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Discrete Pseudo Lindley Distribution: Properties, Estimation and Application on INAR(1) Process

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Abstract: In this paper, we introduce a discrete version of the Pseudo Lindley (PsL) distribution, namely, the discrete Pseudo Lindley (DPsL) distribution, and systematically study its mathematical properties. Explicit forms gathered for the properties such as the probability generating function, moments, skewness, kurtosis and stress–strength reliability made the distribution favourable. Two different methods are considered for the estimation of unknown parameters and, hence, compared with a broad simulation study. The practicality of the proposed distribution is illustrated in the first-order integer-valued autoregressive process. Its empirical importance is proved through three real datasets.

Keywords: Pseudo Lindley distribution; survival discretization method; over dispersion; moments; simulation; maximum likelihood estimation

MSC: 62E15; 62E20; 62E17

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1. Introduction

Count data reflect the non-negative integers which represent the frequency of occurrence of a discrete event. Such datasets can be observed in numerous fields, such as actuarial science, finance, medical, sports, etc. For instance, the yearly number of destructive floods, the number of sports people injured in a month and the hourly number of COVID-19 vaccinations given are some examples of count data. Increasing the utilization of discrete distributions for modelling such datasets influenced researchers to propose more flexible distributions by reducing the estimation errors. Discretizing continuous distributions by survival discretization is one of the widely followed methods for introducing discrete distributions. The most famous discretization technique is described below. Assume that X is a continuous lifetime random variable with the survival function (sf) $S(x) = \Pr(X > x)$. Then, the probability mass function (pmf) dealing with X is given by:

$$\Pr(X = x) = S(x) - S(x+1), \quad x = 0, 1, 2, \ldots \tag{1}$$

Some of the recently introduced discrete distributions based on this survival discretization method are as follows: Discrete Lindley distribution by [1], discrete inverse Weibull distribution by [2], discrete Pareto distribution by [3], discrete Rayleigh distribution by [4], two-parameter discrete Lindley distribution by [5], exponentiated discrete Lindley distribution by [6], discrete Burr–Hatke distribution by [7], discrete Bilal distribution [8], discrete three-parameter Lindley distribution by [9], etc. Recently, Ref. [10] proposed a discrete version of Ramos–Louzada distribution [11] for asymmetric and over-dispersed data with a leptokurtic shape.

Furthermore, count datasets arising in time series can be seen in many applied research areas. Examples include modelling and predicting the number of claims for next month for the insurance sector in a company, predicting the number of deaths from disasters, etc. The first-order integer-valued autoregressive process, or INAR(1), is appropriate for such cases. The authors of [12,13] independently developed the pioneer works of INAR(1) with Poisson innovations. Furthermore, since time series of counts mainly display over-dispersion (i.e., empirical mean is less than empirical variance), Poisson for innovation distribution is less efficient (since equi-dispersed). Hence, researchers have assembled many approaches concerning innovations in modelling over-dispersed time series count datasets. The INAR(1) process with geometric innovations (INAR(1)G) by [14], INAR(1) process with Poisson–Lindley innovations (INAR(1)PL) by [15], INAR(1) process with a new Poisson weighted exponential innovation ((INAR(1)NPWE)) by [16], INAR(1) process with discrete three-parameter Lindley as innovation by [9], INAR(1) process with discrete Bilal as innovation by [8], INAR(1) process with Poisson quasi Gamma innovations (INAR(1)PQX) by [17] and the INAR(1) process with Bell innovations (INAR(1)BL) by [18] are some of the recently developed over-dispersed INAR(1) processes.

Even though these processes provide better solutions to over-dispersed time series count datasets, they have some limitations that can sometimes cause computing difficulties. Even if a model has one parameter, the inclusion of special functions in the pmf, cumulative distribution function (cdf) and other statistical properties makes it difficult to obtain explicit expressions and, hence, for estimation procedures to generate them (see, e.g., [9,19]).

Hence, the main objective of the present work is to introduce a two-parameter discrete distribution, the discrete Pseudo Lindley (DPsL) distribution, which can serve as a model to analyse under as well as over-dispersed datasets, having a simple pmf and cdf. The main peculiarity of the proposed distribution is that it has closed-form expressions for its statistical properties such as a hazard rate function (hrf), probability-generating function (pmf), moments, skewness, kurtosis, mean past lifetime (mpl), mean residual lifetime (mrl), stress–strength reliability, etc. We embellish the importance of the DPsL distribution in the INAR(1) process by applying the DPsL distribution as an innovation process.

The remaining parts of the paper are organized as follows: Section 2 defines the proposed distribution and various properties such as moments, mean residual lifetime, mean past lifetime and stress–strength reliability,. Section 3 contains estimation methods and their simulation study. The INAR(1) process with DPsL innovations is developed in Section 4 with its parameter estimation and simulation study. In Section 5, three datasets are analysed by the DPsL distribution, and some other competitive and well-referenced distributions, in order to prove its applicability. Final remarks are provided in Section 6.

2. The Discrete Pseudo Lindley Distribution

2.1. Some Basics

A discrete analogue of the PsL distribution is derived in this section, namely, the DPsL distribution by using the survival discretization method. First of all, let us briefly present the work of [20], which introduced the Pseudo Lindley (PsL) distribution by mixing two independent random variables: one having the Exponential (θ) distribution, and the other having the Gamma (2,θ) distribution, with mixing probabilities $\frac{\beta-1}{\beta}$ and $\frac{1}{\beta}$, respectively. Assume that X is a continuous random variable having the PsL distribution; then, its probability density function (pdf) and sf are given by:

$$f_{\text{PsL}}(x;\theta,\beta) = \begin{cases} \dfrac{\theta(\beta-1+\theta x)e^{-\theta x}}{\beta}, & x>0 \\ 0 & , \text{otherwise} \end{cases}$$

and

$$S_{\text{PsL}}(x;\theta,\beta) = \begin{cases} \dfrac{(\beta+\theta x)e^{-\theta x}}{\beta}, & x>0 \\ 1 & , \text{otherwise} \end{cases}, \tag{2}$$

respectively, where $\beta \geq 1$ and $\theta > 0$. Using the survival discretization technique as described in (1) by using (2), the pmf of the DPsL distribution can be derived as:

$$P_{\text{DPsL}}(x;\theta,\beta) = \frac{(\beta+\theta x)e^{-\theta x} - (\beta+\theta(x+1))e^{-\theta(x+1)}}{\beta},\ x = 0,1,2,\ldots. \tag{3}$$

The parameter β can be considered as a shape parameter and θ as a scale parameter. The DPsL distribution can sometimes be denoted by the DPsL (θ,β) distribution to indicate the parameters.

The corresponding cdf and sf are given by:

$$F_{\text{DPsL}}(x;\theta,\beta) = 1 - \frac{e^{-\theta(1+x)}(\beta+(x+1)\theta)}{\beta}$$

and

$$S_{\text{DPsL}}(x;\theta,\beta) = \frac{e^{-\theta(1+x)}(\beta+(x+1)\theta)}{\beta}, \tag{4}$$

respectively. As a first property, the pmf given in (3) is log concave, since:

$$\frac{P_{\text{DPsL}}(x+1;\theta,\beta)}{P_{\text{DPsL}}(x;\theta,\beta)} = \frac{\beta+\theta+x\theta-e^{-\theta}(\beta+(2+x)\theta)}{\beta(e^{\theta}-1)+\theta((e^{\theta}-1)x-1)}$$

is a decreasing function in x for every possible value of the parameters.

The possible pmf shapes plotted for different values of the parameters of the DPsL distribution are displayed in Figure 1.

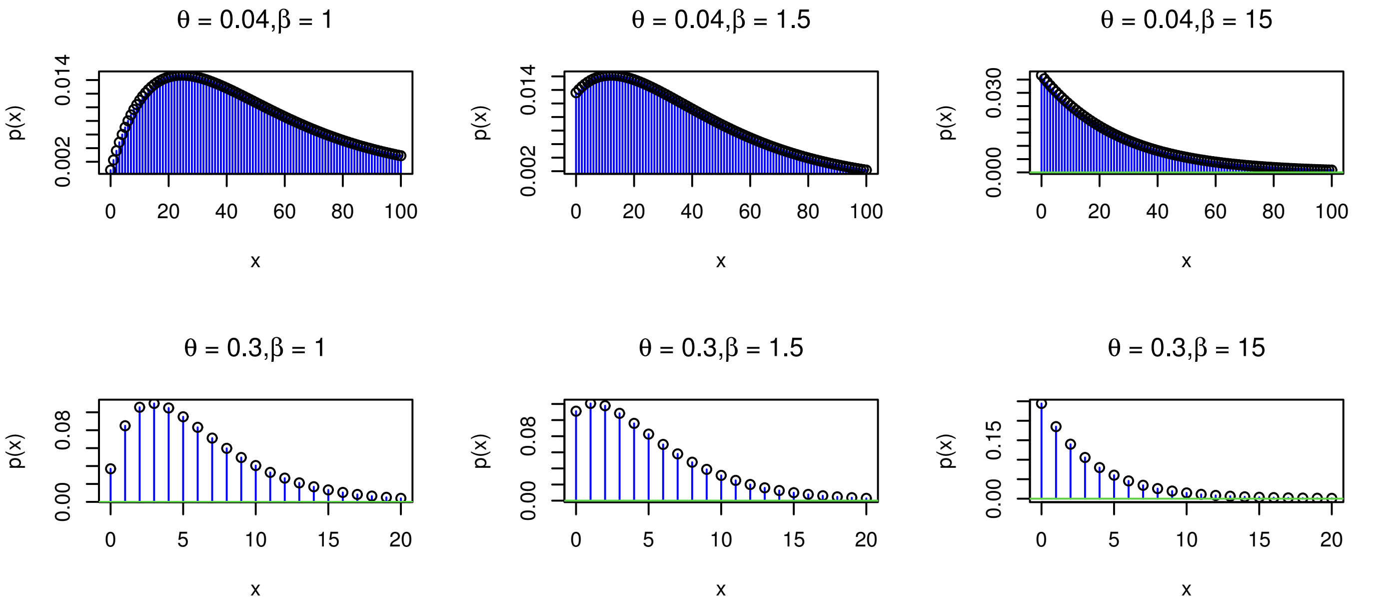

Figure 1. The pmf plots of the DPsL distribution for some set of values for θ and β.

