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Article

A Finite Difference Method for Solving the Wave Equation with Fractional Damping

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Abstract: In this paper, we develop a finite difference method for solving the wave equation with fractional damping in 1D and 2D cases, where the fractional damping is given based on the Caputo fractional derivative. Firstly, based on the weighted method, we propose a new numerical approximation for the Caputo fractional derivative and apply it for the 1D case to obtain a time-stepping method. We then develop an alternating direction implicit (ADI) scheme for the 2D case. Using the discrete energy method, we prove that the proposed difference schemes are unconditionally stable and convergent in both 1D and 2D cases. Finally, several numerical examples are given to verify the theoretical results.

Keywords: finite difference method; fractional damping; wave equation; stability; convergence

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1. Introduction

Recently, the study of damping effects has been a hot research topic because it appears in a variety of the dynamic processes of complex systems, including electromagnetic shunt [1], extensible beams [2], swelling porous elastic [3], vibration [4], and so on [5–13]. In this paper, we consider the initial value problem for the wave equation with fractional damping as follows:

$$\lambda_1 \frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} + \lambda_2 {}_0^C\mathcal{D}_t^\alpha u(\mathbf{x},t) = D\Delta u(\mathbf{x},t) + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega,\ t \in (0,T], \tag{1}$$

$$u(\mathbf{x},0) = \phi(\mathbf{x}), \quad u_t(\mathbf{x},0) = \psi(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}, \tag{2}$$

$$u(\mathbf{x},t)|_{\mathbf{x}\in\partial\Omega} = \varphi(\mathbf{x},t), \quad t \in [0,T], \tag{3}$$

where $\Omega = (a,b) \subset \mathbf{R}^1$ or $\Omega = (a,b) \times (c,d) \subset \mathbf{R}^2$, $\partial\Omega$ is the boundary of Ω, Δ is the Laplacian operator, $D(>0)$ is the diffusion coefficient, $f(\mathbf{x},t)$, $\psi(\mathbf{x},t)$, $\phi(\mathbf{x})$ and $\varphi(\mathbf{x})$ are given functions, and the operator ${}_0^C\mathcal{D}_t^\alpha$ denotes the Caputo fractional derivative defined by

$${}_0^C\mathcal{D}_t^\alpha u(\mathbf{x},t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(\mathbf{x},s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \quad 1 < \alpha < 2. \tag{4}$$

In an overdamped bistable system, the governing model (1)–(3) presents a double-resonance pattern [14]. The fractional derivative in Equation (1) is referred as a damping term, and can account for the memory effects in the system. The case $\alpha = \frac{3}{2}$ in Equation (1) is known as the Lokshin model, which appears in the study of the propagation of the air inside a duct when taking into account viscothermal losses [15,16]. The case $\alpha = 1$ in Equation (1) is known as the damped Klein–Gordon model, arising in relativistic quantum mechanics [17]. The case $\alpha = 2$ in Equation (1) is known as the classical wave equation. Compared with the above commonly discussed models, Equation (1) is more physically flexible due to the term ${}_0^C\mathcal{D}_t^\alpha u(\mathbf{x},t)$ and the parameter λ. Moreover, the inclusion of the time fractional derivative allows for the simulation of the anomalous diffusion and other non-classical transport phenomena.

Although some linear and special fractional partial differential Equations (FPDEs) can be solved using analytical methods, the analytical solutions of the most generalized FPDEs (e.g., multi-term FPDEs) cannot be worked out. Consequently, the study of efficient numerical algorithms for solving the FPDEs plays a critical role in scientific computing and engineering applications. At present, many numerical methods have been developed for solving the FPDEs, such as finite difference methods [18–25], finite element methods [26–30], and spectral methods [31,32].

To the best of our knowledge, only a few numerical methods for the wave equation with fractional damping in Equations (1)–(3) have been considered. Saffarian and Mohebbi [33] developed and analyzed an ADI spectral element method for solving the time fractional damped nonlinear Klein–Gordon equation, in which the equation includes the second-order derivative and the Caputo fractional derivative of order $\alpha \in (0,1)$ in the temporal direction. The WSGL scheme was used to discretize the Caputo derivative based on the relation between the Reimann–Liouville and the Caputo derivative in [33]. Wu et al. [34] used the reduced-order method to develop a Crank–Nicolson compact difference scheme of the time-fractional damped plate vibration equation, in which the equation includes the second-order derivative and the Caputo fractional derivative of order $\alpha \in (1,2)$ in the temporal direction. Based on an efficient sum-of-exponentials (SOE) approximation for the kernel function $t^{-\beta-1}$ with $\beta \in (0,1)$, Lyu et al. [35] presented a fast linearized finite difference scheme for the 1D nonlinear multi-term time-fractional wave equation, where all fractional orders are in $(1,2)$.

In this paper, we aim to propose a new finite difference method for the one- and two-dimensional wave equations with fractional damping in Equations (1)–(3). Note that a direct application of the classic $L1$ approximation to the Caputo derivative in Equation (1) at $t = t_n$ leads to a numerical scheme with only first-order temporal convergence accuracy (see [36]). The difficulty of obtaining a high-order algorithm for Equations (1)–(3) in time lies in how to design a numerical approximation with high accuracy for the following problem:

$$\lambda_1 u''(t) + \lambda_2 {}_0^C\mathcal{D}_t^\alpha u(t) = f(t), \quad t \in [0,T]. \tag{5}$$

Therefore, we first consider the difference scheme with high accuracy for Equation (5) in this study. Specifically, we consider the Caputo fractional derivative ${}_0^C\mathcal{D}_t^\alpha u(t)$ and the second-order derivative $u''(t)$ as a whole in the numerical discretization, and then take its weighted average at $t = t_{n-1}$ and $t = t_{n+1}$. After careful analysis, a (3–α)-order numerical approximation for Equation (5) is then obtained. This is one key contribution of this paper. In addition, we apply this technique of time discretization to derive numerical algorithms for the wave equation with fractional damping in Equations (1)–(3) in both 1D and 2D cases. We then use the discrete energy method to prove the unconditional stability and convergence of the proposed difference schemes. This is another novelty of our work. For the 2D case, we add some small perturbations to obtain an alternating direction implicit (ADI) scheme in order to reduce the huge computational work and storage. Finally, numerical examples are provided to verify the theoretical analysis, and to show some new diffusion-wave phenomena based on fractional damping.

2. Numerical Discretization for a Fractionally Damped System

In this section, we derive a difference scheme for solving the following fractionally damped system with initial conditions

$$\lambda_1 u''(t) + \lambda_2 {}_0^C\mathcal{D}_t^\alpha u(t) = f(t), \ \ t \in (0,T], \tag{6}$$

$$u(0) = \phi, \quad u'(0) = \psi, \tag{7}$$

where the fractional order is $\alpha \in (1,2)$, and $f(t)$ is a given source term. Here, $u'(t)$ and $u''(t)$ denote the first-order and second-order derivative with respect to t, respectively.

Take a uniform partition of the computing interval $[0, T]$, i.e., $t_n = n\tau,\ 0 \le n \le N$ with the step size $\tau = \frac{T}{N}$, where N is a given positive integer. The numerical accuracy reaches no more than first order if we directly consider the difference scheme of Equation (6) at $t = t_n$. In order to obtain a stable difference scheme with high accuracy for Equation (6), we average the results of Equation (6) at $t = t_{n-1}$ and $t = t_{n+1}$, i.e.,

$$\lambda_1 \frac{u''(t_{n-1}) + u''(t_{n+1})}{2} + \lambda_2 \frac{{}_0^C\mathcal{D}_t^{\alpha} u(t_{n-1}) + {}_0^C\mathcal{D}_t^{\alpha} u(t_{n+1})}{2} = \frac{f(t_{n-1}) + f(t_{n+1})}{2}. \tag{8}$$

For simplicity, we define grid functions $U^n = u(t_n),\ n \ge 0$. Furthermore, we introduce some difference quotient operators

$$\Delta_\tau U^n = \frac{U^{n+1} - U^{n-1}}{2\tau}, \quad \delta_t U^{n-\frac{1}{2}} = \frac{U^n - U^{n-1}}{\tau}, \quad \delta_t^2 U^n = \frac{\delta_t U^{n+\frac{1}{2}} - \delta_t U^{n-\frac{1}{2}}}{\tau}.$$

Firstly, we can obtain a second-order approximation for the weighted average of $u''(t)$ at $t = t_{n-1}$ and $t = t_{n+1}$ by using the Taylor expansion, i.e.,

$$\frac{u''(t_{n-1}) + u''(t_{n+1})}{2} = \delta_t^2 U^n + O(\tau^2). \tag{9}$$

Next, we adopt the reduced-order method to construct our numerical formula for the Caputo derivative ${}_0^C\mathcal{D}_t^{\alpha} u(t)$. Let $\beta = \alpha - 1,\ v(t) = u'(t)$. We transform ${}_0^C\mathcal{D}_t^{\alpha} u(t)$ into its substitute as ${}_0^C\mathcal{D}_t^{\beta} v(t)$, i.e.,

$${}_0^C\mathcal{D}_t^{\beta} v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{v'(s)ds}{(t-s)^{\beta}}.$$

The following lemma states the truncation error of the $L1$-type fractional formula.

