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On a Special Weighted Version of the Odd Weibull-Generated Class of Distributions

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Abstract: In recent advances in distribution theory, the Weibull distribution has often been used to generate new classes of univariate continuous distributions. They find many applications in important disciplines such as medicine, biology, engineering, economics, informatics, and finance; their usefulness is synonymous with success. In this study, a new Weibull-generated-type class is presented, called the weighted odd Weibull generated class. Its definition is based on a cumulative distribution function, which combines a specific weighted odd function with the cumulative distribution function of the Weibull distribution. This weighted function was chosen to make the new class a real alternative in the first-order stochastic sense to two of the most famous existing Weibull generated classes: the Weibull-G and Weibull-H classes. Its mathematical properties are provided, leading to the study of various probabilistic functions and measures of interest. In a consequent part of the study, the focus is on a special three-parameter survival distribution of the new class defined with the standard exponential distribution as a reference. The exploratory analysis reveals a high level of adaptability of the corresponding probability density and hazard rate functions; the curves of the probability density function can be decreasing, reversed N shaped, and unimodal with heterogeneous skewness and tail weight properties, and the curves of the hazard rate function demonstrate increasing, decreasing, almost constant, and bathtub shapes. These qualities are often required for diverse data fitting purposes. In light of the above, the corresponding data fitting methodology has been developed; we estimate the model parameters via the likelihood function maximization method, the efficiency of which is proven by a detailed simulation study. Then, the new model is applied to engineering and environmental data, surpassing several generalizations or extensions of the exponential model, including some derived from established Weibull-generated classes; the Weibull-G and Weibull-H classes are considered. Standard criteria give credit to the proposed model; for the considered data, it is considered the best.

Keywords: Weibull distribution; general class of distributions; statistical model; stochastic ordering; moments; real data analysis

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1. Introduction

This section presents the background, motivations, contributions, and structure of the study.

1.1. Background

General classes of univariate continuous distributions are involved in various facets of statistical modeling. They offer solutions to practitioners who wish to understand and best explain their subject of study from the data observed. In fact, most of these general classes are based on well-known distributions serving as generators. Basic works and recent developments on this topic are discussed in [1–3]. Among the most interesting classes in terms of heterogeneous distributions, are the Weibull-generated-type classes based on the standard Weibull distribution. Formally, they are identified by a cumulative distribution function (cdf) of the following form:

$$F_H(x) = 1 - \exp\left[-\alpha H(x)^\beta\right], \quad x \in (a, b), \tag{1}$$

$F_H(x) = 0$ for $x \le a$, and $F_H(x) = 1$ for $x \ge b$, where $\alpha > 0$ is a scale parameter, $\beta > 0$ is a shape parameter, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in (a, +\infty) \cup \{+\infty\}$, and $H(x)$ is a non-negative monotonically increasing function over (a, b) satisfying $\lim_{x \to a} H(x) = 0$ and $\lim_{x \to b} H(x) = +\infty$. One can notice that $F_H(x) = F_W[H(x)]$, where $F_W(x)$ denotes the cdf of the Weibull distribution with scale parameter α and shape parameter β, showing the role of the Weibull distribution in the definition of the class. Clearly, for each function $H(x)$, we can define a Weibull-generated-type class; the choice of $H(x)$ must be motivated by some theoretical or practical interest. Among the existing Weibull-generated-type classes, there is the Weibull-X class proposed by [4], the Weibull-G class studied by [5], the modified odd Weibull-G (MOW-G) class established by [6], and the new Weibull-X class developed by [7]. On the other hand, more sophisticated extensions of Weibull-generated-type classes include the extended Weibull-generated class proposed by [8], the odd flexible Weibull-H class introduced by [9], the new Weibull-G class developed by [10], the transmuted Weibull-G class established by [11], the generalized odd Weibull-G class constructed by [12], and the flexible Weibull-G class proposed by [13]. Most of these extensions consider an extended Weibull cdf as generator, or a mathematical transformations of the cdf of the Weibull-G class, with the use of one or more parameters. For the use of the Weibull-G class for bivariate modeling, we refer to [14]. A discrete analogue of the the odd Weibull-G class was developed in [15].

The motivation for our study is connected with the classical Weibull-X, Weibull-G, and modified odd Weibull-G classes. A retrospective on these classes is discussed below. When $H(x)$ is determined by the following logarithmic function:

$$H(x) = H_1(x) = -\ln(1 - G(x)),$$

where $G(x)$ denotes the cdf of an arbitrary or targeted reference continuous distribution with support on (a, b), then the cdf given as Equation (2) defines the Weibull-X class by [4]. By considering the logistic distribution as reference, it is proved in [16] that the corresponding Weibull-X distribution is flexible enough to perfectly fit the glass fiber data set of [17]. In particular, the fit of the proposed Weibull logistic model outperforms the adjustment criteria of the logistic, skewed logistic, and skewed logistic "with location" models. Further, when $H(x)$ denotes the following "odd ratio" function:

$$H(x) = H_2(x) = \frac{G(x)}{1 - G(x)},$$

the cdf given as Equation (2) characterizes the Weibull-G class by [5]. The foundational work of [5] shows that the distributions of the Weibull-G class are applicable in various settings. In particular, the Weibull-G models defined with the exponential, log-logistic, and Burr XII models as references are appropriate for the adjustments of the glass fiber data set of [17] and the fatigue time data set of [18]. As a valuable indicator of its usefulness, the Weibull-G class has been cited in more than 400 references. On the other hand, when $H(x)$ denotes the following modified odd ratio function:

$$H(x) = H_3(x) = \frac{G(x)}{1 - G(x)(1 + G(x))/2},$$

the cdf given as Equation (2) defines the MOW-G class by [6]. The MOW-G class is illustrated with the reference distributions: gamma, Weibull, and Lindley distributions. The related models are proved to be efficient for fitting the guinea pig data set by [19], the fatigue time data set by [18], and the failure time data set by [20]. Thanks to their flexible properties, these models are preferable to several competitors based on the same reference models. A deep relationship exists between the Weibull-X, Weibull-G, and MOW-G classes. In particular, they are complementary in the following stochastic dominance sense: For any $x \in \mathbb{R}$, we have

$$F_{H_1}(x) \leq F_{H_3}(x) \leq F_{H_2}(x).$$

Thus, this hierarchical order indicates that, for a given reference distribution and data set, the objectives of the corresponding estimated cdfs of the classes differ somewhat; one estimated cdf may be more adequate than another relative to the empirical cdf of the data.

1.2. Motivations and Contributions

In this paper, we contribute to the subject by proposing a new motivated Weibull-generated-type class, called the weighted odd Weibull-generated class (WOW-G class for short), which has deep connections with the Weibull-X, Weibull-G, and modified odd Weibull-G classes. It respects the definition of the cdf given as Equation (2) with the following weighted odd ratio function:

$$H(x) = H_4(x) = \frac{G(x)}{1 - G(x)} w(x), \quad w(x) = 1 - \frac{1}{2} G(x).$$

Thus, $H_4(x)$ is a weighted version of $H_2(x)$; $H_4(x) = w(x) H_2(x)$, precisely. The WOW-G class is specified by the following cdf:

$$F_{H_4}(x) = 1 - \exp\left\{ -\alpha \left[\frac{G(x)}{1 - G(x)} \right]^{\beta} w(x)^{\beta} \right\}, \quad x \in (a, b), \tag{2}$$

$F_{H_4}(x) = 0$ for $x \leq a$, and $F_{H_4}(x) = 1$ for $x \geq b$. Here, we note specific motivations for considering the particular function $H_4(x)$:

(i) As a basic remark, in relation to $H_1(x)$, $H_2(x)$ and $H_3(x)$, the analytical complexity of $H_4(x)$ is quite acceptable, with the weight function $w(x)$ remaining simple.

(ii) Let us now underline the mathematical connections behind the functions $H_1(x)$, $H_2(x)$, $H_3(x)$, and $H_4(x)$. By virtue of the result in [21], the following logarithmic holds: $\ln(1 - y) \geq -y(1 - y/2)/(1 - y)$ for $y \in [0, 1)$, which implies that $H_1(x) \leq H_4(x)$. On the other side, for any $y \in [0, 1]$, we have $1 - y^2/4 < 1$, which is equivalent to $1 - y/2 < 2/(2 + y)$; so,

$$H_4(x) = \frac{G(x)}{1 - G(x)} w(x) \leq \frac{G(x)}{1 - G(x)} \frac{2}{2 + G(x)} = \frac{G(x)}{1 - G(x)(1 + G(x))/2} = H_3(x).$$

Therefore, the following chain of inequalities holds: $H_1(x) \leq H_4(x) \leq H_3(x) \leq H_2(x)$, implying the following first stochastic dominance result:

$$F_{H_1}(x) \leq F_{H_4}(x) \leq F_{H_3}(x) \leq F_{H_2}(x).$$

In this sense, the proposed WOW-G class can be seen as an alternative to the Weibull-X, Weibull-G, and MOW-G classes. This information is important enough to warrant an investigation of the WOW-G class.