The figure clearly indicates that the DPsL distribution is rightly skewed and has a longer right tail.

A mode of the DPsL distribution, e.g., x_m, is an integer value of x, for which the pmf $P_{\text{DPsL}}(x;\theta,\beta)$ is the maximum. That is $P_{\text{DPsL}}(x;\theta,\beta) \geq P_{\text{DPsL}}(x+1;\theta,\beta)$ and $P_{\text{DPsL}}(x;\theta,\beta) \geq P_{\text{DPsL}}(x-1;\theta,\beta)$, which is equivalent to:

$$\frac{\theta(1+e^{\theta})-\beta(e^{\theta}-1)}{\theta(e^{\theta}-1)} - 1 \leq x_m \leq \frac{\theta(1+e^{\theta})-\beta(e^{\theta}-1)}{\theta(e^{\theta}-1)}.$$

Hence, if $\frac{\theta(1+e^{\theta})-\beta(e^{\theta}-1)}{\theta(e^{\theta}-1)} \geq 0$, and:

1. If $\frac{\theta(1+e^{\theta})-\beta(e^{\theta}-1)}{\theta(e^{\theta}-1)}$ is not an integer, x_m is given as the integer part of $\frac{\theta(1+e^{\theta})-\beta(e^{\theta}-1)}{\theta(e^{\theta}-1)}$;
2. If $\frac{\theta(1+e^{\theta})-\beta(e^{\theta}-1)}{\theta(e^{\theta}-1)}$ is an integer, the DPsL distribution is bimodal, with the modes given by $x_m^{(1)} = \frac{\theta(1+e^{\theta})-\beta(e^{\theta}-1)}{\theta(e^{\theta}-1)}$ and $x_m^{(2)} = \frac{\theta(1+e^{\theta})-\beta(e^{\theta}-1)}{\theta(e^{\theta}-1)} - 1$.

If $\frac{\theta(1+e^{\theta})-\beta(e^{\theta}-1)}{\theta(e^{\theta}-1)} < 0$, the mode of the DPsL distribution is $x_m = 0$.
The hrf of the DPsL distribution can be obtained as:

$$\begin{aligned} h_{\mathrm{DPsL}}(x;\theta,\beta) &= \frac{P_{\mathrm{DPsL}}(x;\theta,\beta)}{1-F_{\mathrm{DPsL}}(x;\theta,\beta)} \\ &= \frac{(\beta+\theta x)e^{-\theta x} - (\beta+\theta(x+1))e^{-\theta(x+1)}}{e^{-\theta(1+x)}(\beta+(x+1)\theta)}. \end{aligned}$$

The hrf of the DPsL distribution was plotted for some set of values for θ and β in Figure 2.

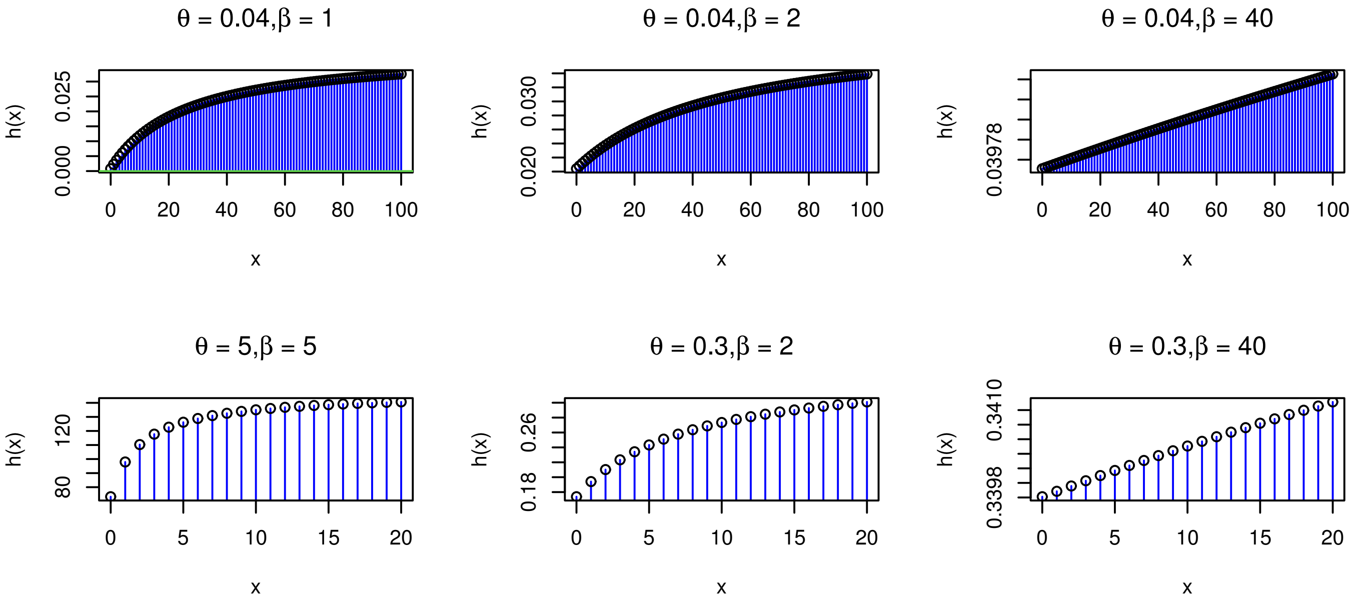

Figure 2. The pmf plots of the DPsL distribution for some set of values for θ and β.

Figure 2 clearly indicates that the hrf of the DPsL distribution is always increasing for different values of the parameters.

2.2. Identifiability

A set of unknown parameters of a model is stated to be identifiable if different sets of parameters give different distributions for a given x. Here, the identifiability property of the DPsL distribution is proved. Let $P_{\mathrm{DPsL}}(x;\lambda_1)$ and $P_{\mathrm{DPsL}}(x;\lambda_2)$ be different pmfs of the DPsL distribution indexed by $\lambda_1 = (\theta_1, \beta_1)$ and $\lambda_2 = (\theta_2, \beta_2)$, respectively. Then, the likelihood ratio is given by:

$$\begin{aligned} \mathrm{U} &= \frac{P_{\mathrm{DPsL}}(x;\lambda_1)}{P_{\mathrm{DPsL}}(x;\lambda_2)} \\ &= \frac{\frac{(\beta_1+\theta_1 x)e^{-\theta_1 x}-(\beta_1+\theta_1(x+1))e^{-\theta_1(x+1)}}{\beta_1}}{\frac{(\beta_2+\theta_2 x)e^{-\theta_2 x}-(\beta_2+\theta_2(x+1))e^{-\theta_2(x+1)}}{\beta_2}} \\ &= \frac{\beta_2}{\beta 1}\frac{(\beta_1+\theta_1 x)e^{-\theta_1 x}-(\beta_1+\theta_1(x+1))e^{-\theta_1(x+1)}}{(\beta_2+\theta_2 x)e^{-\theta_2 x}-(\beta_2+\theta_2(x+1))e^{-\theta_2(x+1)}}. \end{aligned} \tag{5}$$

Taking logarithm of this ratio, we obtained:

$$\begin{aligned} \log \mathrm{U} &= \log\left(\frac{\beta_2}{\beta_1}\right) + \log\Big((\beta_1+\theta_1 x)e^{-\theta_1 x} - (\beta_1+\theta_1(x+1))e^{-\theta_1(x+1)}\Big) \\ &\quad - \log\Big((\beta_2+\theta_2 x)e^{-\theta_2 x} - (\beta_2+\theta_2(x+1))e^{-\theta_2(x+1)}\Big). \end{aligned}$$

