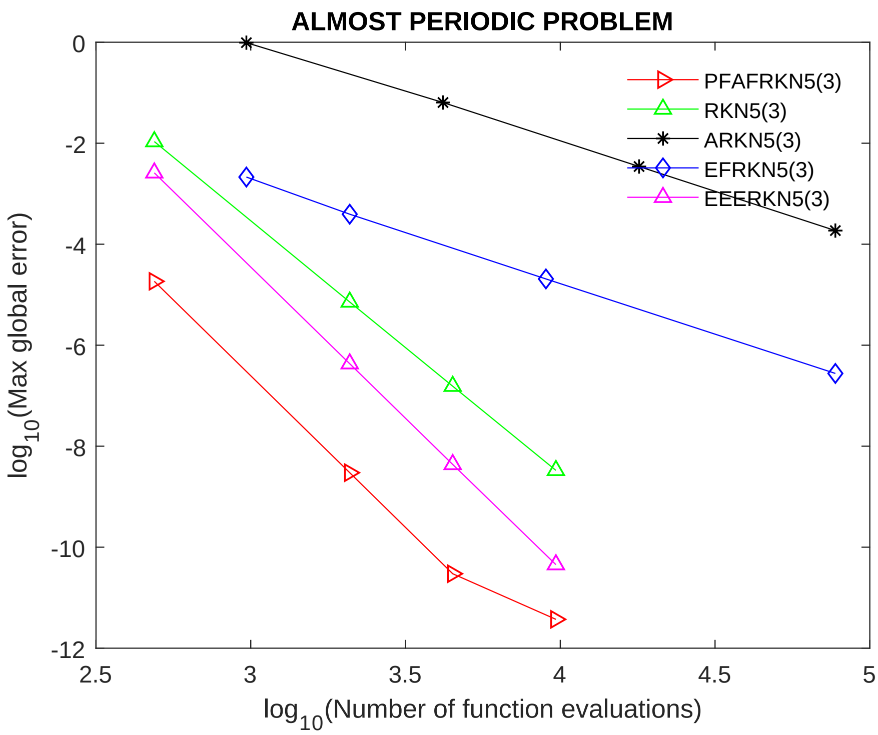

Figure 1. Efficiency curves corresponding to Problem 1.

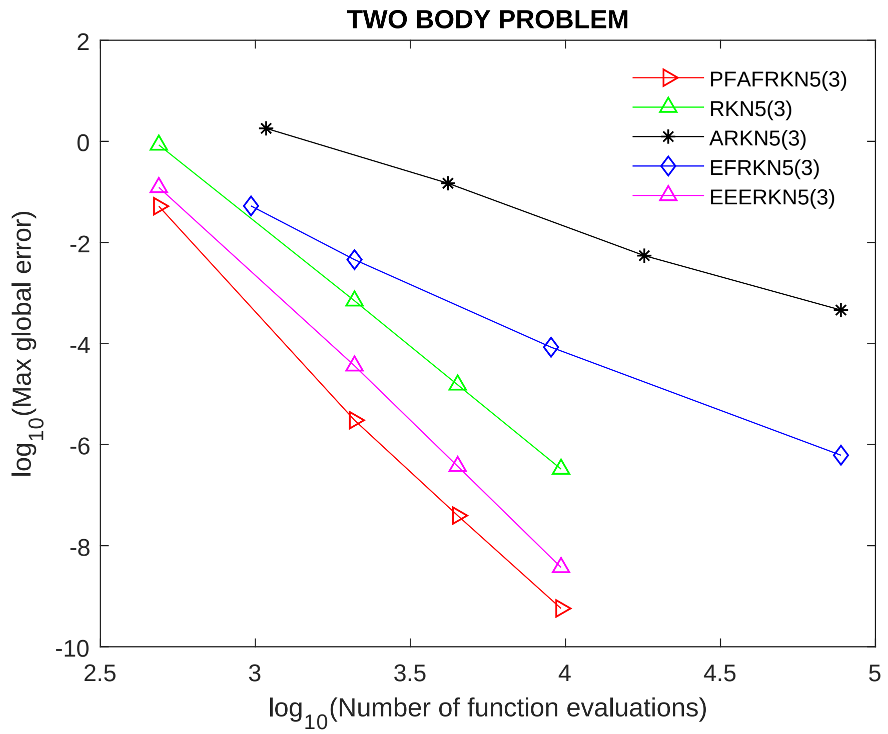

Figure 2. Efficiency curves corresponding to Problem 2.