(iii) In relation to the literature on distribution theory, we notice that $H_4(x)$ can be expressed as

$$H_4(x) = c[H_2(x) + \gamma G(x)],$$

with $c = 1/2$ and $\gamma = 1$. Therefore, the WOW-G class appears to be a subclass of the extended odd-G class championed by [22].

(iv) Last but not least, the inverse function of $H_4(x)$ is quite manageable; after some operations, we find

$$H_4^{-1}(y) = G^{-1}\left(y + 1 - \sqrt{y^2+1}\right), \quad y > 0, \tag{3}$$

where $G^{-1}(y)$ denotes the quantile function (qf) of the reference distribution. This implies that the qf of the WOW-G class has an analytical expression, which will be provided later.

In this study, we derive the theoretical and practical properties of the proposed WOW-G class. We emphasize a special distribution of the class based on the standard exponential distribution as a reference. Among its features, it possesses the following simple cumulative hazard rate function: $R(x) = \alpha[\sinh(\phi x)]^{\beta}$ for $x > 0$, with $\alpha > 0$, $\phi > 0$ and $\beta > 0$, where $\sinh(x)$ denotes the standard hyperbolic sine function: $\sinh(x) = (e^x - e^{-x})/2$. Thus defined, it constitutes a special three-parameter survival distribution that takes advantage of the structure of the WOW-G class to achieve a high level of flexibility. In particular, its probability density function (pdf) exhibits all the asymmetric qualities; the triplet "left skewed-almost symmetrical-right skewed" is observed, as well as various monotonic decreasing shapes and reversed N shapes. Its sister function in terms of modeling, the hazard rate function (hrf), presents increasing concave and convex shapes, decreasing, or almost constant shapes. This flexibility is also observed in diverse moment measures, such as the mean, variance, moment skewness, and moment kurtosis. This new survival distribution is thus adapted for the adjustment of versatile data sets. We illustrate this claim by considering engineering and environmental data, estimating the model parameters using the maximum likelihood (ML) method. By following the standards, we use established statistical tools to prove that the fit of the proposed model surpasses those of several extended exponential models. Statistical comparisons with those derived from the Weibull-X, Weibull-G, and MOW-G classes are discussed. Numerical tables and graphics support the findings.

1.3. Structure of the Paper

Section 2 formally presents the most useful functions of the WOW-G class, as well as the mentioned special distribution. In Section 3, we describe some relevant properties of the class. Statistical considerations are given in Section 4. The application to two practical data sets is given in Section 5. Summary and further research are posed in Section 6.

2. The WOW-G Class

Here, we refine the presentation of the WOW-G class and show what we can call its "distributional richness".

2.1. Presentation

We recall that the WOW-G class is defined by the cdf given as Equation (2), which will be denoted in the next section as $F(x) = F_{H_4}(x)$, to simplify the notations. From $F(x)$, we derive the pdf by differentiation according to x in the almost everywhere sense; the pdf of the WOW-G class is given as

$$f(x) = \frac{\alpha\beta}{2} g(x) \left[1 + \frac{1}{(1-G(x))^2}\right] \left[\frac{G(x)}{1-G(x)}\right]^{\beta-1} w(x)^{\beta-1} \exp\left\{-\alpha \left[\frac{G(x)}{1-G(x)}\right]^{\beta} w(x)^{\beta}\right\}, \tag{4}$$
$$x \in (a,b),$$

where $f(x) = 0$ for $x \notin (a, b)$. We recall that $g(x)$ refers to the pdf of the reference distribution. As an important reliability function, the survival function (sf) of the WOW-G class is given as

$$S(x) = 1 - F(x) = \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta} w(x)^{\beta}\right\}, \quad x \in (a, b),$$

$S(x) = 1$ for $x \leq a$, and $S(x) = 0$ for $x \geq b$.

We can also express the main hazard functions of the WOW-G class, defined by the cumulative hrf and hrf. Thus, the cumulative hrf is expressed as

$$R(x) = -\ln[1 - F(x)] = \alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta} w(x)^{\beta}, \quad x \in (a, b),$$

where $R(x) = 0$ for $x \leq a$, and $R(x) = +\infty$ for $x \geq b$, and the hrf is obtained by differentiation of $R(x)$ according to x in the almost everywhere sense:

$$r(x) = \frac{\alpha\beta}{2} g(x)\left[1 + \frac{1}{(1-G(x))^2}\right]\left[\frac{G(x)}{1-G(x)}\right]^{\beta-1} w(x)^{\beta-1}, \quad x \in (a, b),$$

and $r(x) = 0$ for $x \notin (a, b)$.

2.2. Some Examples

Based on the WOW-G class, diverse distributions can be defined for specific purposes. By considering reference distributions, some relevant ones are listed in Table 1.

Table 1. Special distributions of the WOW-G class defined from important distributions as reference.

WOW-G	**Reference**	**Domain**	$G(x)$	**Num. of Par.**	$S(x)$
WOWU	Uniform	$(0, b)$	$\frac{x}{b}$	3	$\exp\left[-\frac{\alpha}{2^{\beta} b^{\beta}}\left(\frac{x(2b-x)}{b-x}\right)^{\beta}\right]$
WOWTP	Topp–Leone	$(0, 1)$	$x^b(2-x)^b$	3	$\exp\left[-\frac{\alpha}{2^{\beta}}\left(\frac{x^b(2-x)^b[2-x^b(2-x)^b]}{1-x^b(2-x)^b}\right)^{\beta}\right]$
WOWE	Exponential	$(0, +\infty)$	$1 - e^{-\phi x}$	3	$\exp\left[-\alpha[\sinh(\phi x)]^{\beta}\right]$
WOWIE	Inverse exp.	$(0, +\infty)$	$e^{-\theta/x}$	3	$\exp\left[-\frac{\alpha}{2^{\beta}}\left(\frac{2-e^{-\theta/x}}{e^{\theta/x}-1}\right)^{\beta}\right]$
WOWW	Weibull	$(0, +\infty)$	$1 - e^{-\phi x^{\lambda}}$	4	$\exp\left[-\alpha[\sinh(\phi x^{\lambda})]^{\beta}\right]$
WOWLom	Lomax	$(0, +\infty)$	$1 - \left(1 + \frac{x}{\rho}\right)^{-\theta}$	4	$\exp\left[-\frac{\alpha}{2^{\beta}}\left(1+\frac{x}{\rho}\right)^{-\theta\beta}\left[\left(1+\frac{x}{\rho}\right)^{2\theta}-1\right]^{\beta}\right]$
WOWGu	Gumbel	$\mathbb{R}$	$\exp(-e^{-bx})$	3	$\exp\left[-\frac{\alpha}{2^{\beta}}\left(\frac{2-\exp(-e^{-bx})}{\exp(e^{-bx})-1}\right)^{\beta}\right]$
WOWLog	Logistic	$\mathbb{R}$	$(1 + e^{-bx})^{-1}$	3	$\exp\left[-\frac{\alpha}{2^{\beta}}\left(\frac{e^{bx}(e^{bx}+2)}{e^{bx}+1}\right)^{\beta}\right]$

All the distributions presented in Table 1 deserve an in-depth mathematical treatment to reveal their capacities in a modeling context. In this study, we focus on the WOWE distribution—on the one hand, because of the simplicity and originality of this sf, and on the other hand, because of the beautiful results observed in various statistical analyses, which will be presented later.

2.3. The WOWE Distribution

As a basis, the WOWE distribution is derived from the WOW-G class by choosing the exponential distribution as a reference. Here, the exponential distribution is specified by the following cdf and pdf: $G(x) = 1 - e^{-\phi x}$ for $x > 0$ and $G(x) = 0$ for $x \leq 0$, and $g(x) = \phi e^{-\phi x}$ for $x > 0$, and $g(x) = 0$ for $x \leq 0$, respectively, where $\phi > 0$ is a scale parameter. Let us now discuss the cdf of the WOWE distribution. Noticing that

$$H_4(x) = \frac{G(x)}{1 - G(x)} w(x) = \frac{1}{2}\left(\frac{1 - [1 - G(x)]^2}{1 - G(x)}\right) = \frac{1}{2}\left(\frac{1 - e^{-2\phi x}}{e^{-\phi x}}\right) = \sinh(\phi x),$$

it follows from Equation (2) that

$$F(x) = 1 - \exp\left\{-\alpha[\sinh(\phi x)]^{\beta}\right\}, \quad x > 0, \tag{5}$$

and $F(x) = 0$ for $x \leq 0$. From this expression, one can remark that the WOWE distribution is also the distribution of the random $Y = \operatorname{arsinh}(X)/\phi$, where X denotes a random variable following the Weibull distribution with scale parameter α and shape parameter β, and $\operatorname{arsinh}(x)$ denotes the area hyperbolic sine function, defined as the inverse of $\sinh(x)$—that is, $\operatorname{arsinh}(x) = \ln(x + \sqrt{x^2 + 1})$.