Now, by considering x as a continuous variable and taking the partial derivative of log U with respect to x and equating it to 0, we obtained:

$$\frac{\theta_1\big(\theta_1+\beta_1-1+\theta_1 x - e^{\theta_1}(\theta_1 x+\beta_1-1)\big)}{(\beta_1+\theta_1 x)e^{-\theta_1 x}-(\beta_1+\theta_1(x+1))e^{-\theta_1(x+1)}} = \frac{\theta_2\big(\theta_2+\beta_2-1+\theta_2 x - e^{\theta_2}(\theta_2 x+\beta_2-1)\big)}{(\beta_2+\theta_2 x)e^{-\theta_2 x}-(\beta_2+\theta_2(x+1))e^{-\theta_2(x+1)}},$$

which implies that:

$$e^{-(\theta_2-\theta_1)x}\frac{(\beta_2+\theta_2 x)-(\beta_2+\theta_2(x+1))e^{-\theta_2}}{(\beta_1+\theta_1 x)-(\beta_1+\theta_1(x+1))e^{-\theta_1}} = \frac{\theta_2\big(\theta_2+\beta_2-1+\theta_2 x - e^{\theta_2}(\theta_2 x+\beta_2-1)\big)}{\theta_1\big(\theta_1+\beta_1-1+\theta_1 x - e^{\theta_1}(\theta_1 x+\beta_1-1)\big)}.$$

By performing $x \to +\infty$, we obtained $0 = \frac{\theta_2^2(1-e^{\theta_2})}{\theta_1^2(1-e^{\theta_1})}$ or $+\infty = \frac{\theta_2^2(1-e^{\theta_2})}{\theta_1^2(1-e^{\theta_1})}$ according to $\theta_2 > \theta_1$ or $\theta_2 < \theta_1$, respectively, which is impossible since $\theta_1 > 0$ and $\theta_2 > 0$. Therefore, $\theta_1 = \theta_2$. By taking into account this equality, by taking $x = 0$ in (5), we obtained $\frac{\beta_1-(\beta_1+\theta_1)e^{-\theta_1}}{\beta_2-(\beta_2+\theta_1)e^{-\theta_1}} = \frac{\beta_1}{\beta_2}$, which is possible if, and only if, $\beta_1 = \beta_2$. Therefore, we concluded that the DPsL model is identifiable and that the parameters uniquely determine the distribution, that is, $P_{\mathrm{DPsL}}(x;\lambda_1) = P_{\mathrm{DPsL}}(x;\lambda_2) \iff \lambda_1 = \lambda_2$.

2.3. Moments, Skewness and Kurtosis

In the rest of the study, X denotes a random variable that follows the DPsL distribution. Then, the probability generating function (pgf) of X can be derived as:

$$\begin{aligned} G(s) &= \mathrm{E}(s^X) = \sum_{x=0}^{\infty} s^x P_{\mathrm{DPsL}}(x;\theta,\beta) \\ &= \frac{e^{2\theta}\beta - e^{\theta}(\beta+s\beta+\theta-\theta s)+s\beta}{(e^{\theta}-s)^2\beta}, \quad |s| < e^{\theta}. \end{aligned}$$

When s in pgf is substituted by e^t, the moment generating function (mgf) follows as:

$$M(t) = \mathrm{E}(e^{tX}) = \frac{e^{2\theta}\beta - e^{\theta}(\beta+e^t\beta+\theta-\theta e^t)+e^t\beta}{(e^{\theta}-e^t)^2\beta}, \ \ t < \theta.$$

By using the well-known relationship between $M(t)$ and the (standard) moments of X, the first four moments of the DPsL distribution are:

$$\mathrm{E}(X) = \frac{e^{\theta}(\beta+\theta)-\beta}{(e^{\theta}-1)^2\beta}, \tag{6}$$

$$\mathrm{E}(X^2) = \frac{e^{2\theta}\beta+3e^{\theta}\theta+e^{2\theta}\theta-\beta}{(e^{\theta}-1)^3\beta},$$

$$\mathrm{E}(X^3) = \frac{-\beta-3e^{\theta}\beta+3e^{2\theta}\beta+e^{3\theta}\beta+7e^{\theta}\theta+10e^{2\theta}\theta+e^{3\theta}\theta}{\left(e^{\theta}-1\right)^4\beta}$$

and

$$\mathrm{E}(X^4) = \frac{-\beta-10e^{\theta}\beta+10e^{3\theta}\beta+e^{4\theta}\beta+15e^{\theta}\theta+55e^{2\theta}\theta+25e^{3\theta}\theta+e^{4\theta}\theta}{\left(e^{\theta}-1\right)^5\beta}.$$

Based on $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$, the variance of X follows from the Koenig–Huygens formula as:

$$\mathrm{Var}(X) = \frac{e^{\theta}[(e^{\theta}-1)^2\beta^2+(e^{2\theta}-1)\beta\theta-e^{\theta}\theta^2]}{(e^{\theta}-1)^4\beta^2}. \tag{7}$$

Expressions for skewness and kurtosis of the DPsL distribution can be derived explicitly by using the following formulas:

$$\mathrm{Skewness}(X) = \frac{\mathrm{E}\left(X^3\right)-3\mathrm{E}\left(X^2\right)\mathrm{E}(X)+2[\mathrm{E}(X)]^3}{[\mathrm{Var}(X)]^{3/2}}$$

and

$$\mathrm{Kurtosis}(X) = \frac{\mathrm{E}\left(X^4\right)-4\mathrm{E}\left(X^2\right)\mathrm{E}(X)+6\mathrm{E}\left(X^2\right)[\mathrm{E}(X)]^2-3[\mathrm{E}(X)]^4}{[\mathrm{Var}(X)]^2}.$$

2.4. Coefficient of Variation and Dispersion Index

The expressions of the coefficient of variation (CV) and dispersion index (DI) of X are given by:

$$CV(X) = \frac{\sqrt{\mathrm{Var}(X)}}{\mathrm{E}(X)} = \frac{\sqrt{(e^{\theta}-1)^2\beta^2+(e^{2\theta}-1)\beta\theta-e^{\theta}\theta^2}}{\sqrt{e^{\theta}}\left((\beta+\theta)-\beta e^{-\theta}\right)}$$

and

$$DI(X) = \frac{\mathrm{Var}(X)}{\mathrm{E}(X)} = \frac{(e^{\theta}-1)^2\beta^2+(e^{2\theta}-1)\beta\theta-e^{\theta}\theta^2}{(e^{\theta}-1)^2\beta e^{\theta}(\beta+\theta)}, \tag{8}$$

respectively.

In full generality, when the DI is one, the distribution is equi-dispersed, and if DI is greater than (less than) one, the distribution is over-dispersed (under-dispersed). Some numerical values of the mean, variance, DI, skewness and kurtosis for the DPsL distribution for some values of the parameters are presented in Tables 1 and 2.

From the information contained in these tables, it is clear that the DPsL distribution would be an appropriate option for modelling under as well as over-dispersed and positively skewed datasets.

Table 1. Values for some moment measures for the DPsL distribution for $\beta = 1.5$ and different values of θ.

	θ				
Measures	**4**	**5**	**6**	**7**	**8**
Mean	0.06934	0.02955	0.01245	0.00518	0.00213
Variance	0.06901	0.02939	0.01241	0.00517	0.00212
DI	0.99525	0.99447	0.99649	0.99820	0.99911
Skewness	3.77540	5.77052	8.91577	13.86130	21.65950
Kurtosis	17.19030	35.95970	81.94370	194.42800	471.29900

Table 2. Values for some moment measures for the DPsL distribution for $\theta = 2$ and different values of β.

	β					
Measures	**10**	**11**	**12**	**13**	**14**	**15**
Mean	0.19272	0.18943	0.18669	0.18437	0.18238	0.18065
Variance	0.22724	0.22315	0.21972	0.21681	0.21430	0.21212
DI	1.17912	1.17799	1.17694	1.17595	1.17504	1.17420
Skewness	2.84454	2.86632	2.88457	2.90007	2.91340	2.92497
Kurtosis	12.9041	13.05360	13.17890	13.28540	13.3768	13.4562

2.5. Mean Residual Lifetime and Mean Past Lifetime

The mean residual lifetime (mrl) and mean past lifetime (mpl) of a component are two widely used measures to study the ageing behaviour of components. Both measures characterize the distribution uniquely. By assuming that the lifetime of a component is modelled by X, the mrl of X at $i = 0, 1, 2, \ldots$ is defined as:

$$\begin{aligned}\zeta(i) &= \mathrm{E}(X - i \mid X \geq i) \\ &= \frac{1}{1 - F_{\mathrm{DPsL}}(i-1;\theta,\beta)} \sum_{j=i+1}^{\infty} (1 - F_{\mathrm{DPsL}}(j-1;\theta,\beta)).\end{aligned}$$

That is:

$$\begin{aligned}\zeta(i) &= \frac{1}{e^{-\theta i}(\beta + \theta i)} \sum_{j=i+1}^{\infty} e^{-\theta j}(\beta + \theta j) \\ &= \frac{e^{i\theta}\left((e^{\theta} - 1)\beta - i\theta + e^{\theta}(1+i)\theta\right)}{e^{-\theta i}(\beta + \theta i)(e^{\theta} - 1)^2}.\end{aligned}$$