Lemma 1 (see [37])**.** *Suppose* $g(t) \in C^2[0, t_n]$. *Then*

$$\begin{aligned}
&\left| \int_0^{t_n} g'(t) \frac{dt}{(t_n - t)^{\beta}} - \frac{1}{\tau}\left[\tilde{a}_0 g(t_n) - \sum_{k=1}^{n-1} (\tilde{a}_{n-k-1} - \tilde{a}_{n-k}) g(t_k) - \tilde{a}_{n-1} g(t_0) \right] \right| \\
&\le \frac{1}{1-\beta}\left[\frac{1-\beta}{12} + \frac{2^{2-\beta}}{2-\beta} - (1 + 2^{-\beta}) \right] \max_{0 \le t \le t_n} |g''(t)| \tau^{2-\beta},
\end{aligned} \tag{10}$$

where $0 < \beta < 1$, *and*

$$\tilde{a}_l = \int_{t_l}^{t_{l+1}} \frac{dt}{t^{\beta}} = \frac{1}{1-\beta}[(t_{l+1})^{1-\beta} - (t_l)^{1-\beta}] = \frac{\tau^{1-\beta}}{1-\beta}[(l+1)^{1-\beta} - l^{1-\beta}],\ l \ge 0.$$

We assume that the solution $u(t)$ of Equation (6) is smooth. Accordingly, $v(t)$ is smooth. By Lemma 1, we have the following discrete formula at grid point $t = t_n$, i.e.,

$${}_0^C\mathcal{D}_t^{\beta} v(t_n) = \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\left[a_0^{(\beta)} v(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\beta)} - a_{n-k}^{(\beta)}) v(t_k) - a_{n-1}^{(\beta)} v(t_0) \right] + (r_1)^n, \tag{11}$$

where $a_l^{(\beta)} = (l+1)^{1-\beta} - l^{1-\beta},\ l \ge 0$, and there exists a constant c_1 which satisfies $|(r_1)^n| \le c_1 \tau^{2-\beta}$.

Noticing the relations $\alpha = \beta + 1$ and $u'(t) = v(t)$, we then calculate the weighted average of ${}_0^C\mathcal{D}_t^{\alpha} u(t_{n+1})$ and ${}_0^C\mathcal{D}_t^{\alpha} u(t_{n-1})$ to yield

$$\begin{aligned}
\frac{{}_0^C\mathcal{D}_t^{\alpha} u(t_{n-1}) + {}_0^C\mathcal{D}_t^{\alpha} u(t_{n+1})}{2} &= \frac{{}_0^C\mathcal{D}_t^{\beta} v(t_{n-1}) + {}_0^C\mathcal{D}_t^{\beta} v(t_{n+1})}{2} \\
&= \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Bigg[a_0^{(\beta)} \frac{v(t_{n-1}) + v(t_{n+1})}{2} - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\beta)} - a_{n-k}^{(\beta)}) \frac{v(t_{k-1}) + v(t_{k+1})}{2} \\
&\quad - a_{n-1}^{(\beta)} \frac{v(t_0) + v(t_1)}{2} + a_n^{(\beta)} \frac{v(t_1) - v(t_0)}{2}\Bigg] + \frac{(r_1)^{n-1} + (r_1)^{n+1}}{2} \\
&= \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\Bigg[b_0^{(\alpha)} \frac{u'(t_{n-1}) + u'(t_{n+1})}{2} - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\alpha)} - b_{n-k}^{(\alpha)}) \frac{u'(t_{k-1}) + u'(t_{k+1})}{2} \\
&\quad - b_{n-1}^{(\alpha)} \frac{u'(t_0) + u'(t_1)}{2} + b_n^{(\alpha)} \frac{u'(t_1) - u'(t_0)}{2}\Bigg] + \frac{(r_1)^{n-1} + (r_1)^{n+1}}{2},
\end{aligned} \tag{12}$$

where $b_l^{(\alpha)} = (l+1)^{2-\alpha} - l^{2-\alpha}$, $l \geq 0$. Based on the Taylor expansion, the averaging operators in Equation (12) can be approximated using

$$\frac{1}{2}[u'(t_{k-1}) + u'(t_{k+1})] = \Delta_\tau U^k + (r_2)^k, \quad 1 \leq k \leq n, \tag{13}$$

and

$$\frac{1}{2}[u'(t_k) + u'(t_{k-1})] = \delta_t U^{k-\frac{1}{2}} + (r_3)^k, \quad 1 \leq k \leq n, \tag{14}$$

where

$$(r_2)^k = -\frac{\tau^2}{4}\int_0^1 [u'''(t_k - s\tau) + u'''(t_k + s\tau)](1-s)^2 ds, \tag{15}$$

and

$$(r_3)^k = -\frac{\tau^2}{16}\int_0^1 [u'''(t_{k-\frac{1}{2}} - \frac{\tau}{2}s) + u'''(t_{k-\frac{1}{2}} + \frac{\tau}{2}s)](1-s)^2 ds. \tag{16}$$

In Equation (12), we transform $\frac{u'(t_1)-u'(t_0)}{2}$ into the form $\frac{u'(t_1)+u'(t_0)}{2} - u'(t_0)$, and then substitute the approximations in Equations (13) and (14) into Equation (12). This gives

$$\begin{aligned}
\frac{{}_0^C\mathcal{D}_t^{\alpha} u(t_{n-1}) + {}_0^C\mathcal{D}_t^{\alpha} u(t_{n+1})}{2} =& \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\Bigg[b_0^{(\alpha)} \Delta_\tau U^n - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\alpha)} - b_{n-k}^{(\alpha)}) \Delta_\tau U^k \\
& - b_{n-1}^{(\alpha)} \delta_t U^{\frac{1}{2}} + b_n^{(\alpha)} (\delta_t U^{\frac{1}{2}} - \psi)\Bigg] + (r_4)^n,
\end{aligned} \tag{17}$$

where $(r_4)^n$ denotes

$$\begin{aligned}
(r_4)^n =& \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\Bigg[b_0^{(\alpha)} (r_2)^n - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\alpha)} - b_{n-k}^{(\alpha)})(r_2)^k + (b_n^{(\alpha)} - b_{n-1}^{(\alpha)})(r_3)^1\Bigg] \\
& + \frac{(r_1)^{n-1} + (r_1)^{n+1}}{2} = O(\tau^{3-\alpha}).
\end{aligned} \tag{18}$$

For simplicity, we define the operator $\delta_\tau^{\alpha}(U^n, \psi)$ as

$$\delta_\tau^{\alpha}(U^n, \psi) = \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\Bigg[b_0^{(\alpha)} \Delta_\tau U^n - \sum_{k=1}^{n-1} (b_{n-k-1}^{(\alpha)} - b_{n-k}^{(\alpha)}) \Delta_\tau U^k - b_{n-1}^{(\alpha)} \delta_t U^{\frac{1}{2}} + b_n^{(\alpha)} (\delta_t U^{\frac{1}{2}} - \psi)\Bigg]. \tag{19}$$

Hence, based on the above numerical formulas in Equations (9) and (17), we obtain a numerical discretization for Equation (6) from the second level to the N-th level as follows:

$$\lambda_1 \delta_t^2 U^n + \lambda_2 \delta_\tau^{\alpha}(U^n, \psi) = \frac{f(t_{n-1}) + f(t_{n+1})}{2} + R^n, \quad 1 \leq n \leq N, \tag{20}$$

where $R^n = O(\tau^{3-\alpha})$. We now consider the numerical discretization at the first level. Since ${}_0^C\mathcal{D}_t^\alpha u(t_0) = 0$, Equation (6) at $t = t_0$ reads

$$\lambda_1 u''(t_0) = f(t_0). \tag{21}$$

We apply the Taylor expansion to obtain

$$\frac{2\lambda_1}{\tau}\left[\delta_t U^{\frac{1}{2}} - \psi\right] = f(t_0) + R^0, \tag{22}$$

where $R^n = O(\tau)$.