By differentiating $F(x)$ according to x, the pdf of the WOWE distribution follows immediately:

$$f(x) = \alpha\beta\phi \cosh(\phi x)[\sinh(\phi x)]^{\beta - 1} \exp\left\{-\alpha[\sinh(\phi x)]^{\beta}\right\}, \quad x > 0, \tag{6}$$

$f(x) = 0$ for $x \leq 0$. We now illustrate the flexibility of $f(x)$ in terms of curvature in Figure 1; the values of the parameters are selected to show specific shapes for $f(x)$.

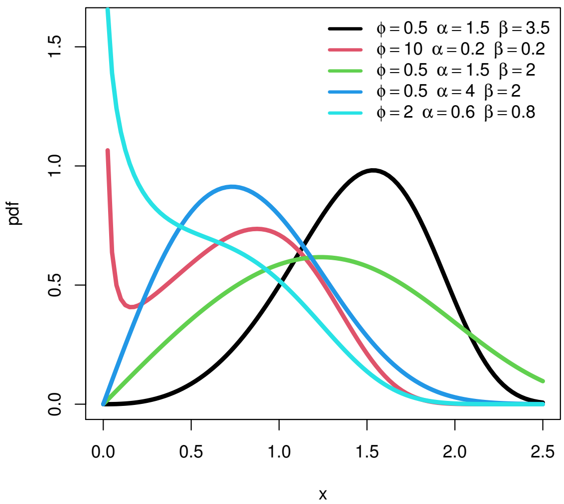

Figure 1. Panels of curves of $f(x)$ for diverse values of the parameters.

Figure 1 displays 5 different types of curves for $f(x)$. The light blue curve shows a decreasing function with small variations; the navy blue curve shows a right-skewed function; the green curve refers to an almost symmetric function; the black curve represents a left-skewed function; and finally, the red curve presents a perfect reversed N shape.

We observe two (local) maxima, one is attained at $x = 0$ and another is attained in the neighborhood of $x = 1$. This panel of curvatures is a strong argument for recommending the WOWE model for the fit of various survival data.

The corresponding sf and cumulative hrf of the WOWE distribution are given as

$$S(x) = \exp\left\{-\alpha[\sinh(\phi x)]^{\beta}\right\}, \quad x > 0,$$

and $S(x) = 1$ for $x \leq 0$, and

$$R(x) = \alpha[\sinh(\phi x)]^{\beta}, \quad x > 0,$$

and $R(x) = 0$ for $x \leq 0$, respectively. The simplicity and originality of these functions are features of the WOWE distribution. By differentiating $R(x)$ according to x, the hrf of the WOWE distribution is obtained by

$$r(x) = \alpha\beta\phi\cosh(\phi x)[\sinh(\phi x)]^{\beta-1}, \quad x > 0, \tag{7}$$

and $r(x) = 0$ for $x \leq 0$. The possible curves of $r(x)$ is informative of the data fitting ability of the WOWE model (see [23]). In this regard, Figure 2 presents various curves for $r(x)$; the values of the parameters for $r(x)$ are chosen to depict specific forms.

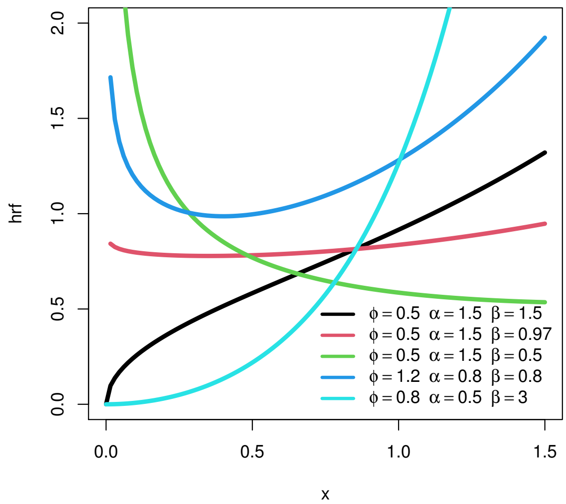

Figure 2. Panels of curves of $r(x)$ for diverse values of the parameters.

Figure 2 reveals 5 different types of curves for $r(x)$. The light blue curve shows an increasing and convex function, the navy blue curve represents a clear bathtub shape, the green curve refers to a decreasing function, the black curve shows an increasing and concave function, and the red curve presents a near-constant shape. These observations support the WOWE model for data fitting purposes.

Other properties of the WOWE distribution will be presented throughout the study.

3. Theoretical Aspects

This section affords some theoretical aspects and properties of the WOW-G class, with applications to the WOWE distribution when possible.

3.1. First-Order Stochastic Dominance

By construction, as already developed in the introductory section, we have $F_{H_1}(x) \leq F_{H_4}(x) \leq F_{H_3}(x) \leq F_{H_2}(x)$. Hence, by using the concept of first-order stochastic (FOS) dominance as presented in [24], the Weibull-X class FOS-dominates the WOW-G class, which itself FOS-dominates the MOW-G class, which itself FOS-dominates the Weibull-G class. In addition, an intrinsic FOS dominance property of the WOW-G class is developed in the next proposition.

Proposition 1. *Let $F(x;\alpha,\beta) = F(x)$. Then, the following inequalities hold:*

- *For $\alpha_1 \leq \alpha_2$, we have $F(x;\alpha_1,\beta) \leq F(x;\alpha_2,\beta)$; the WOW-G class with parameter $\alpha = \alpha_1$ FOS dominates the WOW-G class with the parameter $\alpha = \alpha_2$.*
- *There is no FOS property for the WOW-G class according to β.*

Proof.

- For any $\alpha > 0$, $\beta > 0$ and $x \in \mathbb{R}$, we have

$$\frac{\partial}{\partial\alpha}F(x;\alpha,\beta) = \left[\frac{G(x)}{1-G(x)}\right]^{\beta} w(x)^{\beta} \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta} w(x)^{\beta}\right\} \geq 0,$$

  implying the first result.
- For any $\alpha > 0$, $\beta > 0$ and $x \in \mathbb{R}$, we have

$$\frac{\partial}{\partial\beta}F(x;\alpha,\beta) = \alpha \ln\left\{\frac{G(x)}{1-G(x)}w(x)\right\}\left[\frac{G(x)}{1-G(x)}\right]^{\beta} w(x)^{\beta} \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta} w(x)^{\beta}\right\}.$$

  The sign of this function depends on the sign of the logarithmic term. The function is negative for x such that $G(x) < [1-G(x)]w(x)$ and positive for x such that $G(x) > [1-G(x)]w(x)$; the function $F(x;\alpha,\beta)$ is nonmonotonic with respect to β for varying x; the FOS dominance property does not hold.

This ends the proof of Proposition 1. □

Naturally, further FOS dominances can be established based on the parameter(s) of the reference distribution. For instance, a result of this aspect regarding the WOWE distribution is presented below.

Proposition 2. *In the context of the WOWE distribution, let $F(x;\phi) = F(x)$. Then, for $\phi_1 \leq \phi_2$, we have $F(x;\phi_1) \leq F(x;\phi_2)$; the WOWE distribution with parameter $\phi = \phi_1$ FOS dominates the WOWE distribution with parameter $\phi = \phi_2$.*

Proof. We have

$$\frac{\partial}{\partial\phi}F(x;\phi) = \alpha\beta x \cosh(\phi x)[\sinh(\phi x)]^{\beta-1} \exp\left\{-\alpha[\sinh(\phi x)]^{\beta}\right\} \geq 0.$$

The desired result follows. □

3.2. Quantile Function

In full generality, the qfs are involved in many statistical applications and simulation techniques. The following result determines the qf of the WOW-G class.

Proposition 3. *The qf of the WOW-G class is obtained by inverting $F(x)$. That is,*

$$Q(u) = G^{-1}\left[\left(-\frac{1}{\alpha}\ln(1-u)\right)^{1/\beta} + 1 - \sqrt{\left(-\frac{1}{\alpha}\ln(1-u)\right)^{2/\beta} + 1}\right], \quad u \in (0,1),$$

where $G^{-1}(u)$ denotes the proper qf of the reference distribution.

Proof. In order to determine $Q(u)$, let us investigate the equation $F(x) = u$. Then, the following chain of equivalences holds:

$$F(x) = u \quad \Leftrightarrow \quad H_4(x) = \left(-\frac{1}{\alpha}\ln(1-u)\right)^{1/\beta} \quad \Leftrightarrow \quad u = H_4^{-1}\left[\left(-\frac{1}{\alpha}\ln(1-u)\right)^{1/\beta}\right].$$

The desired result follows immediately from Equation (3). □

From Proposition 3, it follows that the median of the WOW-G class is given by

$$M = Q\left(\frac{1}{2}\right) = G^{-1}\left[\left(\frac{1}{\alpha}\ln 2\right)^{1/\beta} + 1 - \sqrt{\left(\frac{1}{\alpha}\ln 2\right)^{2/\beta} + 1}\right].$$

Further, by considering a random variable U with a uniform distribution over $(0,1)$, $Q(U)$ becomes a random variable with the cdf of the WOW-G class. This distributional property is at the basis of the inverse transform sampling technique, which allows the generation of numerical values from $Q(U)$. In addition, the established qf enables the definitions of various asymmetry and plateness measures, such as the Galton skewness and Moors kurtosis. These aspects are described in detail in [25].