Furthermore, the mpl of X is another reliability measure that corresponds to the time elapsed since the failure of X given that the system has already failed before some i. Thus, the mpl of X at $i = 1, 2, \ldots$ is defined by:

$$\begin{aligned}\zeta^*(i) &= \mathrm{E}(i - X \mid X < i) \\ &= \frac{1}{F_{\mathrm{DPsL}}(i-1;\theta,\beta)} \sum_{m=1}^{i} F_{\mathrm{DPsL}}(m-1;\theta,\beta),\end{aligned}$$

where $\zeta^*(0) = 0$. That is:

$$\begin{aligned}
\zeta^*(i) &= \frac{1}{\beta - e^{-\theta i}(\beta + i\theta)} \sum_{m=1}^{i} \left(\beta - e^{-m\theta}(\beta + m\theta)\right) \\
&= \frac{e^{-i\theta}}{\beta - e^{-\theta i}(\beta + i\theta)(e^{\theta} - 1)^2} \times \\
&\quad e^{-i\theta}\left\{\left[(e^{\theta} - 1)(1 + e^{\theta i}(1+i) - e^{\theta i}(1+i))\right]\beta - \left[e^{\theta(1+i)} + i - e^{\theta}(1+i)\right]\theta\right\}.
\end{aligned}$$

2.6. Stress–Strength Analysis

Stress–strength reliability has wide applications in almost all fields of engineering and machine learning. Let X_{stress} and $X_{strength}$ be random variables that model the stress and strength of a system, respectively. Then, the expected reliability can be calculated by the following formula:

$$Re_{Stress-Strength} = \Pr\left[X_{Stress} \leq X_{Strength}\right] = \sum_{x=0}^{\infty} P_{X_{Stress}}(x) S_{X_{Strength}}(x),$$

where $P_X(x)$ and $S_X(x)$ denote the pmf and sf, respectively, of a random variable X. Suppose that X_{stress} and $X_{strength}$ are two independent random variables following the DPsL (θ_1, β_1) and DPsL (θ_2, β_2) distributions, respectively. Then, from (3) and (4), the expected reliability is obtained in closed form as:

$$\begin{aligned}
&Re_{Stress-Strength} = \\
&\frac{1}{\beta_1\beta_2(e^{\theta_1+\theta_2} - 1)^3}\left\{(e^{\theta_1} - 1)(e^{\theta_1+\theta_2} - 1)\beta_1\left[(e^{\theta_1+\theta_2} - 1)\beta_2 + \theta_2 e^{\theta_1+\theta_2}\right]\right. \\
&\left.-\theta_1 e^{\theta_1}\left[(e^{\theta_2} - 1)(e^{\theta_1+\theta_2} - 1)\beta_2 + e^{\theta_2}(1 - 2e^{\theta_1} + e^{\theta_1\theta_2})\theta_2\right]\right\}.
\end{aligned}$$

Some numerical values for $Re_{Stress-Strength}$ for different values of the parameters are given in Tables 3–5.

From Tables 3 and 4, it is clear that the expected reliability increases (decreases) as $\beta_1 \to \infty$ ($\beta_2 \to \infty$). In addition, from Table 5, the expected reliability (decreases) as $\theta_1 \to \infty$ ($\theta_2 \to \infty$).

Table 3. Numerical values of $Re_{Stress-Strength}$ associated with the DPsL distribution at $\theta_1 = 0.3$, $\theta_2 = 0.1$ for different values of β_1 and β_2.

	$\boldsymbol{\theta_1 = 0.3, \theta_2 = 0.1}$			
$\boldsymbol{\beta_1 \to}$ $\boldsymbol{\beta_2 \downarrow}$	**1**	**2**	**3**	**7**
1	0.82926	0.87819	0.89449	0.91314
2	0.6227	0.75075	0.77358	0.79967
3	0.63327	0.70827	0.73327	0.76184
7	0.57728	0.65972	0.68721	0.71862

Table 4. Numerical values of $Re_{Stress-Strength}$ associated with the DPsL distribution at $\theta_1 = 0.6$, $\theta_2 = 0.01$ for different values of β_1 and β_2.

	$\theta_1 = 0.6, \theta_2 = 0.01$			
$\beta_1 \rightarrow$ $\beta_2 \downarrow$	**1**	**2**	**3**	**7**
1	0.99903	0.99933	0.99943	0.99955
2	0.98084	0.98488	0.98623	0.98777
3	0.97478	0.98007	0.98183	0.98384
7	0.96785	0.97456	0.97679	0.97935

Table 5. Numerical values of $Re_{Stress-Strength}$ associated with the DPsL distribution at $\beta_1 = 1$, $\beta_2 = 1.5$ for different values of θ_1 and θ_2.

	$\beta_1 = 1, \beta_2 = 1.5$			
$\theta_1 \rightarrow$ $\theta_2 \downarrow$	**0.1**	**0.5**	**0.7**	**0.9**
0.1	0.40431	0.82936	0.87387	0.89879
0.5	0.04949	0.35792	0.45733	0.52947
0.7	0.02667	0.24765	0.33651	0.40671
0.9	0.01619	0.17754	0.25273	0.31061

2.7. Generating Random Values from the DPsL Distribution

Random values from the DPsL distribution can be generated by following the algorithm given below.

1. Generate u as a realization of a random variable U with the $U(0,1)$ distribution.
2. With the expression of the quantile function of the PsL distribution in mind, compute:

$$y = -\frac{\beta}{\theta} - \frac{1}{\theta} W_{-1}(e^{-\beta}\beta(u-1)),$$

   where $W_{-1}(x)$ denotes the negative branch of Lambert–W function.
3. Then, $x = \lfloor y \rfloor$ represents a realization of a random variable with the DPsL distribution.

To generate a random sample of size n, repeat the algorithm n times.

3. Estimation Methods

The estimation of unknown parameters of a distribution is critical in accurately determining the behaviour of this distribution. Here, we use classical methods of estimation such as the method of maximum likelihood (mle) and weighted least square (wls) estimation for this purpose.

3.1. Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample taken from the DPsL (θ, β) distribution, and $x_1, x_2, \ldots, x_n$ be observations of this random sample. The likelihood function is given by:

$$\mathrm{L} = \left(\frac{1}{\beta}\right)^n \left\{ \prod_{i=1}^{n} \left[(\beta + \theta x_i) e^{-\theta x_i} - (\beta + \theta(x_i + 1)) e^{-\theta(x_i+1)} \right] \right\}$$

and the log likelihood function is given by:

$$\log \mathrm{L} = -n \log \beta + \sum_{i=1}^{n} \log \Big[(\beta + \theta x_i) e^{-\theta x_i} - (\beta + \theta (x_i + 1)) e^{-\theta (x_i + 1)} \Big].$$

Then, the maximum likelihood estimates (MLEs) of θ and β were obtained by maximizing L or $\log \mathrm{L}$ with respect to these parameters. They can also be determined as the solutions of the normal equations given by:

$$\frac{\partial \log \mathrm{L}}{\partial \theta} = 0 \implies \sum_{i=1}^{n} \frac{e^{-\theta(2x_i+1)} [e^{\theta x_i}(x_i+1)(\theta x_i + \theta + \beta - 1) - e^{\theta(x_i+1)} x_i (\theta x_i + \beta - 1)]}{(\beta + \theta x_i) e^{-\theta x_i} - (\beta + \theta(x_i+1)) e^{-\theta(x_i+1)}} = 0 \tag{9}$$

and

$$\frac{\partial \log \mathrm{L}}{\partial \beta} = 0 \implies -\frac{n}{\beta} + \sum_{i=1}^{n} \frac{e^{-\theta x_i} - e^{\theta(x_i+1)}}{(\beta + \theta x_i) e^{-\theta x_i} - (\beta + \theta(x_i+1)) e^{-\theta(x_i+1)}} = 0. \tag{10}$$

Equations (9) and (10) can be solved by numerical optimization techniques using mathematical software such as MATHEMATICA, MATHCAD and R.

3.2. Weighted Least Squares Estimation

Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the order statistics of a random sample taken from the DPsL (θ, β) distribution, and $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be observations of these random variables. The weighted least squares estimates (WLEs) of the parameters θ and β of the DPsL distribution were obtained by maximizing the following function with respect to θ and β:

$$W = \sum_{i=1}^{n} \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F_{\mathrm{DPsL}}\left(x_{(i)}; \theta, \beta \right) - \frac{i}{n+1} \right]^2.$$

3.3. Simulation Study

The current section deals with examining the efficiency of two estimation methods for estimating the parameters of the DPsL distribution using simulation. Estimates were calculated for different values of parameters $((\theta = 0.5, \beta = 1)$ and $(\theta = 2.2, \beta = 1.5))$ for various sample sizes $(25, 50, 75, 100)$ using the two estimation methods discussed and, thus, compared. Then, $N = 1000$ samples of values from the DPsL distribution using methods discussed in Section 2.7 were generated. The indices such as values of the estimates, mean square errors (MSEs), average absolute biases (Bias) and average mean relative estimates (MREs) were calculated in R software using the following formulas:

$$\mathrm{MSE} = \frac{1}{N} \sum_{i=1}^{N} (\hat{\zeta}_i - \zeta)^2, \quad \mathrm{Bias} = \frac{1}{N} \sum_{i=1}^{N} |\hat{\zeta}_i - \zeta|,$$
$$\mathrm{MRE} = \frac{1}{N} \sum_{i=1}^{N} \frac{|\hat{\zeta}_i - \zeta|}{\zeta},$$

where $\zeta = \theta$ or β, and the index i refers to the ith sample. Simulation results, including values of estimates, Bias, MSEs and MREs for the two parameters θ and β of the DPsL distribution using the estimation approaches discussed, are reported in Tables 6 and 7.