Let u^n be the approximate solution of $u(t_n)$. We drop the truncation error R^n in Equation (20) and the truncation error R^0 in Equation (22) and obtain a difference scheme for Equation (6) as

$$\begin{cases} \lambda_1 \delta_t^2 u^n + \lambda_2 \delta_\tau^\alpha(u^n, \psi) = \frac{f(t_{n-1}) + f(t_{n+1})}{2}, & n \geq 1, \\ \frac{2\lambda_1}{\tau}\left[\delta_t u^{\frac{1}{2}} - \psi\right] = f(t_0) \end{cases} \tag{23}$$

with $u^0 = \phi$.

To verify the accuracy of the above scheme, we present an example here.

Example 1. *We take $\lambda_1 = \lambda_2 = 1$, $T = 1$, $f(t) = 12t^2 + \frac{24}{\Gamma(5-\alpha)} t^{4-\alpha}$ in Equation (6).*

The exact solution of Example 1 is $u(t) = t^4$. In our computation, we use the difference scheme (23) to solve this problem. The numerical error was computed by

$$E_\infty(\tau) = \max_{0 \leq n \leq N} |u(t_n) - u^n|,$$

and the corresponding convergence order was calculated by Order$= \log_2(\frac{E_\infty(\tau)}{E_\infty(\frac{\tau}{2})})$.

We chose $\alpha = 1.3, 1.5, 1.7$, respectively, to test the convergence accuracy. Table 1 shows the numerical errors and the corresponding convergence accuracy for each fixed α. As expected, our scheme (23) generates the numerical approximation accuracy with $\mathcal{O}(\tau^{3-\alpha})$.

Table 1. Numerical approximation error and convergence accuracy in Example 1.

α	τ	$E_\infty(\tau)$	Order
1.3	$1/2000$	7.0435×10^{-7}	-
	$1/4000$	2.4095×10^{-7}	1.5476
	$1/8000$	8.0200×10^{-8}	1.5871
	$1/16{,}000$	2.6193×10^{-8}	1.6144
	$1/32{,}000$	8.4323×10^{-9}	1.6352
	$1/64{,}000$	2.6448×10^{-9}	1.6728
1.5	$1/2000$	6.4252×10^{-6}	-
	$1/4000$	2.3176×10^{-6}	1.4711
	$1/8000$	8.3087×10^{-7}	1.4799
	$1/16{,}000$	2.9662×10^{-7}	1.4860
	$1/32{,}000$	1.0558×10^{-7}	1.4903
	$1/64{,}000$	3.7468×10^{-8}	1.4946
1.7	$1/2000$	3.9280×10^{-5}	-
	$1/4000$	1.6027×10^{-5}	1.2933
	$1/8000$	6.5273×10^{-6}	1.2959
	$1/16{,}000$	2.6555×10^{-6}	1.2975
	$1/32{,}000$	1.0796×10^{-6}	1.2985
	$1/64{,}000$	4.3871×10^{-7}	1.2992

3. 1D Wave Equation with Fractional Damping

In this section, we propose a finite difference method for solving Equations (1)–(3) in a one-dimensional case, i.e.,

$$\lambda_1 \frac{\partial^2 u(x,t)}{\partial t^2} + \lambda_2 {}_0^C\mathcal{D}_t^\alpha u(x,t) = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad x \in (a,b), \quad t \in (0,T], \tag{24}$$

$$u(a,t) = \varphi_l(t),\ u(b,t) = \varphi_r(t), \quad t \in (0,T], \tag{25}$$

$$u(x,0) = \phi(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad x \in [a,b]. \tag{26}$$

We then analyze the unconditional stability and convergence of the present difference method.

3.1. Derivation of the Finite Difference Method

We first consider the spatial semi-discretization for Equation (24). To this end, we design a set of the grid points as $\Omega_h = \{x_i \mid x_i = a + ih,\ 0 \le i \le M\}$, where the step size $h = (b-a)/M$ and M is a positive integer. Furthermore, we define a space as

$$\mathcal{U}_h = \{u = (u_0, u_1, \ldots, u_M) \text{ is a grid function on } \Omega_h \text{ and } u_0 = u_M = 0\}.$$

For simplicity, we introduce some difference operators as

$$\delta_x u_{i-\frac{1}{2}} = \frac{1}{h}(u_i - u_{i-1}), \quad \delta_x^2 u_i = \frac{1}{h}(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}), \quad \forall u \in \mathcal{U}_h.$$

Suppose the solution $u(x,t) \in C_{x,t}^{(4,3)}([a,b]\times[0,T])$ of Equations (24)–(26). Here, we employ a simple second-order difference scheme for Equation (24) in spatial discretization. For simplicity, we denote $U_i(t) = u(x_i,t)$, $0 \le i \le M$. Thus, at the point (x_i,t), we obtain a spatial semi-discretization scheme for Equation (24) as follows:

$$\lambda_1 \frac{\partial^2 U_i(t)}{\partial t^2} + \lambda_2 {}_0^C\mathcal{D}_t^\alpha U_i(t) = D\delta_x^2 U_i(t) + f(x_i,t) + (R_x)_i(t), \quad 1 \le i \le M-1, \tag{27}$$

where $(R_x)_i(t) = O(h^2)$.

Next, we apply the difference method in Equation (23) to derive the temporal discretization for Equation (24). For $n = 1, \cdots, N$, we average Equation (27) at $t = t_{n-1}$ and t_{n+1} to obtain

$$\begin{aligned} &\frac{\lambda_1}{2}\left(\frac{\partial^2 U_i(t_{n-1})}{\partial t^2} + \frac{\partial^2 U_i(t_{n+1})}{\partial t^2}\right) + \frac{\lambda_2}{2}\left({}_0^C\mathcal{D}_t^\alpha U_i(t_{n-1}) + {}_0^C\mathcal{D}_t^\alpha U_i(t_{n+1})\right) \\ &= \frac{D}{2}\left(\delta_x^2 U_i(t_{n-1}) + \delta_x^2 U_i(t_{n+1})\right) + \frac{f(x_i,t_{n-1}) + f(x_i,t_{n+1})}{2} + \frac{(R_x)_i(t_{n-1}) + (R_x)_i(t_{n+1})}{2}. \end{aligned} \tag{28}$$

For simplicity, we denote the grid points $U_i^n = u(x_i,t_n)$, $0 \le i \le M$, $0 \le n \le N$ and introduce a symbol $U^{\bar{n}} = \frac{1}{2}(U^{n-1} + U^{n+1})$. Applying Equations (9) and (17) to Equation (28), we have

$$\lambda_1 \delta_t^2 U_i^n + \lambda_2 \delta_\tau^\alpha (U_i^n, \psi_i) = D\delta_x^2 U_i^{\bar{n}} + f_i^{\bar{n}} + R_i^n, \quad 1 \le i \le M-1,\ 1 \le n \le N, \tag{29}$$

where the truncation error R_i^n satisfies

$$|R_i^n| \le c_2(\tau^{3-\alpha} + h^2), \quad 1 \le i \le M-1, \quad 1 \le n \le N \tag{30}$$

with c_2 being a positive constant.

We consider Equation (27) at $t = t_0$, and then transform the result into the following equation:

$$\lambda_1 \frac{\partial^2 U_i(t_0)}{\partial t^2} + \lambda_2 {}_0^C\mathcal{D}_t^\alpha U_i(t_0) = D\delta_x^2 U_i^{\frac{1}{2}} + f_i^0 - \frac{D}{2}(\delta_x^2 U_i(t_1) - \delta_x^2 U_i(t_0)) + (R_x)_i(t_0), \quad (31)$$

where $U_i^{n-\frac{1}{2}} = \frac{1}{2}(U_i^n + U_i^{n-1})$. We notice ${}_0^C\mathcal{D}_t^\alpha U_i(t_0) = 0$ and apply the Taylor expansion to the first term in Equation (31). This gives

$$\frac{2\lambda_1}{\tau}\left[\delta_t U_i^{\frac{1}{2}} - \psi_i\right] = D\delta_x^2 U_i^{\frac{1}{2}} + f_i^0 + R_i^0, \quad (32)$$

where the truncation error R_i^0 satisfies

$$|R_i^0| \le c_3(\tau + h^2), \quad 1 \le i \le M-1, \quad (33)$$

with c_3 being a positive constant.