As a concrete example of application, the qf of the WOWE distribution follows from Proposition 3 with $G^{-1}(u) = -\ln(1-u)/\phi$. That is, after some algebra, we obtain

$$Q(u) = \frac{1}{\phi}\operatorname{arsinh}\left[\left(-\frac{1}{\alpha}\ln(1-u)\right)^{1/\beta}\right], \quad u \in (0,1).$$

The median of the WOWE distribution is given by

$$M = \frac{1}{\phi}\operatorname{arsinh}\left[\left(\frac{1}{\alpha}\ln 2\right)^{1/\beta}\right].$$

The other quartiles, as well as Galton skewness and Moors kurtosis, can be expressed in a similar manner.

3.3. Asymptotic and Form Analysis

Let us now examine the asymptotic behavior and form analysis of $f(x)$ and $r(x)$, as already sketched graphically in Figures 1 and 2. Such asymptotic analysis is useful to highlight the roles of the parameters on the possible explosion or not of these functions. In the case of $G(x) \to 0$, the following equivalences hold:

$$f(x) \sim \alpha\beta g(x)G(x)^{\beta-1}, \quad r(x) \sim \alpha\beta g(x)G(x)^{\beta-1}.$$

Moreover, in the case of $G(x) \to 1$, it follows

$$f(x) \sim \frac{\alpha\beta}{2^\beta}g(x)[1-G(x)]^{-\beta-1}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^\beta w(x)^\beta\right\}, \quad r(x) \sim \frac{\alpha\beta}{2^\beta}g(x)[1-G(x)]^{-\beta-1}.$$

We see that α and β have a strong impact on the asymptotic behavior of the functions, especially in the case of $G(x) \to 1$. These asymptotic results can be completed by an analytical study. First, the local maximum(a) of $f(x)$ are of modeling interest, representing possible "peak(s) of values" behind an observed phenomenon. Further, a local maximum corresponds to a mode of the WOW-G class. Such a local maxima—e.g., x_m—satisfies the following equation: $df(x)/dx \mid_{x=x_m}= 0$, or equivalently, $d\ln[f(x)]/dx \mid_{x=x_m}= 0$, where

$$\frac{d}{dx}\ln[f(x)] = \frac{dg(x)/dx}{g(x)} + 2\frac{g(x)}{(1-G(x))[(1-G(x))^2+1]} + (\beta-1)\frac{g(x)}{G(x)}$$
$$+ (\beta-1)\frac{g(x)}{1-G(x)} - (\beta-1)\frac{g(x)}{2w(x)} - \frac{\alpha\beta}{2}g(x)\left[1+\frac{1}{(1-G(x))^2}\right]\left[\frac{G(x)}{1-G(x)}\right]^{\beta-1} w(x)^{\beta-1}.$$

In addition, as a local maximum, x_m satisfies $d\ln[f(x)]/dx > 0$ for $x < x_m$ and $d\ln[f(x)]/dx < 0$ for $x > x_m$. Clearly, the definition and number of modes depend on $g(x)$ and the parameters α and β.

Similarly, a critical point for $r(x)$ satisfies $dr(x)/dx = 0$, or equivalently, $d\ln[r(x)]/dx = 0$, where

$$\frac{d}{dx}\ln[r(x)] = \frac{dg(x)/dx}{g(x)} + 2\frac{g(x)}{(1-G(x))[(1-G(x))^2+1]} + (\beta-1)\frac{g(x)}{G(x)}$$
$$+ (\beta-1)\frac{g(x)}{1-G(x)} - (\beta-1)\frac{g(x)}{2w(x)}.$$

The sign of this function at a solution point can indicate its nature. Nevertheless, from the general equations above, the expressions of the solutions are difficult to exhibit.

As a more concrete study, let us now investigate the asymptotics, modes, and shapes of the pdf and hrf of the WOWE distribution. In the case of $x \to 0$, we have

$$f(x) \sim \alpha\beta\phi^\beta x^{\beta-1}, \quad r(x) \sim \alpha\beta\phi^\beta x^{\beta-1}.$$

Both of $f(x)$ and $r(x)$ explode to $+\infty$ when $\beta < 1$, are equal to $\alpha\beta\phi^\beta$ when $\beta = 1$, and tend to 0 when $\beta > 1$. Further, in the case of $x \to +\infty$, it follows that

$$f(x) \sim \frac{\alpha\beta\phi}{2^\beta}e^{\phi\beta x}\exp\Big\{-\alpha[\sinh(\phi x)]^\beta\Big\}, \quad r(x) \sim \frac{\alpha\beta\phi}{2^\beta}e^{\phi\beta x}.$$

Thus, for all the values of the parameters, $f(x)$ tends to 0 whereas $r(x)$ explodes to $+\infty$. Now, the derivative of the logarithmic transformation of the corresponding pdf is obtained as

$$\frac{d}{dx}\ln[f(x)] = \phi\coth(\phi x)\Big(-\alpha\beta[\sinh(\phi x)]^\beta + \beta + [\tanh(\phi x)]^2 - 1\Big).$$

We already know that $x = 0$ is a local maximum for $f(x)$ when $\beta < 1$. In this case or for $\beta \geq 1$, another possible mode—e.g., x_m—satisfies $-\alpha\beta[\sinh(\phi x_m)]^\beta + \beta + [\tanh(\phi x_m)]^2 - 1 = 0$. Further, we have

$$\frac{d}{dx}\ln[r(x)] = \phi\coth(\phi x)\Big([\tanh(\phi x)]^2 + \beta - 1\Big).$$

If $\beta > 1$, this function is positive, implying that $\ln[r(x)]$ and $r(x)$ are increasing. If $\beta < 1$, a critical point is given as the solution of the following equation: $x = \operatorname{atanh}(\sqrt{1-\beta})/\phi$. These mathematical facts confirm some observations made in Figures 1 and 2.

3.4. Mixture Representation

Following the approach of [5], mixture representations of the main functions of the WOW-G class are proved in the next proposition.

Proposition 4. *The cdf and pdf of the WOW-G class can be expanded in the following manner:*

$$F(x) = \sum_{k=1,\ell,m=0}^{+\infty} \sum_{p=0}^{\beta k+\ell+m} \theta_{k,\ell,m,p} K(x)^p,$$

where $K(x) = 1 - G(x)$ *and*

$$\theta_{k,\ell,m,p} = \frac{1}{k!}(-1)^{k+\ell+m+p+1}\alpha^k 2^{-\ell}\binom{\beta k}{\ell}\binom{-\beta k}{m}\binom{\beta k+\ell+m}{p}.$$

Similarly, assuming that the interchange of the derivative and sum signs is valid in the mathematical sense, we have

$$f(x) = \sum_{k=1,\ell,m=0}^{+\infty} \sum_{p=1}^{\beta k+\ell+m} \eta_{k,\ell,m,p}\{pg(x)K(x)^{p-1}\},$$

where $\eta_{k,\ell,m,p} = -\theta_{k,\ell,m,p}$.

Proof. Based on the standard exponential series expansion, we obtain

$$F(x) = \sum_{k=1}^{+\infty} \frac{1}{k!}(-1)^{k+1}\alpha^k \left[\frac{G(x)}{1-G(x)}\right]^{\beta k} w(x)^{\beta k}.$$

Moreover, by applying the generalized binomial formula twice, we obtain

$$\begin{aligned} F(x) &= \sum_{k=1}^{+\infty} \frac{1}{k!}(-1)^{k+1}\alpha^k G(x)^{\beta k} \sum_{\ell=0}^{+\infty}\binom{\beta k}{\ell}(-1)^{\ell}2^{-\ell}G(x)^{\ell} \sum_{m=0}^{+\infty}\binom{-\beta k}{m}(-1)^m G(x)^m \\ &= \sum_{k=1,\ell,m=0}^{+\infty} \frac{1}{k!}(-1)^{k+\ell+m+1}\alpha^k 2^{-\ell}\binom{\beta k}{\ell}\binom{-\beta k}{m}G(x)^{\beta k+\ell+m}. \end{aligned}$$

By expressing $G(x)$ as $G(x) = 1 - K(x)$ and using the standard binomial formula, it results as follows:

$$F(x) = \sum_{k=1,\ell,m=0}^{+\infty} \sum_{p=0}^{\beta k+\ell+m} \frac{1}{k!}(-1)^{k+\ell+m+p+1}\alpha^k 2^{-\ell}\binom{\beta k}{\ell}\binom{-\beta k}{m}\binom{\beta k+\ell+m}{p}K(x)^p,$$

which corresponds to the announced result. When differentiating with respect to x, the expansion of $f(x)$ follows. □

One can remark that $K(x)$ is literally the sf of the reference distribution. Thus, Proposition 4 shows that the cdf and pdf of the WOW-G class can be expressed in terms of crucial functions of the exponentiated reference distribution. This relationship can be used for further mathematical developments.