Table 6. Simulation results of our estimation approaches for the DPsL distribution with $\theta = 0.5, \beta = 1$.

n	Indices	MLE		WLSE	
		θ	β	θ	β
25	Estimates	0.4902	1.1126	0.4289	1.0049
	Bias	0.0098	0.1126	0.0710	0.0049
	MSE	0.0069	0.1217	0.0448	7.1204×10^{-5}
	MRE	0.1319	0.1326	0.2599	0.0049
50	Estimates	0.4904	1.0808	0.4243	1.0035
	Bias	0.0096	0.0808	0.0757	0.0035
	MSE	0.0033	0.0379	0.0444	3.26×10^{-5}
	MRE	0.0908	0.0868	0.2444	0.0035
75	Estimates	0.4920	1.0614	0.4247	1.0030
	Bias	0.0079	0.0614	0.0753	0.0030
	MSE	0.0019	0.0160	0.0429	2.217×10^{-5}
	MRE	0.0704	0.0614	0.2328	0.0030
100	Estimates	0.4926	1.0553	0.4225	1.0028
	Bias	0.0074	0.0553	0.0775	0.0028
	MSE	0.0015	0.0119	0.0427	1.904×10^{-5}
	MRE	0.0634	0.0553	0.2350	0.0028

Table 7. Simulation results of our estimation approaches for the DPsL distribution with $\theta = 2.2$, $\beta = 1.5$.

n	Indices	MLE		WLSE	
		θ	β	θ	β
25	Estimates	2.3027	1.2005	1.7547	1.3939
	Bias	0.1027	0.2995	0.4452	0.1060
	MSE	1.5197	0.2979	0.2564	0.0154
	MRE	0.2509	0.3328	0.2079	0.0734
50	Estimates	2.1843	1.2621	1.8200	1.3993
	Bias	0.0157	0.2378	0.3799	0.1007
	MSE	0.2381	0.2774	0.1932	0.0134
	MRE	0.1829	0.3193	0.1749	0.0681
75	Estimates	2.1853	1.3217	1.8370	1.4066
	Bias	0.0147	0.1783	0.3629	0.0934
	MSE	0.1565	0.2519	0.1750	0.0118
	MRE	0.1457	0.2949	0.1689	0.0639
100	Estimates	2.2052	1.4245	1.8489	1.4133
	Bias	0.0052	0.0755	0.3511	0.0867
	MSE	0.0993	0.2468	0.1627	0.0105
	MRE	0.1154	0.2784	0.1642	0.0598

From the above tables, it is clear that, for estimating θ, the corresponding MLE performed well, and for β, the corresponding WLSE outperformed the MLE.

4. INAR(1) Process with DPsL Innovations

Numerous fields, such as agriculture, epidemiology, actuarial science, finance, etc., have come across certain time series of counts. Analysing these kinds of datasets using the INAR(1) process was first applied using Poisson innovations by [12,13]. Suppose that $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are the innovations, so are independent and identically distributed (iid) random variables, with $\mathrm{E}(\varepsilon_t) = \mu_\varepsilon$ and variance $\mathrm{Var}(\varepsilon_t) = \sigma_\varepsilon^2$. A stochastic process $\{X_t\}_{t\in\mathbb{Z}}$ defined as:

$$X_t = p \circ X_{t-1} + \varepsilon_t,$$

with $0 \le p < 1$, is stated to be an INAR(1) process. The symbol $\circ$ is called as binomial thinning operator, which can be described as:

$$p \circ X_{t-1} = \sum_{j=1}^{X_{t-1}} U_j,$$

where $\{U_j\}_{j\in\mathbb{Z}}$ is a sequence of iid Bernoulli random variables with parameter p. The one step transition probability of the INAR(1) process is given by:

$$\Pr(X_t = k \mid X_{t-1} = l) = \sum_{i=1}^{\min(k,l)} \Pr(B = i)\Pr(\varepsilon_t = k - i),\ k, l \ge 0,$$

where B denotes a random variable following the Binomial (n, p) distribution. The mean, variance and dispersion index (DI) of $\{X_t\}_{t\in\mathbb{Z}}$ are given by [21]. They are:

$$\mathrm{E}(X_t) = \frac{\mu_\varepsilon}{1-p}, \tag{11}$$

$$\mathrm{Var}(X_t) = \frac{p\mu_\varepsilon + \sigma_\varepsilon^2}{1-p^2} \tag{12}$$

and

$$\mathrm{DI}(X_t) = \frac{\mathrm{DI}_\varepsilon + p}{1+p}, \tag{13}$$

where μ_ε, σ_ε^2 and DI_ε are the mean, variance and DI of the innovation distribution. The results of [12,13] influenced us to propose a new INAR(1) process with DPsL innovations, which are capable of modelling over as well as under-dispersed count datasets. Suppose that $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ follow a DPsL distribution; then, the one step transition probability matrix of the corresponding process is:

$$\begin{aligned}
&\Pr(X_t = k \mid X_{t-1} = l) = \\
&\sum_{i=1}^{\min(k,l)} \binom{l}{i} p^i (1-p)^{l-i} \\
&\times \frac{(\beta + \theta(k-i))e^{-\theta(k-i)} - (\beta + \theta((k-i)+1))e^{-\theta((k-i)+1)}}{\beta},
\end{aligned}$$

which hereafter is called the INAR(1)DPsL process. By substituting μ_ε, σ_ε^2, and DI_ε in (11)–(13) with (6)–(8), the mean, variance and DI of the INAR(1)DPsL process could be attained. The conditional expectation and variance of the INAR(1)DPsL process are given by:

$$\mathrm{E}(X_t \mid X_{t-1}) = pX_{t-1} + \mu_\varepsilon, \tag{14}$$

and

$$\mathrm{Var}(X_t \mid X_{t-1}) = p(1-p)X_{t-1} + \sigma_\varepsilon^2, \tag{15}$$

respectively, where μ_ε and σ_ε^2 are given in (6) and (7), respectively (see [13,21]).

4.1. Estimation

Here, the inference of the INAR(1)DPsL process was examined using two estimation methods: the conditional maximum likelihood (CML) and Yule–Walker (YW) methods. A simulation study was performed to assess the efficiency of the two methods.

4.1.1. Conditional Maximum Likelihood

Let $X_1, X_2, \ldots, X_T$ be a random sample taken from the INAR(1)DPsL process, and $x_1, x_2, \ldots, x_T$ be observations of this random sample. Then, the conditional log likelihood function of the INAR(1)DPsL process is given by:

$$\begin{aligned}\ell(\Theta) &= \sum_{t=2}^{T} \log[\Pr(X_t = x_t \mid X_{t-1} = x_{t-1})] \\ &= \sum_{t=2}^{T} \log\Bigg[\sum_{i=1}^{\min(x_t, x_{t-1})} \binom{x_{t-1}}{i} p^i (1-p)^{x_{t-1}-i} \\ &\qquad \frac{(\beta + \theta(x_t - i))e^{-\theta(x_t - i)} - (\beta + \theta(x_t - i + 1))e^{-\theta(x_t - i + 1)}}{\beta}\Bigg],\end{aligned} \tag{16}$$

where $\Theta = (\theta, \beta, p)$ is the vector of unknown parameters to be estimated. Maximizing (16) with respect to Θ yields the CML estimates (CMLEs). In this regard, we used the optim-function in R software for the same. In addition, the fdHess function in R was used to obtain the observed information matrix and, hence, the standard errors (SE) of estimates of parameters in the INAR(1)DPsL process.

4.1.2. Yule–Walker

The YW estimates (YWEs) of the INAR(1)DPsL process were computed by solving simultaneous equations of sample and theoretical moments. Since the autocorrelation function (ACF) of the INAR(1) process at lag h was $\rho_x(h) = p^h$, the YWE of p is given by:

$$\hat{p}_{YW} = \frac{\sum_{t=2}^{T} (x_t - \bar{x})(x_{t-1} - \bar{x})}{\sum_{t=1}^{T} (x_t - \bar{x})^2}.$$

Now, the YWEs for θ and β were obtained by solving the equations of sample mean equals theoretical mean and sample dispersion equals theoretical dispersion of the process. Here, by denoting as $\hat{\theta}_{YW}$ and $\hat{\beta}_{YW}$ the YWEs of θ and β, respectively, the following relationship holds:

$$\hat{\beta}_{YW} = \frac{\hat{\theta}_{YW} e^{\hat{\theta}_{YW}}}{\overline{x}(1 - \hat{p}_{YW})(e^{\hat{\theta}_{YW}} - 1)^2 - (e^{\hat{\theta}_{YW}} - 1)}, \tag{17}$$

where $\overline{x} = \sum_{t=1}^{T} x_t / N$. Substituting $\hat{\beta}_{YW}$ with (17) in (13) and equating (13) to sample dispersion, we obtained $\hat{\theta}_{YW}$.