Using the boundary conditions in Equation (25)

$$U_0^n = \varphi_l(t_n), \quad U_M^n = \varphi_r(t_n), \quad 1 \le n \le N, \quad (34)$$

and the initial condition in Equation (26)

$$U_i^0 = \phi(x_i), \quad 0 \le i \le M, \quad (35)$$

dropping the small term R_i^n in Equation (29) and the small term R_i^0 in Equation (32), and replacing U_i^n with its numerical approximation u_i^n, we obtain a difference scheme for Equations (24)–(26) as

$$\lambda_1 \delta_t^2 u_i^n + \lambda_2 \delta_\tau^\alpha(u_i^n, \psi_i) = D\delta_x^2 u_i^{\bar{n}} + f_i^{\bar{n}}, \quad 1 \le i \le M-1,\ 1 \le n \le N, \quad (36)$$

$$\frac{2\lambda_1}{\tau}\left[\delta_t u_i^{\frac{1}{2}} - \psi_i\right] = D\delta_x^2 u_i^{\frac{1}{2}} + f_i^0, \quad 1 \le i \le M-1, \quad (37)$$

$$u_0^n = \varphi_l(t_n),\ u_M^n = \varphi_r(t_n), \quad 1 \le n \le N, \quad (38)$$

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M-1, \quad (39)$$

where the truncation error is $\mathcal{O}(\tau + h^2)$ when $n = 1$, and $\mathcal{O}(\tau^{3-\alpha} + h^2)$ when $n = 2, \cdots N$.

3.2. Analysis of the Difference Scheme

We now analyze the optimal error estimate of the difference scheme (36)–(39). To this end, we first define some discrete inner products and norms. For $u, v \in \mathcal{U}_h$, we introduce the following inner products

$$(u, v) = h\sum_{i=1}^{M-1} u_i v_i, \quad \langle \delta_x u, \delta_x v \rangle = h\sum_{i=1}^{M}(\delta_x u_{i-\frac{1}{2}})\delta_x v_{i-\frac{1}{2}},$$

and the corresponding norms:

$$\| u \| = \sqrt{(u, u)}, \quad |u|_1 = \sqrt{\langle \delta_x u, \delta_x u \rangle}.$$

From the definition of $\{b_n^{(\alpha)}\}_{n=0}^\infty$ in Equation (12), we readily obtain that the properties $b_n^{(\alpha)} > 0$, $b_{n-1}^{(\alpha)} > b_n^{(\alpha)}$, $n \ge 1$. The following lemma gives an estimate of $\delta_\tau^\alpha(\Delta_\tau v^n, \psi)$ defined in Equation (19), which plays a crucial role in the analysis.

Lemma 1. *Let $\delta_\tau^\alpha(\Delta_\tau v^n,\psi)$ be defined in Equation (19). Then, we have*

$$\begin{aligned}&(\delta_\tau^\alpha(\Delta_\tau v^n,\psi))\Delta_\tau v^n\\&\quad\geq\frac{1}{\tau}\left[\frac{\tau^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{k=1}^{n}b_{n-k}^{(\alpha)}(\Delta_\tau v^k)^2-\frac{\tau^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{k=1}^{n-1}b_{n-k-1}^{(\alpha)}(\Delta_\tau v^k)^2\right]\\&\quad\quad+\frac{b_{n-1}^{(\alpha)}\tau^{1-\alpha}}{4\Gamma(3-\alpha)}(\Delta_\tau v^n)^2-\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[(b_{n-1}^{(\alpha)}-b_n^{(\alpha)})(\delta_t v^{\frac{1}{2}})^2+b_n^{(\alpha)}\psi^2\right].\end{aligned}\tag{40}$$

Proof. We re-write $\delta_\tau^\alpha(\Delta_\tau v^n,\psi)$ as the following form

$$\delta_\tau^\alpha(\Delta_\tau v^n,\psi)=\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[b_0^{(\alpha)}\Delta_\tau v^n-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\alpha)}-b_{n-k}^{(\alpha)})\Delta_\tau v^k-(b_{n-1}^{(\alpha)}-b_n^{(\alpha)})\delta_t v^{\frac{1}{2}}-b_n^{(\alpha)}\psi\right].\tag{41}$$

Multiplying Equation (41) by $\Delta_\tau v^n$ and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&(\delta_\tau^\alpha(\Delta_\tau v^n,\psi))\Delta_\tau v^n\\&\quad=\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\Big[b_0^{(\alpha)}(\Delta_\tau v^n)^2-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\alpha)}-b_{n-k}^{(\alpha)})(\Delta_\tau v^k)\cdot\Delta_\tau v^n\\&\quad\quad-(b_{n-1}^{(\alpha)}-b_n^{(\alpha)})\cdot(\delta_t v^{\frac{1}{2}})\cdot\Delta_\tau v^n-b_n^{(\alpha)}\cdot\psi\cdot\Delta_\tau v^n\Big]\\&\quad\geq\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\Big[b_0^{(\alpha)}(\Delta_\tau v^n)^2-\sum_{k=1}^{n-1}(b_{n-k-1}^{(\alpha)}-b_{n-k}^{(\alpha)})\frac{(\Delta_\tau v^k)^2+(\Delta_\tau v^n)^2}{2}\\&\quad\quad-(b_{n-1}^{(\alpha)}-b_n^{(\alpha)})(\frac{1}{4}(\Delta_\tau v^n)^2+(\delta_t v^{\frac{1}{2}})^2)-b_n^{(\alpha)}(\frac{1}{4}(\Delta_\tau v^n)^2+\psi^2)\Big]\\&\quad\geq\frac{1}{\tau}\left[\frac{\tau^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{k=1}^{n}b_{n-k}^{(\alpha)}(\Delta_\tau v^k)^2-\frac{\tau^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{k=1}^{n-1}b_{n-k-1}^{(\alpha)}(\Delta_\tau v^k)^2\right]\\&\quad\quad+\frac{b_{n-1}^{(\alpha)}\tau^{1-\alpha}}{4\Gamma(3-\alpha)}(\Delta_\tau v^n)^2-\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[(b_{n-1}^{(\alpha)}-b_n^{(\alpha)})(\delta_t v^{\frac{1}{2}})^2+b_n^{(\alpha)}\psi^2\right].\end{aligned}\tag{42}$$

Hence, we have completed the proof. □

We now derive a prior estimate of the scheme (36)–(39).

Lemma 2. *Suppose that $\{v_i^n\}$ is the solution of the scheme*

$$\lambda_1\delta_t^2 v_i^n+\lambda_2\delta_\tau^\alpha(v_i^n,\tilde{\psi}_i)=D\delta_x^2 v_i^{\bar{n}}+g_i^{\bar{n}},\quad 1\leq i\leq M-1,\ 1\leq n\leq N,\tag{43}$$

$$\frac{2\lambda_1}{\tau}\left[\delta_t v_i^{\frac{1}{2}}-\tilde{\psi}_i\right]=D\delta_x^2 v_i^{\frac{1}{2}}+g_i^0,\quad 1\leq i\leq M-1,\tag{44}$$

$$v_0^n=0,\ v_M^n=0,\quad 1\leq n\leq N,\tag{45}$$

$$v_i^0=\tilde{\phi}_i,\quad 0\leq i\leq M-1.\tag{46}$$

Then, it holds that

$$|v^n|_1^2\leq C(D,\lambda_1,\lambda_2,T,\alpha)(|v^0|_1^2+\|\tilde{\psi}\|^2+\tau^2\|g^0\|^2+\tau\sum_{m=1}^{n}\|g^{\bar{m}}\|^2).\tag{47}$$

Proof. We take an inner product of Equation (44) with $\delta_t v^{\frac{1}{2}}$ to obtain

$$\frac{2\lambda_1}{\tau}(\delta_t v^{\frac{1}{2}}-\tilde{\psi},\delta_t v^{\frac{1}{2}})=D(\delta_x^2 v^{\frac{1}{2}},\delta_t v^{\frac{1}{2}})+(g^0,\delta_t v^{\frac{1}{2}}).\tag{48}$$

Multiplying both sides of Equation (48) by τ, we obtain

$$2\lambda_1\|\delta_t v^{\frac{1}{2}}\|^2 = D\tau(\delta_x^2 v^{\frac{1}{2}}, \delta_t v^{\frac{1}{2}}) + 2\lambda_1(\tilde{\psi}, \delta_t v^{\frac{1}{2}}) + \tau(g^0, \delta_t v^{\frac{1}{2}}). \tag{49}$$

Applying the summation by parts to the first term on the right-hand side of Equation (49) leads to

$$D\tau(\delta_x^2 v^{\frac{1}{2}}, \delta_t v^{\frac{1}{2}}) = -\frac{D}{2}(|v^1|_1^2 + |v^0|_1^2) + D|v^0|_1^2. \tag{50}$$