For the WOWE distribution, based on this proposition, the corresponding cdf and pdf become expressed in terms of inverse exponential functions as follows:

$$F(x) = \sum_{k=1,\ell,m=0}^{+\infty} \sum_{p=0}^{\beta k+\ell+m} \theta_{k,\ell,m,p} e^{-\phi p x}, \quad f(x) = \sum_{k=1,\ell,m=0}^{+\infty} \sum_{p=0}^{\beta k+\ell+m} \eta_{k,\ell,m,p}\phi p e^{-\phi p x}.$$

Under some integrability conditions, one infers that, for a function $h(x)$ defined on $(0, +\infty)$, we have

$$\int_0^{+\infty} h(x) f(x) dx = \sum_{k=1,\ell,m=0}^{+\infty} \sum_{p=0}^{\beta k+\ell+m} \eta_{k,\ell,m,p} \phi p \mathcal{T}(h)(\phi p),$$

where $\mathcal{T}(h)(t) = \int_0^{+\infty} h(x) e^{-tx} dx$ denotes the Laplace transform of $h(x)$. This expansion can serve for computational purposes for various probabilistic measures.

3.5. Moments

We now work with a random variable X whose distribution belongs to the WOW-G class. Let us denote by E the standard expectation. Then, if the r^{th} order moment about the origin of X converges in the mathematical sense, it is given by

$$\begin{aligned} m_r &= E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx \\ &= \frac{\alpha\beta}{2} \int_{-\infty}^{\infty} x^r g(x) \left[1 + \frac{1}{(1-G(x))^2}\right] \left[\frac{G(x)}{1-G(x)}\right]^{\beta-1} w(x)^{\beta-1} \exp\left\{-\alpha \left[\frac{G(x)}{1-G(x)}\right]^{\beta} w(x)^{\beta}\right\} dx. \end{aligned}$$

After some changes of variable based on the involved qfs, it can be re-expressed as

$$m_r = \frac{\alpha\beta}{2} \int_0^1 [G^{-1}(u)]^r \left[1 + \frac{1}{(1-u)^2}\right] \left[\frac{u(1-u/2)}{1-u}\right]^{\beta-1} \exp\left\{-\alpha \left[\frac{u(1-u/2)}{1-u}\right]^{\beta}\right\} du,$$

or, based on Proposition 3,

$$m_r = \int_0^1 [Q(u)]^r du = \int_0^1 \left\{ G^{-1}\left[\left(-\frac{1}{\alpha}\ln(1-u)\right)^{1/\beta} + 1 - \sqrt{\left(-\frac{1}{\alpha}\ln(1-u)\right)^{2/\beta} + 1}\right]\right\}^r du.$$

In all cases, these integrals are not simply developable; if the reference distribution is known, along with the parameters α and β, numerical treatments of them are possible via mathematical software such as Matlab, Mathematica, R, Python, etc. Alternatively, one can use the expansion of $f(x)$ in Proposition 4 for deriving an expansion of m_r, as in [5].

In the setting of the WOWE distribution, the r^{th}-order moment about the origin of X converges in the mathematical sense and the above integrals can be applied. As a remark, by introducing a random variable W following the Weibull distribution with scale parameter α and shape parameter β, we have $m_r = E\{[\text{arsinh}(W)]^r\}/\phi^r$. From these moments about the origin, one can derive the moments about the mean, variance, moment skewness, and moment kurtosis of X. They are defined by, respectively, $m_r^c = E[(X-m_1)^r] = \sum_{k=0}^{r} \binom{r}{k} (-1)^k m_1^k m_{r-k}$, $\sigma^2 = m_2^c$, $\mathcal{S} = m_3^c/\sigma^3$ and $\mathcal{K} = m_4^c/\sigma^4$. The mean is a central measure, the variance is a dispersion measure, the moment skewness is an asymmetry measure, and the moment kurtosis is a peakedness/flatness measure.

The behavior of the WOWE distribution regarding these measures is illustrated in Table 2. The values of the parameters are chosen in such a way as to illustrate the overall moment versatility phenomenon.

Table 2. Moments, variance, skewness, and kurtosis of the WOWE distribution for some parameter values.

(ϕ, α, β)	m_1	m_2	m_3	m_4	σ^2	$\mathcal{S}$	$\mathcal{K}$
$(0.1, 0.1, 0.1)$	7.7459	545.4638	42424.6700	3478367	22.0332	2.8681	9.3350
$(0.1, 0.5, 0.1)$	18.12071	1206.8810	90740.4800	7265826	29.6398	1.4221	1.4814
$(0.1, 0.5, 0.5)$	16.6674	481.3052	16555.54	630853.9	14.2654	0.6027	4.6662
$(0.1, 2.5, 0.5)$	2.5418	23.3260	328.6222	5764.03	4.1067	2.6507	10.2090
$(0.2, 2.5, 0.5)$	1.2709	5.8315	41.0777	360.2519	2.0533	2.6507	11.45969
$(0.2, 2.5, 2.5)$	2.8451	9.2991	33.4134	128.8816	1.0974	0.0775	71.6438
$(0.3, 3, 3)$	1.9195	4.0841	9.3549	22.6830	0.6319	−0.0721	242.1780
$(0.3, 6, 6)$	6.3961	42.1459	284.6251	1962.7610	1.1113	−0.5442	864.7704

From Table 2, we see that m_1 and σ^2 can vary a lot; for the considered values of the parameters, we have $m_1 \in [1.2709, 18.12071]$ and $\sigma^2 \in [0.6319, 29.6398]$. Furthermore, $\mathcal{S}$ can be negative as for $(\phi, \alpha, \beta) = (0.3, 3, 3)$, almost equal to 0 as for $(\phi, \alpha, \beta) = (0.2, 2.5, 2.5)$, and positive as for $(\phi, \alpha, \beta) = (0.1, 0.1, 0.1)$; the WOWE distribution is confirmed to be possibly left-skewed, almost symmetric, and right-skewed. For the moment kurtosis, all the possible cases are here: $\mathcal{K}$ can be inferior to 3 as for $(\phi, \alpha, \beta) = (0.1, 0.5, 0.1)$, near equal to 3 as for $(\phi, \alpha, \beta) = (0.1, 0.5, 0.5)$, and superior to 3 as for $(\phi, \alpha, \beta) = (0.3, 6, 6)$. Consequently, the WOWE distribution can be of the three following regimes: platykurtic, mesokurtic, and leptokurtic. These observations are in favor of the use of the WOWE model for data fitting aims.

4. Inferential Aspect

In this section, we briefly outline the ML method, which allows for efficient estimation of the model parameters. A simulation study illustrates this efficiency.

4.1. Parameter Estimation

Suppose that we deal with n observations $x_1, \ldots, x_n$ from a characteristic that can be modeled by a random variable with a distribution belonging to the WOW-G class. Then, by denoting $(\alpha, \beta, \boldsymbol{\zeta})$ the vector of the unknown model parameters, including those of the reference model denoted by $\boldsymbol{\zeta}$, a ML estimate (MLE) of $(\alpha, \beta, \boldsymbol{\zeta})$ is given as

$$(\hat{\alpha}, \hat{\beta}, \hat{\boldsymbol{\zeta}}) = \operatorname{argmax}_{(\hat{\alpha}, \hat{\beta}, \hat{\boldsymbol{\zeta}})} \mathcal{L}(\alpha, \beta, \boldsymbol{\zeta}),$$

where $\mathcal{L}(\alpha, \beta, \boldsymbol{\zeta})$ denotes the likelihood function defined by

$$\begin{aligned} \mathcal{L}(\alpha, \beta, \boldsymbol{\zeta}) &= \prod_{i=1}^{n} f(x_i) \\ &= \prod_{i=1}^{n} \left\{ \frac{\alpha\beta}{2} g(x_i) \left[1 + \frac{1}{(1 - G(x_i))^2} \right] \left[\frac{G(x_i)}{1 - G(x_i)} \right]^{\beta - 1} w(x_i)^{\beta - 1} \exp\left\{ -\alpha \left[\frac{G(x_i)}{1 - G(x_i)} \right]^{\beta} w(x_i)^{\beta} \right\} \right\}. \end{aligned}$$

Here, we assume that the MLE exists and is unique. A component of $(\hat{\alpha}, \hat{\beta}, \hat{\boldsymbol{\zeta}})$ is called the MLE of the corresponding component of $(\alpha, \beta, \boldsymbol{\zeta})$. Under some configurations, it is easier to find the MLE of $(\alpha, \beta, \boldsymbol{\zeta})$ through the maximization of the log-likelihood function given as

$$\begin{aligned}
\ell(\alpha, \boldsymbol{\zeta}) &= \ln[\mathcal{L}(\alpha, \beta, \boldsymbol{\zeta})] = \sum_{i=1}^{n} \ln[f(x_i)] \\
&= n \ln\left(\frac{\alpha\beta}{2}\right) + \sum_{i=1}^{n} \ln[g(x_i)] + \sum_{i=1}^{n} \ln\left[1 + \frac{1}{(1-G(x_i))^2}\right] + (\beta-1)\sum_{i=1}^{n} \ln[G(x_i)] \\
&\quad - (\beta-1)\sum_{i=1}^{n} \ln[1-G(x_i)] + (\beta-1)\sum_{i=1}^{n} \ln[w(x_i)] - \alpha \sum_{i=1}^{n} \left[\frac{G(x_i)}{1-G(x_i)}\right]^{\beta} w(x_i)^{\beta}.
\end{aligned}$$