4.2. Simulation of INAR(1)DPsL Process

Here, a simulation study was conducted to comprehensively determine the performance of CMLEs and YWEs of the parameters of the INAR(1)DPsL process. In this regard, we generated $N = 1000$ samples each of sizes $n = 25, 50, 100$ from the proposed distribution for two sets of parameter values ($\theta = 0.1, \beta = 1.1$ and $\theta = 3, \beta = 4$). For each n, average absolute bias, MSE and MRE for the parameters were calculated for the two methods. The simulation results are presented in Table 8.

Table 8. Simulation results of the INAR(1)DPsL process.

$\theta = 0.1, \beta = 1.1$							
Sample Size (n)	**Parameters**	**CML**			**YW**		
		Bias	**MSE**	**MRE**	**Bias**	**MSE**	**MRE**
25	θ	0.0183	0.0019	0.3271	0.0644	0.0047	0.6443
	β	0.2067	1.8986	0.9959	0.1305	0.2778	0.1186
	p	0.0449	0.0248	0.4289	0.6456	0.2627	2.1519
50	θ	0.0035	0.0007	0.1758	0.0633	0.0043	0.6330
	β	0.0916	0.3807	0.4131	0.0687	0.0881	0.0624
	p	0.0113	0.1187	0.2345	0.0232	0.0255	0.0773
100	θ	0.0014	0.0001	0.0841	0.0623	0.0040	0.6225
	β	0.0657	0.0178	0.0732	0.0369	0.0351	0.0336
	p	0.0096	0.0072	0.1812	0.0200	0.0019	0.0668
$\theta = 3, \beta = 4$							
Sample Size (n)	**Parameters**	**CML**			**YW**		
		Bias	**MSE**	**MRE**	**Bias**	**MSE**	**MRE**
25	θ	0.7181	0.0853	1.6194	0.6708	1.1252	0.2236
	β	0.5259	0.2878	0.6254	0.1634	0.0276	0.0408
	p	0.0344	0.0502	0.2546	0.3809	0.5484	0.5441
50	θ	0.5244	0.0824	1.2841	0.5281	0.9221	0.1764
	β	0.0461	0.0434	0.4046	0.1609	0.0263	0.0402
	p	0.0054	0.0382	0.2157	0.2889	0.5318	0.4128
100	θ	0.0709	0.0816	0.3019	0.2791	0.1449	0.0930
	β	0.0363	0.0241	0.2953	0.1606	0.0260	0.0402
	p	0.0032	0.0282	0.1813	0.2553	0.0624	0.3647

From the above table, we observed that the average biases, MSEs and MREs of CMLEs tended to zero quicker than those of YWEs, making them efficient for small as well as large sample sizes. Therefore, the CML estimation was preferred to attain unknown parameters of the INAR(1)DPsL process.

5. Empirical Study

Three real datasets were used in this section to illustrate the performance of the DPsL distribution over some competitive distributions. The capability of the fitted distributions was compared using the goodness of fit criterion with its corresponding p-value.

5.1. Failure Times

The data of failure times for a sample of 15 electronic components in an acceleration life test (see [22]) were considered here. These data were based on the discretization concept. Adopting a data analysis setting, we compared the DPsL, discrete three-parameter Lindley (DTPL) (see [9]), discrete log-logistic (DLL) (see [23]), discrete inverse Weibull (DIW) (see [2]), discrete Burr–Hutke (DBH) (see [6]), discrete Pareto (DP) (see [3]), Poisson (P) and geometric (G) distributions. The MLEs with standard errors (SEs) and confidence intervals (CIs) for the parameter(s), estimated $-$log Likelihood ($-$L), Akaike information criterion (AIC), Bayesian information criterion (BIC) and goodness of fit statistic (Kolmogorov statistic (K-S) and p-value) of these distributions for this dataset are given in Table 9.

Table 9. The MLEs, CIs, $-$L, AIC, BIC, K-S and p-values of all the fitted distributions for the failure times data.

	Model			
Statistic	**DPsL**	**DTPL**	**DLL**	**DIW**
θ MLE (SE)	0.0623 (0.0043)	0.5084 (0.8277)	21.4627 (1.392)	0.0077 (0.0032)
CI	(0.0538, 0.0707)	($-$1.1139, 2.1307)	(18.7344, 24.1909)	(0.0013, 0.0140)
β MLE (SE)	1.3427 (0.1572)	0.0924 (0.1506)	1.7906 (0.1001)	0.7111 (0.0343)
CI	(1.0331, 1.6492)	(0.0629, 0.1219)	(1.5943, 1.9868)	(0.6439, 0.7782)
λ MLE (SE)	–	0.9397 (0.0040)	–	–
CI	–	(0.0845, 0.1003)	–	–
$-$L	64.2790	64.2790	65.6904	70.4214
AIC	132.558	134.558	135.3809	144.8427
BIC	133.9741	136.6822	136.797	146.2588
K-S value	0.1114	0.1116	0.1351	0.2194
p-value	0.9819	0.9816	0.9133	0.4068
	Model			
Statistic	**DBH**	**DP**	**P**	**G**
θ MLE (SE)	0.999 (0.0019)	0.7202 (0.0158)	27.535 (0.3498)	0.035 (0.0023)
CI	(0.9953, 1.0030)	(0.6893, 0.7511)	(26.8495, 28.2208)	(0.0305, 0.0395)
$-$L	91.3684	77.4023	151.2064	66.0001
AIC	184.7368	156.8047	304.4129	133.0002
BIC	185.4448	157.5127	305.1209	134.7083
K-S value	0.7912	0.4053	0.3815	0.1766
p-value	1.582×10^{-10}	0.0097	0.0179	0.6743

From Table 9, it is evident that, besides the DPsL distribution, the DTPL, G and DLL distributions also performed quite well, but it is clear that the DPsL distribution was the best among them, since it had the lowest K-S, AIC and BIC, with a higher p-value. In order to illustrate this claim, Figure 3 provides the probability–probability (P–P) plots, and Figure 4 displays the estimated cdfs of the fitted distributions.

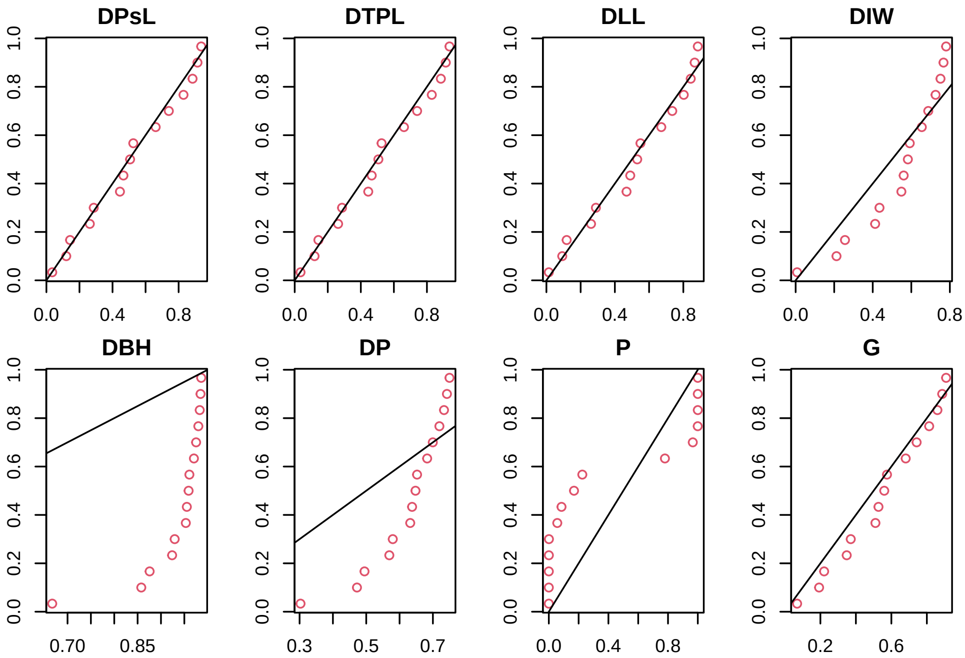

Figure 3. The P–P plots for the fitted distributions using the failure times data.