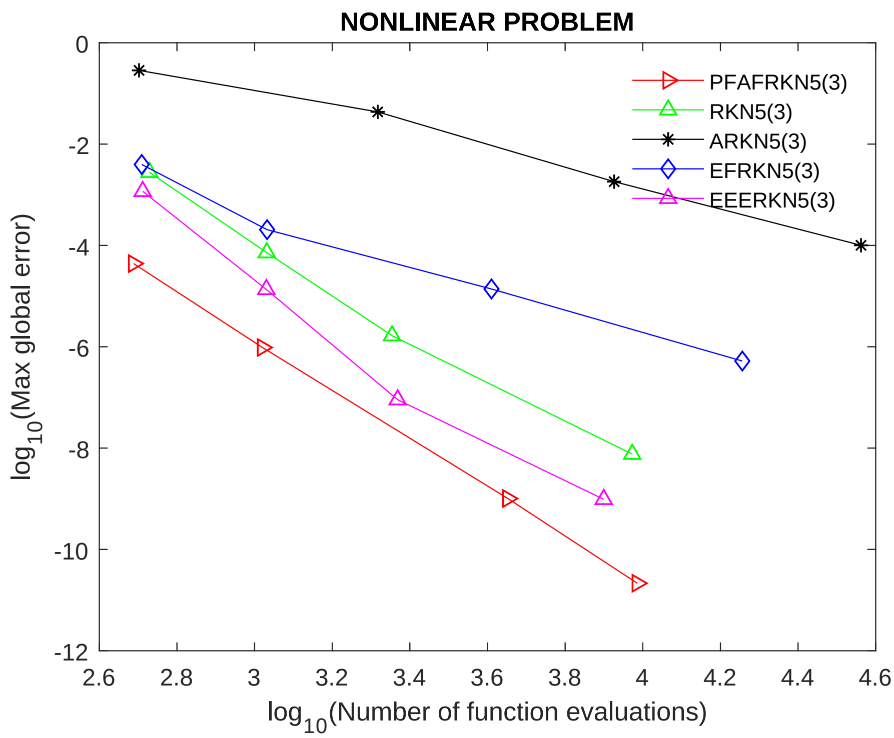

Figure 3. Efficiency curves corresponding to Problem 3.