If the log-likelihood function is differentiable, the desired MLEs are solutions of the following system of equations:

$$\frac{\partial}{\partial \alpha} \ell(\alpha, \beta, \boldsymbol{\zeta}) = \frac{n}{\alpha} - \sum_{i=1}^{n} \left[\frac{G(x_i)}{1-G(x_i)}\right]^{\beta} w(x_i)^{\beta} = 0,$$

$$\begin{aligned}
\frac{\partial}{\partial \beta} \ell(\alpha, \beta, \boldsymbol{\zeta}) &= \frac{n}{\beta} + \sum_{i=1}^{n} \ln[G(x_i)] - \sum_{i=1}^{n} \ln[1-G(x_i)] + \sum_{i=1}^{n} \ln[w(x_i)] \\
&\quad - \alpha \sum_{i=1}^{n} \left[\frac{G(x_i)}{1-G(x_i)}\right]^{\beta} w(x_i)^{\beta} \ln\left\{\left[\frac{G(x_i)}{1-G(x_i)}\right] w(x_i)\right\} = 0,
\end{aligned}$$

and, by denoting $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_p)$, for $j = 1, \ldots, p$,

$$\begin{aligned}
\frac{\partial}{\zeta_j} \ell(\alpha, \beta, \boldsymbol{\zeta}) &= \sum_{i=1}^{n} \frac{\partial g(x_i)/\partial \zeta_j}{g(x_i)} + 2\sum_{i=1}^{n} \frac{\partial G(x_i)/\partial \zeta_j}{(1-G(x_i))[(1-G(x_i))^2+1]} + (\beta-1)\sum_{i=1}^{n} \frac{\partial G(x_i)/\partial \zeta_j}{G(x_i)} \\
&\quad + (\beta-1)\sum_{i=1}^{n} \frac{\partial G(x_i)/\partial \zeta_j}{1-G(x_i)} - (\beta-1)\sum_{i=1}^{n} \frac{\partial G(x_i)/\partial \zeta_j}{2w(x_i)} \\
&\quad - \frac{\alpha\beta}{2} \sum_{i=1}^{n} \frac{\partial G(x_i)}{\partial \zeta_j} \left[1 + \frac{1}{(1-G(x_i))^2}\right] \left[\frac{G(x_i)}{1-G(x_i)}\right]^{\beta-1} w(x_i)^{\beta-1} = 0.
\end{aligned}$$

Due to the analytical complexity of these equations, in most cases, closed form expressions for the MLEs are not possible. We can solve them numerically by using iterative procedures such as the Newton–Raphson-type algorithms. The standard errors (SEs) of the MLEs can be determined by the diagonal elements of J^{-1}, where J denotes the $(p+2) \times (p+2)$ matrix defined by

$$J = \left\{-\frac{\partial^2}{\partial u \partial v} \ell(\alpha, \beta, \boldsymbol{\zeta})\right\}_{(u,v) \in \{\alpha, \beta, \zeta_1, \ldots, \zeta_p\}^2} \Bigg|_{(\alpha, \beta, \boldsymbol{\zeta}) = (\hat{\alpha}, \hat{\beta}, \hat{\boldsymbol{\zeta}})},$$

whose elements can be evaluated numerically. The setting above can be applied directly to the WOWE distribution, as illustrated in the next section. The background on the general theory and practice of the ML method can be found in [26].

4.2. Simulation

Here, based on the WOWE model, a simulation study is performed in order to investigate the behavior of the ML estimation. We start by generating $N = 10{,}000$ samples from the model, each of sample size $n \in \{35, 50, 100, 200, 300, 400, 500\}$, for some selected parameter values. The assessment considers the average value of the ML estimates (AE) and the root-mean-square error (RMSE) defined by

$$AE(\psi) = \frac{1}{N}\sum_{i=1}^{N}\hat{\psi}_i, \quad \mathrm{RMSE}(\psi) = \sqrt{\frac{1}{N}\sum_{i=1}^{n}(\hat{\psi}_i - \psi)^2},$$

respectively, where $\psi = \alpha, \beta$, or ϕ. We employ the package named `AdequacyModel` and developed in [27] of the `R` software (see [28]). The results are given in Table 3.

Table 3. Actual values, AEs, and RMSEs of the simulated data from the WOWE distribution for the following parameter configurations with order (ϕ, α, β): $(1.3, 3.2, 2.4)$, $(1.1, 2.0, 0.8)$, $(1.0, 1.0, 1.0)$, $(1.5, 1.1, 0.2)$, $(0.5, 1.5, 0.5)$, and $(0.6, 0.8, 0.4)$.

Sample Size	Actual Values			Average Estimate			Root-Mean-Square Error		
n	ϕ	α	β	$\hat{\phi}$	$\hat{\alpha}$	$\hat{\beta}$	$RMSE(\hat{\phi})$	$RMSE(\hat{\alpha})$	$RMSE(\hat{\beta})$
35	1.3	3.2	2.4	1.2811	4.4673	2.6784	5.1833	3.9974	0.8393
50				1.2842	4.1085	2.5299	4.4961	3.8963	0.7740
100				1.1629	3.3056	2.4306	0.6595	1.0978	0.6055
200				1.2056	3.2039	2.3994	0.5860	0.7956	0.5456
300				1.2439	3.1723	2.3765	0.5324	0.6722	0.5382
400				1.2819	3.1295	2.3384	0.5148	0.5818	0.5072
500				1.2917	3.1314	2.3424	0.4666	0.5356	0.4764
35	1.1	2.0	0.8	1.84938	2.9949	0.9004	0.8098	1.7134	0.2838
50				0.9228	2.4632	0.8454	0.7457	1.5873	0.2459
100				0.9688	2.0734	0.8018	0.6778	0.5467	0.1942
200				0.9901	2.0240	0.7975	0.6095	0.4079	0.1718
300				1.0162	2.0047	0.7929	0.5681	0.3475	0.1584
400				1.0364	1.9872	0.7880	0.5364	0.3013	0.1510
500				1.0421	1.9859	0.7904	0.4932	0.2768	0.1396
35	1.0	1.0	1.0	1.0292	1.6786	1.2355	0.9120	0.656O	0.4432
50				0.7724	1.1494	1.0896	0.7139	0.4497	0.3299
100				0.8465	1.0294	1.0152	0.6871	0.2006	0.2584
200				0.8718	1.0119	1.0045	0.6354	0.1536	0.2312
300				0.9051	1.0022	0.9937	0.6097	0.1295	0.2182
400				0.9298	0.9955	0.9852	0.5828	0.1119	0.2096
500				0.9328	0.9945	0.9876	0.5453	0.1029	0.1957
35	1.5	1.1	0.2	1.3456	1.4467	0.3359	0.9688	0.7997	0.1097
50				0.9736	1.2720	0.2660	0.7394	0.5391	0.0915
100				1.1755	1.1355	0.2354	0.6736	0.2359	0.0721
200				1.2573	1.1111	0.2260	0.6151	0.1775	0.0650
300				1.3222	1.1011	0.2193	0.5559	0.1498	0.0602
400				1.3628	1.0966	0.2150	0.5105	0.1309	0.0560
500				1.3763	1.0953	0.2142	0.4779	0.1193	0.0534
35	0.5	1.5	0.5	0.8998	2.2134	0.5577	0.9878	1.9899	0.1567
50				0.6828	1.9401	0.4873	0.7175	1.0610	0.1288
100				0.6177	1.6527	0.4755	0.6509	0.3950	0.0963
200				0.5693	1.6051	0.4789	0.5981	0.2950	0.0822
300				0.5485	1.5815	0.4816	0.5581	0.2481	0.0717
400				0.5330	1.5633	0.4823	0.5341	0.2136	0.0664
500				0.5187	1.5555	0.4845	0.5035	0.1932	0.0592
35	0.6	0.8	0.4	0.8890	1.0999	0.5656	0.7868	0.6857	0.1200
50				0.6169	0.9284	0.4049	0.6858	0.3288	0.1163
100				0.6166	0.8408	0.3855	0.6614	0.1521	0.0877
200				0.5963	0.8261	0.3857	0.6169	0.1163	0.0773
300				0.6008	0.8178	0.3837	0.5953	0.0977	0.0713
400				0.5989	0.8118	0.3837	0.5840	0.0841	0.0685
500				0.5824	0.8096	0.3862	0.5504	0.0768	0.0629

It can be seen from Table 3 that the performance of the ML estimation shows consistency and is quite satisfactory; the AEs converge to the corresponding actual values and the RMSEs decrease as the sample size increases.

5. Data Analysis

In this section, we attempt to apply the WOWE model to two real data sets, comparing its fit behavior with those of famous extended exponential models of the literature.