Using the Cauchy–Schwarz inequality to the last two terms on the right-hand side of Equation (49) gives

$$2\lambda_1(\tilde{\psi}, \delta_t v^{\frac{1}{2}}) \leq \frac{\lambda_1}{2}\|\delta_t v^{\frac{1}{2}}\|^2 + 2\lambda_1\|\tilde{\psi}\|^2, \tag{51}$$

and

$$\tau(g^0, \delta_t v^{\frac{1}{2}}) \leq \frac{\lambda_1}{2}\|\delta_t v^{\frac{1}{2}}\|^2 + \frac{\tau^2}{2\lambda_1}\|g^0\|^2. \tag{52}$$

Inserting Equations (50)–(52) into Equation (49) leads to

$$\lambda_1\|\delta_t v^{\frac{1}{2}}\|^2 + \frac{D}{2}(|v^1|_1^2 + |v^0|_1^2) \leq D|v^0|_1^2 + 2\lambda_1\|\tilde{\psi}\|^2 + \frac{\tau^2}{2\lambda_1}\|g^0\|^2. \tag{53}$$

We take an inner product of Equation (43) with $\Delta_\tau v^n$ as

$$\lambda_1(\delta_t^2 v^n, \Delta_\tau v^n) + \lambda_2(\delta_\tau^\alpha(v^n, \tilde{\psi}), \Delta_\tau v^n) = D(\delta_x^2 v^{\bar{n}}, \Delta_\tau v^n) + (g^{\bar{n}}, \Delta_\tau v^n). \tag{54}$$

After careful analysis, we obtain

$$\lambda_1(\delta_t^2 v^n, \Delta_\tau v^n) = \frac{\lambda_1}{2\tau}\Big(\|\ \delta_t v^{n+\frac{1}{2}}\ \|^2 - \|\ \delta_t v^{n-\frac{1}{2}}\ \|^2\Big). \tag{55}$$

Applying the summation by parts to the first term on the right-hand side of Equation (54) and noticing the boundary conditions in Equation (45), we have

$$D(\delta_x^2 v^{\bar{n}}, \Delta_\tau v^n) = -\frac{D}{4\tau}(|v^{n+1}|_1^2 - |v^{n-1}|_1^2). \tag{56}$$

We denote $E^n = \frac{\lambda_2\tau^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{k=1}^{n} b_{n-k}^{(\alpha)}\|\Delta_\tau v^k\|^2$. By Lemma 1, we obtain

$$\begin{aligned}\lambda_2(\delta_\tau^\alpha(v^n, \tilde{\psi}), \Delta_\tau v^n) \geq& \frac{1}{\tau}(E^n - E^{n-1}) + \frac{\lambda_2 b_{n-1}^{(\alpha)}\tau^{1-\alpha}}{4\Gamma(3-\alpha)}\|\Delta_\tau v^n\|^2 \\ &- \frac{\lambda_2\tau^{1-\alpha}}{\Gamma(3-\alpha)}\Big[(b_{n-1}^{(\alpha)} - b_n^{(\alpha)})\|\delta_t v^{\frac{1}{2}}\|^2 + b_n^{(\alpha)}\|\tilde{\psi}\|^2\Big].\end{aligned} \tag{57}$$

We further apply the Cauchy–Schwarz inequality to the last term on the right-hand side of Equation (54). This gives

$$(g^{\bar{n}}, \Delta_\tau v^n) \leq \frac{\lambda_2 b_{n-1}^{(\alpha)}\tau^{1-\alpha}}{4\Gamma(3-\alpha)}\|\Delta_\tau v^n\|^2 + \frac{\Gamma(2-\alpha)t_n^{\alpha-1}}{\lambda_2}\|\bar{g}\|^2. \tag{58}$$

For simplicity, we denote $F^n = E^n + \frac{\lambda_1}{2}\|\delta_t v^{n+\frac{1}{2}}\|^2 + \frac{D}{4}(|v^{n+1}|_1^2 + |v^n|_1^2)$. Substituting Equations (55)–(58) into Equation (54) leads to

$$\frac{1}{\tau}(F^n - F^{n-1}) \leq \frac{\lambda_2\tau^{1-\alpha}}{\Gamma(3-\alpha)}\Big[(b_{n-1}^{(\alpha)} - b_n^{(\alpha)})\|\delta_t v^{\frac{1}{2}}\|^2 + b_n^{(\alpha)}\|\tilde{\psi}\|^2\Big] + \frac{\Gamma(2-\alpha)t_n^{\alpha-1}}{\lambda_2}\|g^{\bar{n}}\|^2. \tag{59}$$

Multiplying Equation (59) by τ, we obtain

$$F^n \leq F^{n-1} + \frac{\lambda_2 \tau^{2-\alpha}}{\Gamma(3-\alpha)} \Big[(b_{n-1}^{(\alpha)} - b_n^{(\alpha)}) \|\delta_t v^{\frac{1}{2}}\|^2 + b_n^{(\alpha)} \|\tilde{\psi}\|^2 \Big] + \frac{\Gamma(2-\alpha) t_n^{\alpha-1}}{\lambda_2} \tau \|g^{\bar{n}}\|^2. \quad (60)$$

Replacing n with m in Equation (60) and then summing up for m from 1 to n on both sides of the result, we have

$$\begin{aligned} F^n \leq & F^0 + \frac{\lambda_2 \tau^{2-\alpha}}{\Gamma(3-\alpha)} \Big[\sum_{m=1}^{n} (b_{m-1}^{(\alpha)} - b_m^{(\alpha)}) \|\delta_t v^{\frac{1}{2}}\|^2 + \sum_{m=1}^{n} b_m^{(\alpha)} \|\tilde{\psi}\|^2 \Big] \\ & + \frac{\Gamma(2-\alpha)}{\lambda_2} \tau \sum_{m=1}^{n} t_m^{\alpha-1} \|g^{\bar{m}}\|^2. \end{aligned} \quad (61)$$

Using the following relations

$$\sum_{m=1}^{n} (b_{m-1}^{(\alpha)} - b_m^{(\alpha)}) = b_0^{(\alpha)} - b_n^{(\alpha)} < b_0^{(\alpha)} = 1, \quad (62)$$

$$\sum_{m=1}^{n} b_m^{(\alpha)} = \sum_{m=1}^{n} [(m+1)^{2-\alpha} - m^{2-\alpha}] = (n+1)^{2-\alpha} - 1 < (n+1)^{2-\alpha}, \quad (63)$$

and $t_m^{\alpha-1} \leq t_n^{\alpha-1}$, $m = 1, 2, \cdots, n$ for Equation (61), we obtain

$$F^n \leq F^0 + \frac{\lambda_2 \tau^{2-\alpha}}{\Gamma(3-\alpha)} \|\delta_t v^{\frac{1}{2}}\|^2 + \frac{\lambda_2 t_{n+1}^{2-\alpha}}{\Gamma(3-\alpha)} \|\tilde{\psi}\|^2 + \frac{\Gamma(2-\alpha) t_n^{\alpha-1}}{\lambda_2} \tau \sum_{m=1}^{n} \|g^{\bar{m}}\|^2. \quad (64)$$

Applying Equation (53) to estimate F^0 and $\|\delta_t v^{\frac{1}{2}}\|^2$, we have

$$\begin{aligned} F^n \leq & \Big(\frac{D}{2} + \frac{\lambda_2 D \tau^{2-\alpha}}{\lambda_1 \Gamma(3-\alpha)} \Big) |v^0|_1^2 + \Big(\lambda_1 + \frac{2\lambda_2 \tau^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{\lambda_2 t_{n+1}^{2-\alpha}}{\Gamma(3-\alpha)} \Big) \|\tilde{\psi}\|^2 \\ & + \frac{\lambda_2 \tau^{4-\alpha}}{2\lambda_1^2 \Gamma(3-\alpha)} \|g^0\|^2 + \frac{\Gamma(2-\alpha) t_n^{\alpha-1}}{\lambda_2} \tau \sum_{m=1}^{n} \|g^{\bar{m}}\|^2. \end{aligned} \quad (65)$$

Hence, we have completed the proof. □

Based on Lemma 2, we have the following theorem for stability of the scheme (36)–(39).

Theorem 3 (Stability)**.** *Assume that $\{(u^1)_i^n\}$ and $\{(u^2)_i^n\}$ are two numerical solutions obtained based on the scheme (36)–(39) with same initial and boundary conditions but different source terms $(f^{(1)})_i^n$ and $(f^{(2)})_i^n$. Let $u_i^n = (u^{(1)})_i^n - (u^{(2)})_i^n$ and $f_i^n = (f^{(1)})_i^n - (f^{(2)})_i^n$. Then, it holds that*

$$\|u^n\|_\infty^2 \leq C(D, \lambda_1, \lambda_2, T, \alpha)(\tau^2 \|f^0\|^2 + \tau \sum_{m=1}^{n} \|f^{\bar{m}}\|^2). \quad (66)$$

This implies that the scheme (36)–(39) is unconditionally stable.