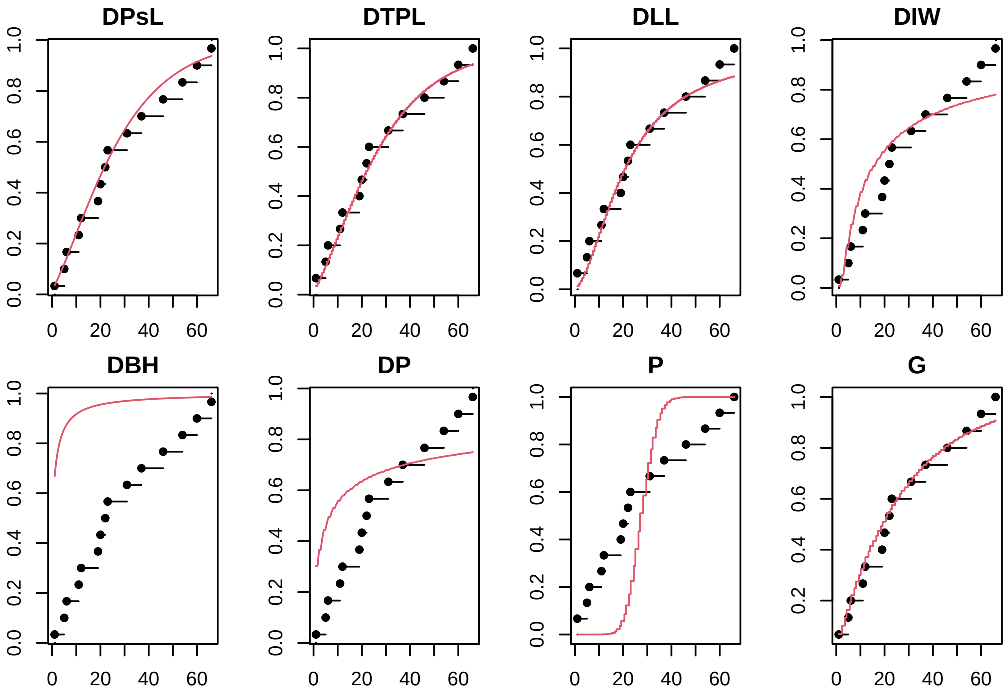

Figure 4. Estimated cdfs of the fitted distributions using the failure times data.

From the above figures, we could infer that the DPsL distribution yielded a better fit among other fitted distributions. Table 10 completes these results by presenting some descriptive measures of the fitted DPsL distribution. Hence, it is evident that the fitted DPsL distribution was over dispersed, moderately right skewed and leptokurtic.

Table 10. Values of some descriptive statistics of the DPsL distribution for the failure times data.

Mean	Variance	DI	Skewness	Kurtosis
27.8667	395.5822	14.1955	0.7020	2.3149

5.2. Numbers of Borers

The second dataset was the biological experiment data, which represented the number of European corn borer (No. ECB) larvae Pyrausta in the field (see [24]). It was an experiment conducted randomly on eight hills in 15 replications, and the experimenter counted the number of borers per hill of corn. The fits of the DPsL distribution were compared together with some competitive distributions which were the new Poisson weighted exponential (NPWE) (see [16]), DIW, discrete Burr-XII (DBXII) (see [23]), discrete Bilal (DBl) (see [8]), DP, DBH and Poisson (P) distributions. The MLEs with their corresponding SEs, CIs under the form (lower bound of the CI (LCI), upper bound of the CI (UCI)) for the parameter(s) and goodness of fit test for the numbers of borers dataset are reported in Table 11.

Table 11. The MLE, LCI, UCI, −L, AIC, BIC, χ^2 and p-values for the one parameter distributions considered using the number of borers dataset.

X	Observed Frequency	Expected Frequency							
		DPsL	**NPWE**	**DIW**	**DBXII**	**DBl**	**DP**	**DBH**	**P**
0	43	44.62	48.32	41.37	43.84	32.74	64.45	68.07	27.22
1	35	30.46	28.86	41.85	39.61	39.59	20.15	21.97	40.38
2	17	19.07	17.24	15.42	15.62	24.27	9.69	10.51	29.95
3	11	11.34	10.29	7.17	7.20	12.50	5.65	5.98	14.81
4	5	6.51	6.15	3.94	3.91	5.97	3.68	3.75	5.49
5	4	3.65	3.67	2.42	2.37	2.74	2.58	2.51	1.63
6	1	2.01	2.19	1.61	1.59	1.23	1.90	1.75	0.40
7	2	1.09	1.31	1.13	1.09	0.54	1.46	1.26	0.09
8	2	1.25	1.94	5.09	4.80	0.24	1.15	0.93	0.02
Total	120	120	120	120	120	120	120	120	120
θ	MLE	0.7219	0.1434	0.345	0.519	0.6565	0.3292	0.8654	1.4834
	SE	0.0122	0.2945	0.043	0.051	0.0017	0.0031	0.0035	0.0101
	LCI	0.6980	0	0.261	0.419	0.6532	0.3232	0.8585	1.4635
	UCI	0.7459	0.4339	0.429	0.619	0.6599	0.3352	0.8723	1.5033
β	MLE	2.4635	0.5896	1.541	2.358				
	SE	0.1367	1.3706	0.156	0.3656				
	LCI	2.1956	0	1.235	1.641				
	UCI	2.7315	3.2760	1.847	3.074				
−L		200.4152	200.8774	204.812	204.293	204.6753	220.6182	214.0490	219.1879
AIC		404.8303	405.7548	413.624	412.586	411.3505	443.2363	430.0979	440.3759
BIC		410.4053	411.3297	419.199	418.161	414.138	446.0238	432.8854	443.1634
χ^2		1.4445	2.1591	5.511	4.664	10.0780	26.645	25.795	38.583
Degrees of freedom		3	3	3	3	4	4	4	4
p-value		0.9194	0.8267	0.138	0.198	0.0731	<0.001	<0.001	<0.001

From the above table, it is evident that, besides the DPsL distribution, the NPWE distribution also performed quite well, but it is clear that the DPsL distribution was the best among them, since it had the lowest −L, AIC, BIC and χ^2 value with the highest p-value.

From Figure 5, we could infer that the DPsL distribution yielded a better fit among other fitted distributions. To complete this, Table 12 contains some descriptive measures of the fitted DPsL distribution. Hence, here also, it is evident that the fitted DPsL distribution was over-dispersed, moderately right skewed and leptokurtic.

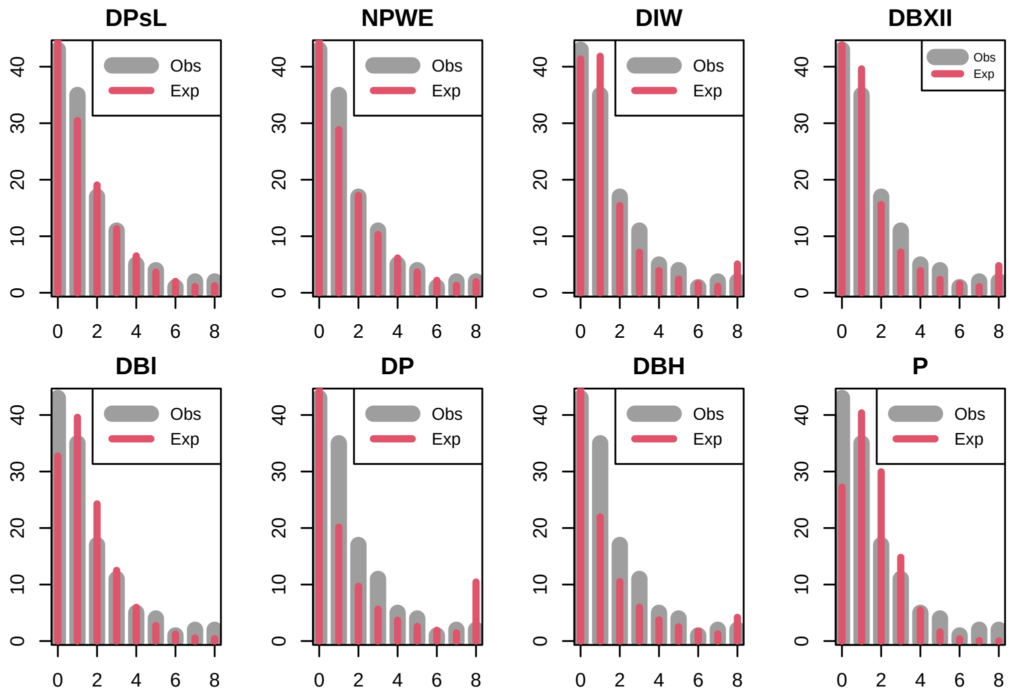

Figure 5. The estimated pmfs of the fitted distributions for the number of borers dataset.

Table 12. Values of some descriptive statistics of the DPsL distribution for the number of borers dataset.