5.1. Method

The following eight models are considered:

(i) the proposed WOWE model as described by the cdf and pdf given in Equations (5) and (6), respectively;

(ii) modified odd Weibull exponential (MOWE) model by [6];
(iii) Weibull exponential (WE) model by [5];
(iv) Weibull-X exponential (WXE) model by [4,16];
(v) Lomax exponential (LxE) model by [29];
(vi) beta exponential (BeE) model by [30];
(vii) length bias exponential (LbE) model by [31];
(viii) exponential (E) model.

We discriminate these models by considering the following standard statistical measures:

(i) Akaike information criterion (AIC);
(ii) Bayesian information criterion (BIC);
(iii) consistent AIC (CAIC);
(iv) Hannan–Quinn information criterion (HQIC);
(v) Cramér-von Mises (W) statistic;
(vi) Anderson–Darling (A) statistic;
(vii) Kolmogorov–Smirnov (KS);
(viii) p-value of the KS test.

Conventionally, the best model is the one with the smallest value for AIC, BIC, W, CAIC, HQIC, A, KS and the largest value for the p-value related to the KS test. It is worth mentioning that the AIC, BIC, CAIC, and HQIC depend on the minus maximal estimated log-likelihood that we denote as $-\hat{\ell}$. Further details on these measures, including their mathematical definitions, can be found in [32]. As for the simulation study, the results are obtained by employing the `AdequacyModel` package of the `R` software.

5.2. Fit of the Drilling Machine Data

The first data set named "drilling machine data" refers to the 50 observations with holes and sheets of a certain thickness. It is extracted from [33]. The data are presented as follows: 0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

For these data, the MLEs of the model parameters and the related SEs are presented in Table 4.

Table 4. MLEs and SEs of the parameters of the considered models for the drilling machine data.

Models	MLEs with Related SEs in (.)		
WOWE	6.6212	0.4624	1.5122
(ϕ, α, β)	(3.0716)	(0.4375)	(0.4013)
MOWE	0.2663	179.1942	1.7132
(ϕ, λ, θ)	(0.1012)	(6.7116)	(0.1830)
WE	1.1042	18.6634	1.9670
(ϕ, α, β)	(0.2272)	(0.2837)	(0.1928)
WXE	0.8830	2.1191	0.1622
(ϕ, γ, c)	(4.2139)	(0.2462)	(0.7741)
LxE	139.3275	20.4673	10.6675
(a, k, s)	(2.0930)	(6.9044)	(2.6622)
BeE	3.0295	67.1880	0.2722
(a, b, λ)	(0.5757)	(8.9695)	(2.2320)
LbE	67.5728	0.9904	-
(α, θ)	(4.4556)	(0.7415)	-
E	6.1274	-	-
(ϕ)	(0.8665)	-	-

In particular, for the WOWE model, we note that the MLEs of ϕ, α, and β are $\hat{\phi} = 6.6212$, $\hat{\alpha} = 0.4624$, and $\hat{\beta} = 1.5122$, respectively. Based on Equations (5) and (6) and the plug-in technique, the corresponding estimated cdf and pdf are given as

$$\hat{F}(x) = 1 - \exp\left\{-\hat{\alpha}\left[\sinh(\hat{\phi}x)\right]^{\hat{\beta}}\right\}, \quad x > 0,$$

and

$$\hat{f}(x) = \hat{\alpha}\hat{\beta}\hat{\phi}\cosh(\hat{\phi}x)\left[\sinh(\hat{\phi}x)\right]^{\hat{\beta}-1}\exp\left\{-\hat{\alpha}\left[\sinh(\hat{\phi}x)\right]^{\hat{\beta}}\right\},$$

respectively. Other estimated functions and measures of the WOWE model can be presented in a similar manner.

The values of the $-\hat{\ell}$, AIC, BIC, CAIC, HQIC, W, A, KS, and p-value of the KS test of all the models are given in Table 5.

Table 5. Fit measures for the drilling machine data.

Models	$-\hat{\ell}$	**AIC**	**BIC**	**CAIC**	**HQIC**	**W**	**A**	**KS**	**p-Value**
WOWE	−57.5115	−109.0230	−103.2869	−108.5013	−106.8387	0.0723	0.4303	0.0953	0.7542
MOWE	−50.6530	−95.2832	−89.5471	−94.7844	−93.12181	0.1259	0.7675	0.1529	0.1928
WE	−56.4905	−106.7723	−101.0362	−106.4593	−104.7967	0.0922	0.5626	0.1106	0.5733
WXE	−55.8918	−105.7836	−100.0476	−105.2619	−103.5993	0.1052	0.6435	0.1099	0.5811
LxE	−57.0479	−108.0958	−102.3598	−107.5741	−105.9115	0.0770	0.4609	0.1008	0.6888
BeE	−53.3708	−100.7417	−95.0056	−100.2200	−98.5574	0.1816	1.0902	0.1540	0.1864
LbE	−55.1966	−106.4177	−102.5936	−106.1380	−104.9371	0.1232	0.7526	0.1407	0.2753
E	−40.6389	−79.2778	−77.3658	−79.1945	−78.5497	0.1831	1.0985	0.2806	0.00075

In Table 5, according to the considered criteria, the best results are obtained for the proposed WOWE model. The main challenger is the LxE model. Figure 3 displays four different plots illustrating the nice fit of the WOWE model, which are the probability–probability (P–P) plot, the estimated pdf $\hat{f}(x)$ superposed to the histogram of the data, the estimated cdf $\hat{F}(x)$ superposed to the empirical cdf of the data, and the quantile–quantile (Q–Q) plot.

In each of these plots, we see that the empirical object is well-adjusted by the estimated object. In particular, on the plot of the estimated pdf, we see that the kurtosis flexibility of the WOWE model is determinant; the round shape of the histogram of the data is perfectly fitted.

We complete this data analysis by comparing some descriptive statistics with the estimated ones obtained from the WOWE model. The results are presented in Table 6.

Table 6. Descriptive versus estimated statistics results for the drilling machine data.

Nature	**Mean**	**Standard Deviation**	**Skewness**	**Kurtosis**
Empirical	0.1632	0.0810	0.0701	−0.8711
Estimated	0.1629	0.0821	0.0712	−0.8392

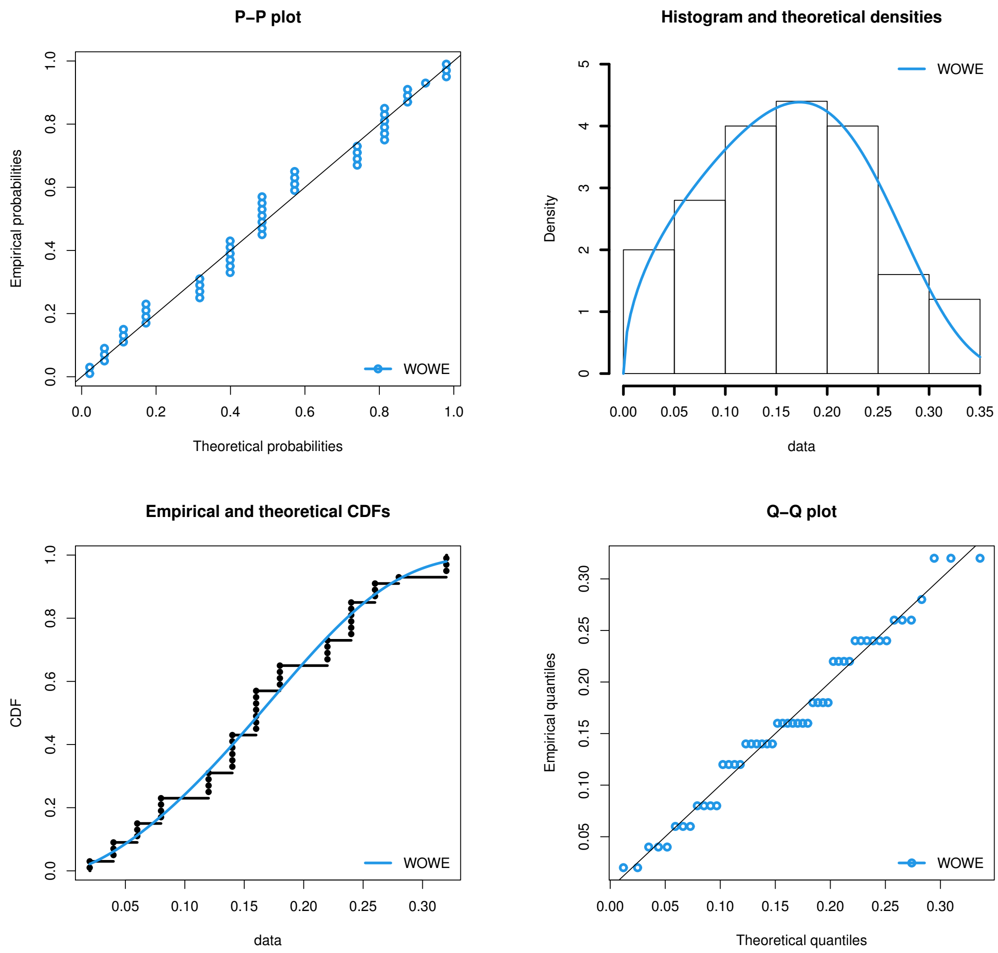

Figure 3. Plots of four estimated objects for the drilling machine data.