We now give the error estimate of the difference scheme (36)–(39). Let $e_i^n = U_i^n - u_i^n$, $0 \leq i \leq M$, $0 \leq n \leq N$. Subtracting Equations (36)–(39) from Equations (29), (32), (34) and (35), we obtain error equations as follows:

$$\lambda_1 \delta_t^2 e_i^n + \lambda_2 \delta_\tau^\alpha (e_i^n, 0) = D\delta_x^2 e_i^{\bar{n}} + R_i^n, \quad 1 \le i \le M-1,\ 1 \le n \le N, \tag{67}$$

$$\frac{2\lambda_1}{\tau} \delta_t e_i^{\frac{1}{2}} = D\delta_x^2 e_i^{\frac{1}{2}} + R_i^0, \quad 1 \le i \le M-1, \tag{68}$$

$$e_0^n = 0,\ e_M^n = 0, \quad 1 \le n \le N, \tag{69}$$

$$e_i^0 = 0, \quad 0 \le i \le M-1. \tag{70}$$

Theorem 4 (Convergence)**.** *Suppose that the solution $u(x,t) \in C_{x,t}^{(4,3)}([a,b] \times [0,T])$ of 1D wave equation with fractional damping (24)–(26), and $\{u_i^n \mid 0 \le i \le M,\ 0 \le n \le N\}$ is the solution of the difference scheme (36)–(39). Then, the following optimal error estimate holds*

$$\|U^n - u^n\|_\infty \le C(\tau^{3-\alpha} + h^2), \quad 1 \le n \le N, \tag{71}$$

which implies that the numerical solution is convergent to the analytical solution with the error $O(\tau^{3-\alpha} + h^2)$.

Proof. We take an inner product of Equation (68) with $\delta_t e^{\frac{1}{2}}$ and Equation (67) with $\Delta_\tau e^n$, respectively. This gives

$$\frac{2\lambda_1}{\tau}(\delta_t e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) = D(\delta_x^2 e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) + (R^0, \delta_t e^{\frac{1}{2}}), \tag{72}$$

and

$$\lambda_1(\delta_t^2 e^n, \Delta_\tau e^n) + \lambda_2(\delta_\tau^\alpha(e^n, 0), \Delta_\tau e^n) = D(\delta_x^2 e^{\bar{n}}, \Delta_\tau e^n) + (R^n, \Delta_\tau e^n). \tag{73}$$

Adopting the analysis strategy in Lemma 2, we have

$$\|e^n\|_\infty^2 \le C(D, \lambda_1, \lambda_2, T, \alpha)(\tau^2 \|R^0\|^2 + \tau \sum_{m=1}^n \|R^m\|^2). \tag{74}$$

Using the estimates of the truncation errors in Equations (30) and (33) for Equation (74), we have completed the proof. □

4. 2D Wave Equation with Fractional Damping

4.1. Derivation of the Difference Scheme

We derive an alternating direction implicit (ADI) scheme for solving Equations (1)–(3) in two dimensions

$$\lambda_1 \frac{\partial^2 u(x,y,t)}{\partial t^2} + \lambda_2 {}_0^C\mathcal{D}_t^\alpha u(x,y,t) = D\Delta u + f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \tag{75}$$

$$u(x,y,t) = \varphi(x,y,t), \quad (x,y,t) \in \partial\Omega \times (0,T], \tag{76}$$

$$u(x,y,0) = \phi(x,y), \quad \frac{\partial u(x,y,0)}{\partial t} = \psi(x,y), \quad (x,y) \in \bar{\Omega}, \tag{77}$$

where $\Omega = (a,b) \times (c,d)$. We take two positive integers M_1 and M_2, and design a set of the grid points as $\bar{\Omega}_h = \{(x_i, y_j) \mid x_i = a + ih_1,\ y_j = c + jh_2,\ 0 \le i \le M_1,\ 0 \le j \le M_2\}$ with step sizes $h_1 = (b-a)/M_1$ and $h_2 = (d-c)/M_2$, and $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$. For $v = \{v_{i,j} \mid 0 \le i \le M_1, 0 \le j \le M_2\}$, we denote

$$\delta_x v_{i-\frac{1}{2},j} = \frac{1}{h_1}(v_{i,j} - v_{i-1,j}), \quad \delta_x^2 v_{i,j} = \frac{1}{h_1}(\delta_x v_{i+\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j}),$$
$$\delta_y v_{i,j-\frac{1}{2}} = \frac{1}{h_2}(v_{i,j} - v_{i,j-1}), \quad \delta_y^2 v_{i,j} = \frac{1}{h_2}(\delta_y v_{i,j+\frac{1}{2}} - \delta_y v_{i,j-\frac{1}{2}}).$$

We further denote the grid function as

$$U_{i,j}^n = u(x_i, y_j, t_n), \quad (x_i, y_j) \in \bar{\Omega}_h, \quad 0 \le n \le N.$$

Suppose the solution $u(\mathbf{x}, t) \in C_{\mathbf{x},t}^{(4,3)}(\bar{\Omega} \times [0, T])$ of Equations (75)–(77) with $\mathbf{x} = (x, y)$. Following the derivation of the difference scheme in one-dimensional case, we construct

$$\lambda_1 \delta_t^2 U_{i,j}^n + \lambda_2 \delta_\tau^\alpha (U_{i,j}^n, \psi_{i,j}) = D(\delta_x^2 U_{i,j}^{\bar{n}} + \delta_y^2 U_{i,j}^{\bar{n}}) + f_{i,j}^{\bar{n}} + O(\tau^{3-\alpha} + h_1^2 + h_2^2), \ (x_i, y_j) \in \Omega_h, \ 1 \le n \le N, \tag{78}$$

$$\frac{2\lambda_1}{\tau}\left[\delta_t U_{i,j}^{\frac{1}{2}} - \psi_{i,j}\right] = D(\delta_x^2 U_{i,j}^{\frac{1}{2}} + \delta_y^2 U_{i,j}^{\frac{1}{2}}) + f_{i,j}^0 + O(\tau + h_1^2 + h_2^2), \quad (x_i, y_j) \in \Omega_h. \tag{79}$$

We define $\omega = \lambda_1 + \lambda_2 \frac{\tau^{2-\alpha}}{2\Gamma(3-\alpha)}$. Adding small perturbations $\frac{D^2\tau^3}{4\omega}\Delta_\tau \delta_x^2 \delta_y^2 U_{i,j}^n$ and $\frac{D^2\tau^3}{8\lambda_1}\delta_t \delta_x^2 \delta_y^2 U_{i,j}^{\frac{1}{2}}$ to both sides of Equations (78) and (79), respectively, we have

$$\lambda_1 \delta_t^2 U_{i,j}^n + \lambda_2 \delta_\tau^\alpha (U_{i,j}^n, \psi_{i,j}) + \frac{D^2\tau^3}{4\omega}\Delta_\tau \delta_x^2 \delta_y^2 U_{i,j}^n = D(\delta_x^2 U_{i,j}^{\bar{n}} + \delta_y^2 U_{i,j}^{\bar{n}}) + f_{i,j}^{\bar{n}} + R_{i,j}^n, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N, \tag{80}$$

$$\frac{2\lambda_1}{\tau}\left[\delta_t U_{i,j}^{\frac{1}{2}} - \psi_{i,j}\right] + \frac{D^2\tau^3}{8\lambda_1}\delta_t \delta_x^2 \delta_y^2 U_{i,j}^{\frac{1}{2}} = D(\delta_x^2 U_{i,j}^{\frac{1}{2}} + \delta_y^2 U_{i,j}^{\frac{1}{2}}) + f_{i,j}^0 + R_{i,j}^0, \quad (x_i, y_j) \in \Omega_h, \tag{81}$$

where there exists a positive constant c_4 satisfying

$$|R_{i,j}^n| \le c_4(\tau^{3-\alpha} + h_1^2 + h_2^2), \quad |R_{i,j}^0| \le c_4(\tau + h_1^2 + h_2^2). \tag{82}$$