Mean	Variance	DI	Skewness	Kurtosis
1.5917	2.6249	1.6491	0.8172	2.6435

5.3. Numbers of Claims

In this part, a comparison of the performance of the INAR(1)DPsL process with the INAR(1)DTPL (see [7]), INAR(1)NPWE (see [16]), INAR(1)DPLi (see [15]) and INAR(1)G (see [14]) processes was conducted. The one-step translation probabilities of the competitive INAR(1) processes were given as follows:

1. For the INAR(1)DPLi process:

$$\Pr(X_t = k \mid X_{t-1} = l) = \sum_{i=0}^{\min(k,l)} \binom{l}{i} p^i (1-p)^{l-i} \frac{\theta^2 (k - i + \theta + 2)}{(\theta + 1)^{k-i+3}}, \ \theta > 0.$$

2. For the INAR(1)DTPL process:

$$\begin{aligned} &\Pr(X_t = k \mid X_{t-1} = l) = \\ &\sum_{i=1}^{\min(k,l)} \binom{l}{i} p^i (1-p)^{l-i} \\ \times \ &\frac{\lambda^{k-i}\{\beta(\lambda(\log(\lambda) - 1) + 1) + (\lambda - 1)\log(\lambda)(\alpha + \beta(k-i))\}}{\beta - \alpha \log(\lambda)}, \\ &0 < \lambda < 1, \alpha\theta + \beta > 0, \theta = -\log(\lambda). \end{aligned}$$

3. For the INAR(1)NPWE process:

$$\Pr(X_t = k \mid X_{t-1} = l) = \sum_{i=0}^{\min(k,l)} \binom{l}{i} p^i (1-p)^{l-i} \alpha(1+\theta)(1+\alpha+\alpha\theta)^{-(k-i)-1},$$
$$\alpha > 0, \theta > 0.$$

4. For the INAR(1)G process:

$$\Pr(X_t = k \mid X_{t-1} = l) = \sum_{i=1}^{\min(k,l)} \binom{l}{i} p^i (1-p)^{l-i} \left[\alpha(1-\alpha)^{k-i}\right],$$
$$0 < \alpha < 1.$$

The third data we used here were to illustrate the application of the DPsL distribution in the INAR(1) process. Originally, the data were studied by [25], which consisted of 67 monthly claims for short-term disability benefits made by injured workers to the B.C. Workers' Compensation Board (WCB). These data were reported from the BC Center, Richmond, for the period of 10 years from 1985 to 1994. The mean, variance, and DI of the dataset were 8.6042, 11.2392 and 1.3062, respectively. To check whether the data considered had statistically significant over-dispersion, the hypothesis test proposed by [26] was applied. The value test statistic was 51.971 with a p-value less than 0.001, which showed the data had significant over-dispersion. Figure 6 displays the plots of the autocorrelation function (ACF), partial ACF (PACF), histogram and time series plots, and in the PACF plot the unique first lag significance indicated that these data could be used for modelling the INAR(1) process.

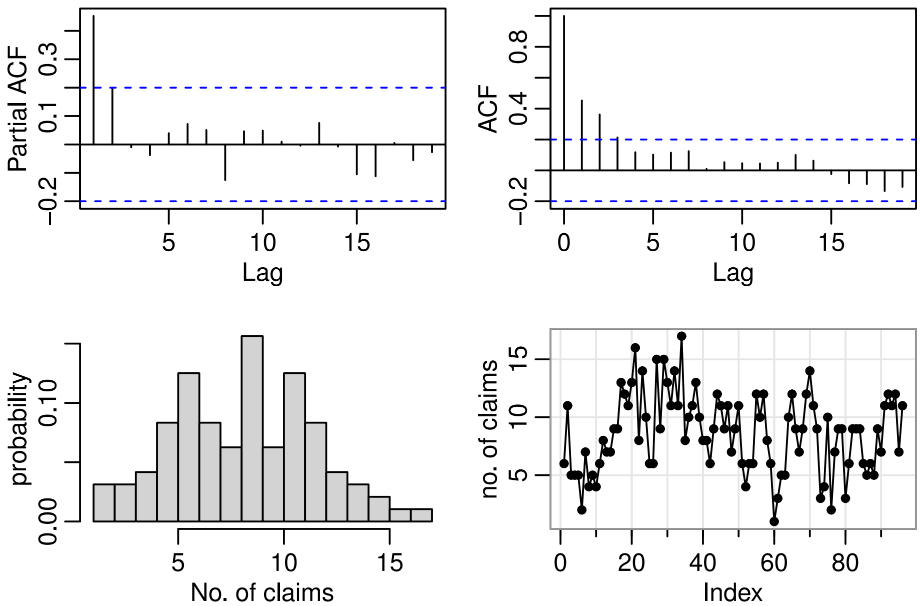

Figure 6. PACF, ACF, histogram and time series plot for the number of claims dataset.

The parameter estimates, modelling adequacy criteria, theoretical mean, variance and DI of the fitted INAR(1) process were recorded in Table 13. Since the INAR(1)DPsL process had lesser values for -L, AIC and BIC statistics than those of the INAR(1)DTPL, INAR(1)NPWE, INAR(1)PL and INAR(1)G processes, the INAR(1)DPsL process provided better fits than the competitors. Additionally, the obtained DI value of the INAR(1)DPsL process was very near the empirical one. It is conclusive that the INAR(1)DPsL process impressively explained the characteristics of the dataset.

Table 13. The estimates and modelling adequacy statistics of the fitted distributions for the number of claims dataset.

Model	Parameters	Estimates (SE)	$-$L	AIC	BIC	μ_x	σ_x^2	DI_x
INAR(1)DPsL	θ	0.4835(0.0526)	245.3344	496.6687	504.3618	8.7812	15.9626	1.8178
	β	1.9214(0.1254)						
	p	0.5620(0.0439)						
INAR(1)DTPL	θ	$-$0.1211(0.3067)	245.3344	498.6687	508.9261	8.7604	16.2473	1.8546
	β	0.4834(0.1903)						
	λ	0.7477(0.0324)						
	p	0.5619(0.0439)						
INAR(1)NPWE	θ	0.1729(0.8221)	252.3457	510.6913	518.3844	8.3542	18.4417	2.2075
	β	0.2738(0.1919)						
	p	0.6432(0.0338)						
INAR(1)DPL	θ	0.4938(0.0583)	248.6185	501.237	506.3657	9.375	23.1842	2.4729
	p	0.6139(0.0381)						
INAR(1)G	θ	0.2431(0.0263)	252.3457	508.6913	513.82	9.0417	31.4719	3.4808
	p	0.6432(0.0338)						
Empirical						8.6042	11.2392	1.3062

The residual analysis was conducted to check whether the fitted INAR(1)DPsL process was accurate. For that, Pearson residuals for the INAR(1)DPsL process were calculated through the following formula:

$$r_t = \frac{x_t - \mathrm{E}(X_t \mid X_{t-1} = x_{t-1})}{\mathrm{Var}(X_t \mid X_{t-1} = x_{t-1})^{1/2}},$$

where $\mathrm{E}(X_t \mid X_{t-1} = x_{t-1})$ and $\mathrm{Var}(X_t \mid X_{t-1} = x_{t-1})$ were derived from (14) and (15), respectively. When the fitted INAR(1) process was statistically valid, the Pearson residual had to be uncorrelated and should have had zero mean and unit variance [27]. Here, we obtained the mean and variance of the Pearson residuals of the INAR(1)DPsL process as 0.035 and 0.967, respectively, which were very close to the desired values. According to the results of [28], the INAR(1)DPsL process for the data was

$$X_t = 0.5620 \circ X_{t-1} + \varepsilon_t,$$

where the innovation process was such that ε_t follows the DPsL (0.4835, 1.9214) distribution. Predicted values of the monthly number of claims dataset and the ACF plot of the Pearson residuals via this process were displayed in Figure 7.

Based on this figure, the ACF plot of the Pearson residuals specified that there was no presence of autocorrelation for the Pearson residuals.

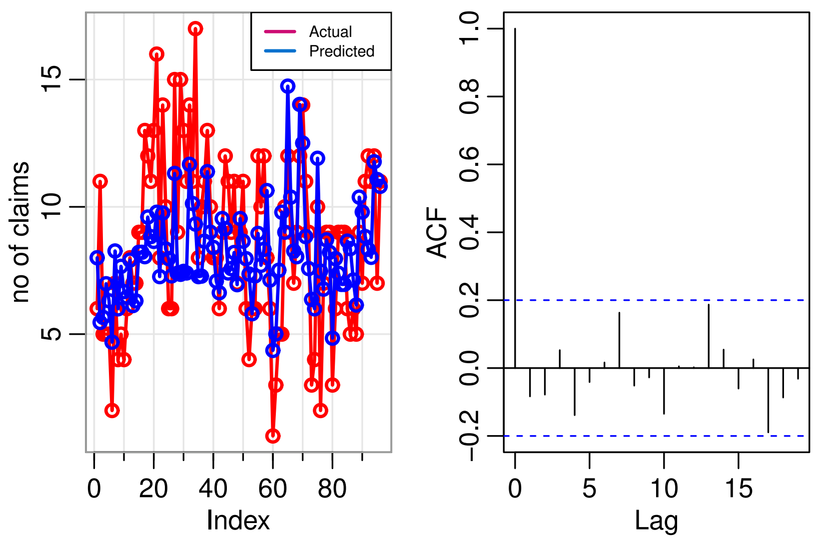

Figure 7. The predicted values of the number of claims dataset (**left**) and the ACF plot of the Pearson residuals (**right**).

6. Concluding Remarks

In this paper, a two-parameter discrete distribution, namely, the discrete Pseudo Lindley (DPsL) distribution, was proposed. Its primary motivation is the ability to model various phenomena with under- and over-dispersed observed values. Various statistical properties, almost all having a closed form, revealed the flexibility and simplicity of the distribution. The estimation of the unknown parameters was performed using two different methods. They conducted an extensive simulation study to reveal the finite sample performance of the distribution. Crucially, a new INAR(1) process with DPsL innovations was developed and studied in detail. Three real-life datasets were considered to prove the efficiency of the proposed distribution. As a future work, we could consider other methods of discretization for the PsL distribution, which would then provide better properties than the survival discretization method. Furthermore, we can attempt to extend it to bivariate models. We hope that the DPsL distribution, as well as the related modelling strategy, will be an interesting alternative to modelling count data, especially in modelling the over-dispersed count data.

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