As expected, from Table 6, we see that the estimated statistics are close to the empirical ones. This is one more evidence of the high performance of the WOWE model.

5.3. Fit of the Daily Precipitation Data

The second data set, named "daily precipitation", is the average of the maximum daily precipitation for 30 years at 35 stations in central and western Peninsular Malaysia. It is extracted by [34]. The data are presented as follows: 1.134, 1.196, 1.181, 1.178, 1.048, 1.077, 0.835, 1.163, 0.880, 1.056, 1.164, 0.914, 1.141, 1.068, 1.007, 1.027, 1.298, 0.842, 0.991, 0.955, 0.703, 0.953, 1.018, 1.003, 1.106, 1.110, 1.249, 1.092, 1.187, 1.047, 0.989, 0.955, 1.234, 0.937, 0.933.

For these data, the MLEs of the model parameters and the associated SEs are indicated in Table 7.

Table 7. MLEs and SEs of the parameters of the considered models for the daily precipitation data.

Models	MLEs with Related SEs in (.)		
WOWE	0.5808	22.7414	8.3177
(ϕ, α, β)	(0.4112)	(3.1489)	(1.6592)
MOWE	0.1314	101.6124	2.4243
(ϕ, λ, θ)	(0.0311)	(3.7921)	(0.2997)
WE	0.4535	24.2825	7.4265
(ϕ, α, β)	(0.4394)	(9.1518)	(1.8564)
WXE	1.4962	9.4037	1.6514
(ϕ, γ, c)	(2.7011)	(1.2307)	(2.8471)
LxE	320.8721	1.8285	5.0865
(a, k, s)	(8.8329)	(0.8373)	(0.6011)
BeE	65.1220	67.3550	0.6490
(a, b, λ)	(9.7973)	(10.2652)	(1.4117)
LbE	81.3686	6.4630	-
(α, θ)	(7.2559)	(3.2448)	-
E	0.9544	-	-
(ϕ)	(0.1613)	-	-

In particular, for the WOWE model, we note that the MLEs of ϕ, α, and β are $\hat{\phi} = 0.5808$, $\hat{\alpha} = 22.7414$, and $\hat{\beta} = 8.3177$, respectively. The corresponding estimated cdf and pdf follow from Equations (5) and (6), respectively, combined with the plug-in estimation technique.

The values of the $-\hat{\ell}$, AIC, BIC, W, CAIC, HQIC, A, KS, and p-value of the considered models are given in Table 8.

Table 8. Fit measures for the daily precipitation data.

Models	$-\hat{\ell}$	AIC	BIC	CAIC	HQIC	W	A	KS	p-Value
WOWE	−22.2542	−38.5084	−33.8424	−37.7342	−36.8977	0.0301	0.1839	0.0635	0.9989
MOWE	18.6300	42.9953	47.6614	44.0342	44.8707	0.1200	0.4728	0.3556	0.0003
WE	−22.2442	−38.4885	−33.8222	−37.7143	−36.8778	0.0304	0.1853	0.0639	0.9978
WXE	−22.3357	−38.6715	−34.0054	−37.8973	−37.0608	0.0269	0.1666	0.0624	0.9992
LxE	−9.3234	−12.5441	−7.8780	−11.8727	−11.0362	0.1790	0.4377	0.2859	0.0065
BeE	−21.2628	−36.5216	−31.8556	−35.7514	−34.9149	0.0276	0.2375	0.0685	0.9866
LbE	14.7886	33.5091	36.6198	33.9523	34.6511	0.2290	0.8986	0.4365	3.21×10^{-6}
E	36.6323	75.2646	76.8200	75.3858	75.8015	0.2720	0.9343	0.5207	1.42×10^{-8}

Table 8 indicates that the WXE model is the best, followed by the proposed WOWE model. The WOWE model is not the best, but remains a solid alternative, as it surpasses the six other models. Figure 4 displays four different plots, allowing the visual illustration of the fit of the WOWE model.

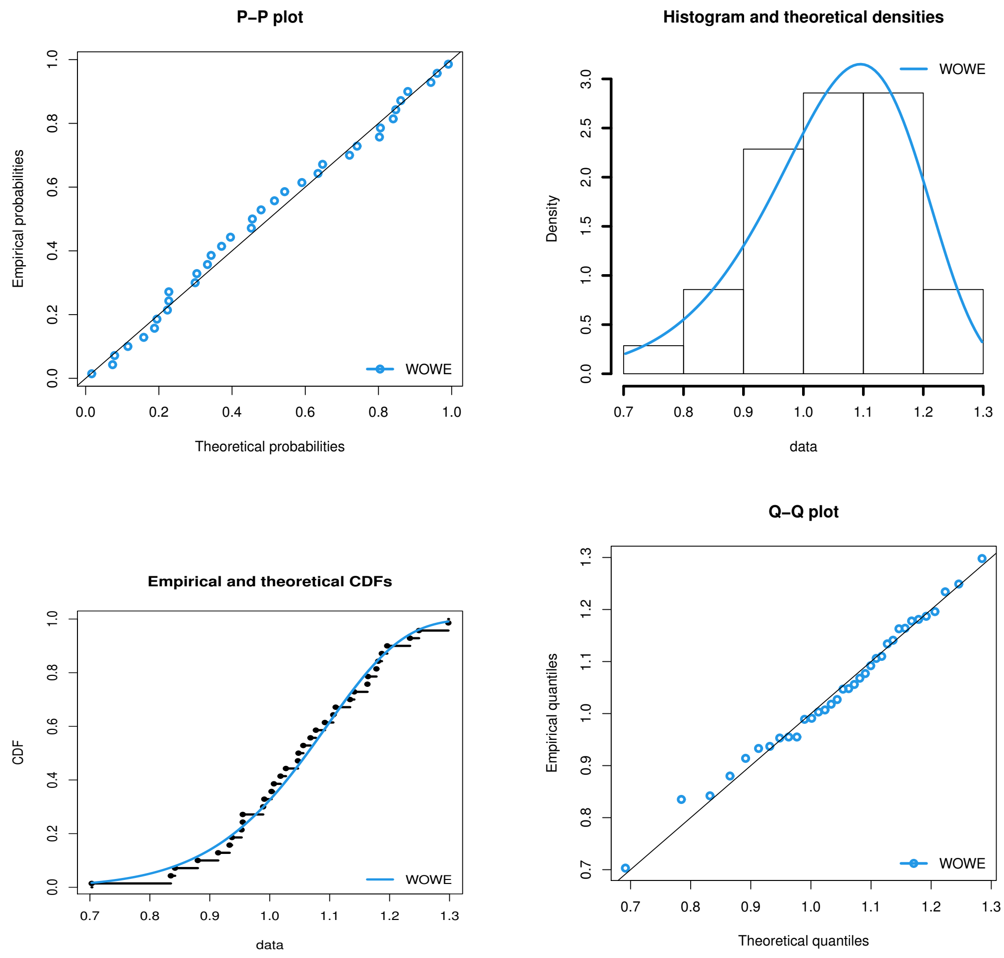

Figure 4. Plots of four estimated objects for the daily precipitation data.

From Figure 4, the observed adjustments are quite satisfactory. In particular, on the plot of the estimated pdf, we see that the left-skewed property of the WOWE model has played an important role in capturing the form of the histogram.

We finish up our data analysis by comparing some descriptive statistics to the estimated ones based on the WOWE model. Table 9 shows the results.

Table 9. Descriptive versus estimated statistics results for the daily precipitation data.

Nature	Mean	Standard Deviation	Skewness	Kurtosis
Empirical	1.0477	0.1309	−0.3355	−0.2406
Estimated	1.0411	0.1391	−0.3400	−0.2394

Table 9 shows that the predicted statistics are very close to the empirical ones, as expected. This is just another example of the strong performance of the WOWE model.

6. Conclusions

We have presented a new motivated Weibull-generated-type class of univariate continuous distribution, called the weighted odd Weibull-generated class. Thanks to the use of a coherent weighted version of the odd transformation, it offers a solid alternative to the Weibull-X, Weibull-G, and MOW-G classes. We have explored its main theoretical and practical facets, with a focus on a special distribution based on the exponential distribution.

The faculties of the proposed class make this special distribution particularly flexible in the modeling sense, with pdf and hrf demonstrating heterogeneous skewness and tail weights. We exploit this advantage in a data fitting scenario.

By estimating the model parameters with the maximum likelihood method, we show that our model is efficient enough to fit two data sets of importance: one with engineering data and the other with environmental data. It is proved to outclass several models also based on the exponential model.

The classical Weibull-G class has been successfully applied to several significant disciplines in recent years, including medical, biology, engineering, dependability, economics, computer science, and finance. Detailed examples can be found in [5,35,36], among others. Our new Weibull-generated class can be used in similar scenarios. It may provide a more accurate model owing to its exceptional analytical versatility. As a result, it must be considered for purposes of testing beside the former Weibull-G class.

As a future potential work, we may consider extension of the new Weibull class, including its bivariate extension based on the same approach as [14], or its discrete analogue following the spirit of [15].

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