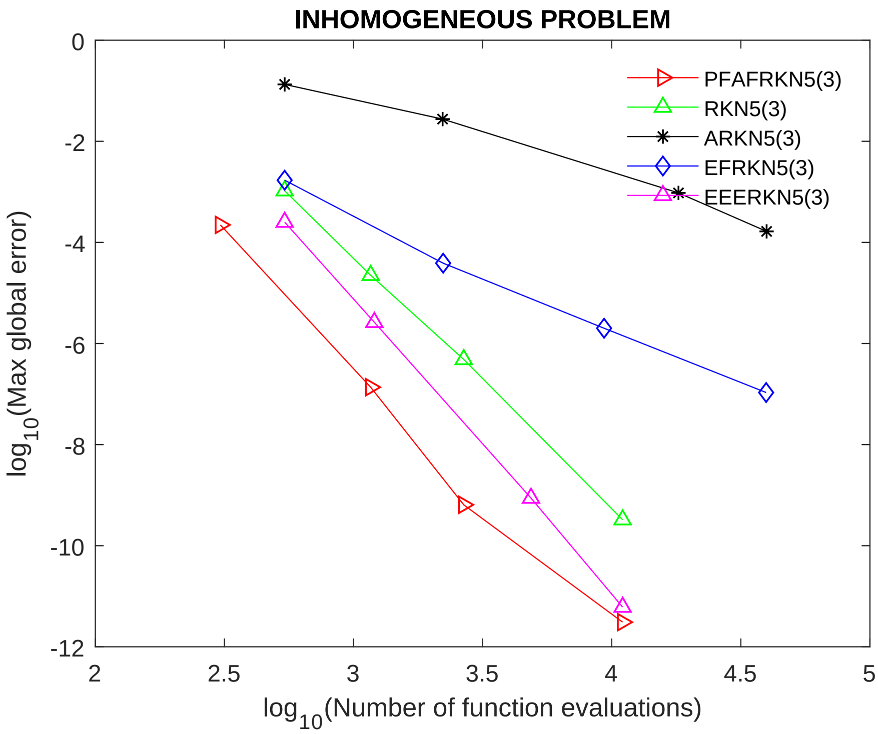

Figure 4. Efficiency curves corresponding to Problem 4.

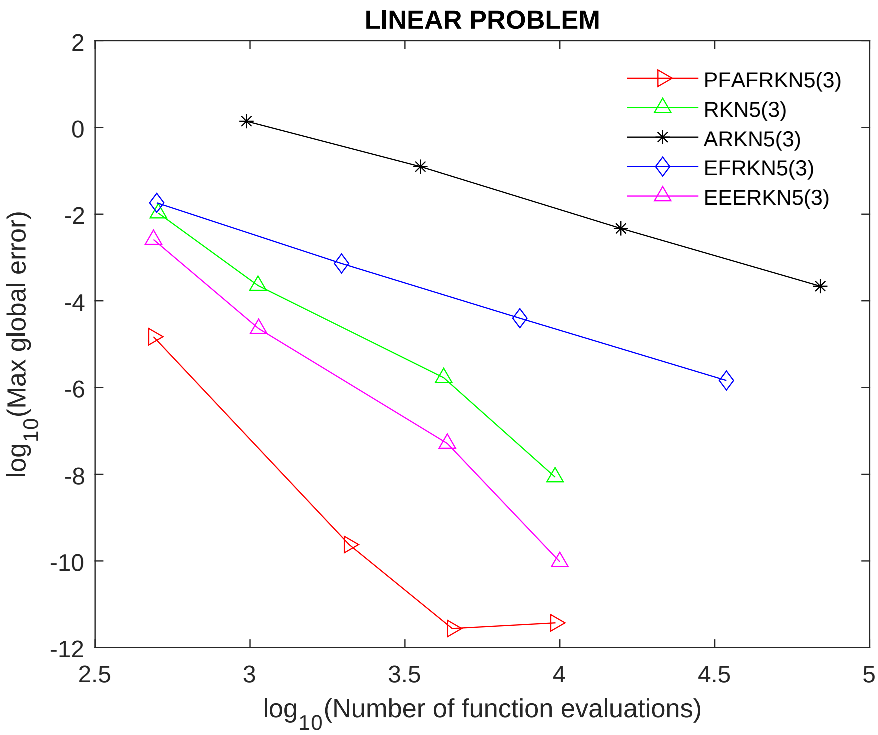

Figure 5. Efficiency curves corresponding to Problem 5.

The new pair PFAFRKN5(3) gives minimum error norm, minimum number of function evaluations and minimum computational cost. Tables 2–6 and Figures 1–5 present evidence that PFAFRKN5(3) is a very efficient scheme. Therefore, we can say that PFAFRKN5(3) is more appropriate for solving the type of problems in (1) than other existing embedded 5(3) pairs of RKN methods with four stages in the literature.

5. Conclusions

In this study, we have used the methodology for constructing phase-fitted and amplification-fitted methods to develop a new efficient explicit phase-fitted and amplification-fitted embedded RKN pair based on the 5(3) RKN pair of Van de Vyver in [2]. The newly developed pair has four variable coefficients depending on the parameter $\mu = wh$, which is usually known as the parameter frequency [24,25]. We computed the local truncation error for both the higher and lower order methods in the new pair PFAFRKN5(3), confirming that the algebraic orders of convergence of the underlying pair are maintained. In addition, the stability intervals for both the higher and lower order methods have been obtained. The numerical results obtained clearly show that PFAFRKN5(3) is more accurate and efficient than other 5(3) RKN pairs in the literature.

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