Using the boundary condition in Equation (76)

$$U_{i,j}^n = \varphi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \le n \le N, \tag{83}$$

and the initial condition in Equation (77)

$$U_{i,j}^0 = \phi(x_i, y_j), \quad (x_i, y_j) \in \bar{\Omega}_h, \tag{84}$$

and dropping the small term $R_{i,j}^n$ in Equation (80) and the small term $R_{i,j}^0$ in Equation (81), and then replacing $U_{i,j}^n$ with its numerical approximation $u_{i,j}^n$, we obtain a difference scheme as

$$\lambda_1 \delta_t^2 u_{i,j}^n + \lambda_2 \delta_\tau^\alpha (u_{i,j}^n, \psi_{i,j}) + \frac{D^2\tau^3}{4\omega}\Delta_\tau \delta_x^2 \delta_y^2 u_{i,j}^n = D(\delta_x^2 u_{i,j}^{\bar{n}} + \delta_y^2 u_{i,j}^{\bar{n}}) + f_{i,j}^{\bar{n}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N, \tag{85}$$

$$\frac{2\lambda_1}{\tau}\left[\delta_t u_{i,j}^{\frac{1}{2}} - \psi_{i,j}\right] + \frac{D^2\tau^3}{8\lambda_1}\delta_t \delta_x^2 \delta_y^2 u_{i,j}^{\frac{1}{2}} = D(\delta_x^2 u_{i,j}^{\frac{1}{2}} + \delta_y^2 u_{i,j}^{\frac{1}{2}}) + f_{i,j}^0, \quad (x_i, y_j) \in \Omega_h, \tag{86}$$

$$u_{i,j}^n = \varphi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \le n \le N, \tag{87}$$

$$u_{i,j}^0 = \phi(x_i, y_j), \quad (x_i, y_j) \in \bar{\Omega}_h. \tag{88}$$

We multiply Equations (85) and (86) by τ^2, and then factorize the results. This gives

$$\left(\sqrt{w}\mathcal{I} - \frac{D\tau^2}{2\sqrt{w}}\delta_x^2\right)\left(\sqrt{w}\mathcal{I} - \frac{D\tau^2}{2\sqrt{w}}\delta_y^2\right)u_{i,j}^{n+1} = F_{i,j}^n, \ (x_i, y_j) \in \Omega_h, \ 1 \le n \le N, \tag{89}$$

$$\left(\sqrt{2\lambda_1}\mathcal{I} - \frac{D\tau^2}{2\sqrt{2\lambda_1}}\delta_x^2\right)\left(\sqrt{2\lambda_1}\mathcal{I} - \frac{D\tau^2}{2\sqrt{2\lambda_1}}\delta_y^2\right)u_{i,j}^1 = F_{i,j}^0, \ (x_i, y_j) \in \Omega_h, \tag{90}$$

where

$$
\begin{aligned}
F^n =& 2\lambda_1 u_{i,j}^n + \Big(\frac{\lambda_2 \tau^{2-\alpha}}{2\Gamma(3-\alpha)} - \lambda_1\Big) u_{i,j}^{n-1} + \frac{D\tau^2}{2}(\delta_x^2 u_{i,j}^{n-1} + \delta_y^2 u_{i,j}^{n-1}) + \tau^2 f_{i,j}^{\bar{n}} \\
&+ \frac{\lambda_2 \tau^{3-\alpha}}{\Gamma(3-\alpha)}\Big[\sum_{k=1}^{n-1}(b_{n-k-1}^{(\alpha)} - b_{n-k-1}^{(\alpha)})\Delta_\tau u_{i,j}^k + b_{n-1}^{(\alpha)}\delta_t u_{i,j}^{\frac{1}{2}} - b_n^{(\alpha)}(\delta_t u_{i,j}^{\frac{1}{2}} - \psi_{i,j})\Big]; \quad (91)
\end{aligned}
$$

$$
F^0 = 2\lambda_1 u_{i,j}^0 + 2\lambda_1 \tau \psi_{i,j} + \frac{D\tau^2}{2}(\delta_x^2 u_{i,j}^0 + \delta_y^2 u_{i,j}^0) + \tau^2 f_{i,j}^0. \quad (92)
$$

We then introduce the intermediate variables $u_{i,j}^*$ and $v_{i,j}^*$ to obtain a D'Yakonov type of ADI scheme:

$$
\Big(\sqrt{w}\mathcal{I} - \frac{D\tau^2}{2\sqrt{w}}\delta_x^2\Big)u_{i,j}^* = F_{i,j}^n,\ (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \quad (93)
$$

$$
\Big(\sqrt{w}\mathcal{I} - \frac{D\tau^2}{2\sqrt{w}}\delta_y^2\Big)u_{i,j}^{n+1} = u_{i,j}^*,\ (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \quad (94)
$$

$$
\Big(\sqrt{2\lambda_1}\mathcal{I} - \frac{D\tau^2}{2\sqrt{2\lambda_1}}\delta_x^2\Big)v_{i,j}^* = F_{i,j}^0,\ (x_i, y_j) \in \Omega_h, \quad (95)
$$

$$
\Big(\sqrt{2\lambda_1}\mathcal{I} - \frac{D\tau^2}{2\sqrt{2\lambda_1}}\delta_y^2\Big)u_{i,j}^1 = v_{i,j}^*,\ (x_i, y_j) \in \Omega_h. \quad (96)
$$

The detailed computation is described as follows:

Firstly, we solve $u_{i,j}^1$ of the scheme (95)–(96) by computing two sets of independent one-dimensional problems. Specially, for fixed $j(1 \le j \le M_2 - 1)$, we obtain $\{v_{i,j}^* \mid (x_i, y_j) \in \Omega_h\}$ by solving the following linear equations

$$
\Big(\sqrt{2\lambda_1}\mathcal{I} - \frac{D\tau^2}{2\sqrt{2\lambda_1}}\delta_x^2\Big)v_{i,j}^* = F_{i,j}^0, \quad 1 \le i \le M_1 - 1, \quad (97)
$$

$$
v_{0,j}^* = \Big(\sqrt{2\lambda_1}\mathcal{I} - \frac{D\tau^2}{2\sqrt{2\lambda_1}}\delta_y^2\Big)u_{0,j}^1,\ v_{M_1,j}^* = \Big(\sqrt{2\lambda_1}\mathcal{I} - \frac{D\tau^2}{2\sqrt{2\lambda_1}}\delta_y^2\Big)u_{M_1,j}^1. \quad (98)
$$

For fixed $i(1 \le i \le M_1 - 1)$, we then obtain $u_{i,j}^1$ by solving the following linear equations

$$
\Big(\sqrt{2\lambda_1}\mathcal{I} - \frac{D\tau^2}{2\sqrt{2\lambda_1}}\delta_y^2\Big)u_{i,j}^1 = v_{i,j}^*,\ 1 \le j \le M_2 - 1, \quad (99)
$$

$$
u_{i,0}^1 = \varphi(x_i, y_0, t_1),\ u_{i,M_2}^1 = \varphi(x_i, y_M, t_1). \quad (100)
$$

Following the above idea, we can solve $u_{i,j}^n$ of the scheme (93)–(94) for $n \ge 2$ by computing two sets of independent one-dimensional problems.

4.2. Analysis of the Difference Scheme

We now give the optimal error estimate of the difference scheme (85)–(88). To this end, we first define a space as

$$
\mathcal{V}_h = \{v \mid v = \{v_{i,j}\} \text{ is a grid function on } \Omega_h \text{ and } v_{i,j} = 0 \text{ if } (x_i, y_j) \in \partial\Omega_h\}.
$$

For $u, v \in \mathcal{V}_h$, we introduce some inner products

$$
(u, v) = h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} u_{i,j} v_{i,j},
$$

$$
\langle \delta_x u, \delta_x v \rangle = h_1 h_2 \sum_{i=1}^{M_1}\sum_{j=1}^{M_2-1}(\delta_x u_{i-\frac{1}{2},j})\delta_x v_{i-\frac{1}{2},j},
$$

$$\langle \delta_y u, \delta_y v \rangle = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} (\delta_x u_{i,j-\frac{1}{2}}) \delta_x v_{i,j-\frac{1}{2}},$$

and the corresponding norms $\|u\| = \sqrt{(u,u)}$ and $|u|_1^2 = \langle \delta_x u, \delta_x u \rangle + \langle \delta_y u, \delta_y u \rangle$.

In the following lemma, we obtain a prior estimate of the scheme (85)–(88).

Lemma 5. *Suppose that $\{v_{i,j}^n\}$ is the solution of the following scheme*

$$\lambda_1 \delta_t^2 v_{i,j}^n + \lambda_2 \delta_\tau^\alpha (v_{i,j}^n, \tilde{\psi}_{i,j}) + \frac{D^2\tau^3}{4\omega} \Delta_\tau \delta_x^2 \delta_y^2 v_{i,j}^n = D(\delta_x^2 v_{i,j}^{\bar{n}} + \delta_y^2 v_{i,j}^{\bar{n}}) + g_{i,j}^{\bar{n}},$$
$$(x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{101}$$

$$\frac{2\lambda_1}{\tau}\left[\delta_t v_{i,j}^{\frac{1}{2}} - \tilde{\psi}_{i,j}\right] + \frac{D^2\tau^3}{8\lambda_1} \delta_t \delta_x^2 \delta_y^2 v_{i,j}^{\frac{1}{2}} = D(\delta_x^2 v_{i,j}^{\frac{1}{2}} + \delta_y^2 v_{i,j}^{\frac{1}{2}}) + g_{i,j}^0, \quad (x_i, y_j) \in \Omega_h, \tag{102}$$

$$v_{i,j}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \le n \le N, \tag{103}$$

$$v_{i,j}^0 = \tilde{\phi}_{i,j}, \quad (x_i, y_j) \in \partial\bar{\Omega}_h. \tag{104}$$

Then, it holds that

$$|v^n|_1^2 \le C(D, \lambda_1, \lambda_2, T, \alpha)(|v^0|_1^2 + \|\tilde{\psi}\|^2 + \tau^2\|g^0\|^2 + \tau \sum_{m=1}^n \|g^{\bar{m}}\|^2). \tag{105}$$

Proof. We take inner products of Equation (102) with $\delta_t v^{\frac{1}{2}}$ and Equation (101) with $\Delta_\tau v^n$. This gives

$$\frac{2\lambda_1}{\tau}(\delta_t v^{\frac{1}{2}} - \tilde{\psi}, \delta_t v^{\frac{1}{2}}) + \frac{D^2\tau^3}{8\lambda_1}(\delta_t \delta_x^2 \delta_y^2 v^{\frac{1}{2}}, \delta_t v^{\frac{1}{2}}) = D(\delta_x^2 v^{\frac{1}{2}} + \delta_y^2 v^{\frac{1}{2}}, \delta_t v^{\frac{1}{2}}) + (g^0, \delta_t v^{\frac{1}{2}}) \tag{106}$$

and

$$\lambda_1(\delta_t^2 v^n, \Delta_\tau v^n) + \lambda_2(\delta_\tau^\alpha(v^n, \tilde{\psi}), \Delta_\tau v^n) + \frac{D^2\tau^3}{4\omega}(\Delta_\tau \delta_x^2 \delta_y^2 v^n, \Delta_\tau v^n)$$
$$= D(\delta_x^2 v^{\bar{n}} + \delta_y^2 v^{\bar{n}}, \Delta_\tau v^n) + (g^{\bar{n}}, \Delta_\tau v^n). \tag{107}$$

Applying the summation by parts to the second term on the left-hand side of Equation (106) and the third term on the left-hand side of Equation (107), respectively, we have

$$\frac{D^2\tau^3}{8\lambda_1}(\delta_t \delta_x^2 \delta_y^2 v^{\frac{1}{2}}, \delta_t v^{\frac{1}{2}}) \ge 0, \quad \frac{D^2\tau^3}{4\omega}(\Delta_\tau \delta_x^2 \delta_y^2 v^n, \Delta_\tau v^n) \ge 0. \tag{108}$$

Using the same analysis in Lemma 2 to the last terms of Equations (106) and (107), one may obtain the estimate (105). □

Based on Lemma 5, we have the following theorem for stability of the scheme (85)–(88).

Theorem 6 (Stability)**.** *Assume that $\{(u^1)_{i,j}^n\}$ and $\{(u^2)_{i,j}^n\}$ are two numerical solutions obtained based on the difference scheme (85)–(88) with same initial and boundary conditions but different source terms $(f^{(1)})_{i,j}^n$ and $(f^{(2)})_{i,j}^n$. Let $u_{i,j}^n = (u^{(1)})_{i,j}^n - (u^{(2)})_{i,j}^n$ and $f_{i,j}^n = (f^{(1)})_{i,j}^n - (f^{(2)})_{i,j}^n$. Then, it holds that*

$$|u^n|_1^2 \le C(D, \lambda_1, \lambda_2, T, \alpha)(\tau^2\|f^0\|^2 + \tau \sum_{m=1}^n \|f^{\bar{m}}\|^2). \tag{109}$$

This implies that the scheme (85)–(88) is unconditionally stable.

We now give the error estimate of the difference scheme (85)–(88). Let $e^n_{i,j} = U^n_{i,j} - u^n_{i,j}$, $(x_i, y_j) \in \Omega_h$, $0 \le n \le N$. Subtracting Equations (85)–(88) from Equations (80), (81), (83) and (84), we obtain error equations as follows:

$$\lambda_1 \delta_t^2 e^n_{i,j} + \lambda_2 \delta_\tau^\alpha (e^n_{i,j}, 0) + \frac{D^2\tau^3}{4\omega} \Delta_\tau \delta_x^2 \delta_y^2 e^n_{i,j} = D(\delta_x^2 e^{\bar{n}}_{i,j} + \delta_y^2 e^{\bar{n}}_{i,j}) + R^n_{i,j},$$
$$(x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{110}$$
$$\frac{2\lambda_1}{\tau} \delta_t e^{\frac{1}{2}}_{i,j} + \frac{D^2\tau^3}{8\lambda_1} \delta_t \delta_x^2 \delta_y^2 e^{\frac{1}{2}}_{i,j} = D(\delta_x^2 e^{\frac{1}{2}}_{i,j} + \delta_y^2 e^{\frac{1}{2}}_{i,j}) + R^0_{i,j}, \quad (x_i, y_j) \in \Omega_h, \tag{111}$$
$$e^n_{i,j} = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \le n \le N, \tag{112}$$
$$e^0_{i,j} = 0, \quad (x_i, y_j) \in \partial\bar{\Omega}_h. \tag{113}$$

Theorem 7 (Convergence). *Suppose that the solution $u(\mathbf{x}, t) \in C^{(4,3)}_{\mathbf{x},t}(\bar{\Omega} \times [0, T])$ of 2D wave equation with fractional damping (75)–(77), and $\{u^n_{i,j} \mid (x_i, y_j) \in \bar{\Omega}_h,\ 0 \le n \le N\}$ is the solution of the scheme (85)–(88). Then, the following optimal error estimate holds:*

$$|U^n - u^n|_1 \le C(\tau^{3-\alpha} + h_1^2 + h_2^2), \quad 1 \le n \le N, \tag{114}$$

which implies that the numerical solution is convergent to the analytical solution with the error $O(\tau^{3-\alpha} + h_1^2 + h_2^2)$.

Proof. We take an inner product of Equation (111) with $\delta_t e^{\frac{1}{2}}$ and Equation (110) with $\Delta_\tau e^n$, respectively. This gives

$$\frac{2\lambda_1}{\tau}(\delta_t e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) + \frac{D^2\tau^3}{8\lambda_1}(\delta_t \delta_x^2 \delta_y^2 e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) = D(\delta_x^2 e^{\frac{1}{2}} + \delta_y^2 e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) + (R^0, \delta_t e^{\frac{1}{2}}), \tag{115}$$

and

$$\lambda_1(\delta_t^2 e^n, \Delta_\tau e^n) + \lambda_2(\delta_\tau^\alpha(e^n, 0), \Delta_\tau e^n) + \frac{D^2\tau^3}{4\omega}(\Delta_\tau \delta_x^2 \delta_y^2 e^n, \Delta_\tau e^n)$$
$$= D(\delta_x^2 e^{\bar{n}} + \delta_y^2 e^{\bar{n}}, \Delta_\tau e^n) + (R^n, \Delta_\tau e^n). \tag{116}$$

Adopting the analysis strategy in Lemma 5, we have

$$|e^n|_1^2 \le C(D, \lambda_1, \lambda_2, T, \alpha)(\tau^2 \|R^0\|^2 + \tau \sum_{m=1}^{n} \|R^m\|^2). \tag{117}$$

Using the estimates of the truncation errors in Equation (82) for Equation (117), we have completed the proof. □

5. Numerical Examples

In this section, we carry out three numerical examples to test the performance and the applicability of our proposed difference schemes for solving the wave equation with fractional damping (1)–(3) in both 1D and 2D cases. All the numerical experiments are implemented using Matlab R2018b on a desktop with Intel(R) Core(TM) i3-7100U CPU @ 2.40 GHz and 8 GB.

In our computation, we denote τ as the time step size. For the 2D case, we take same step sizes in x and y spatial directions, i.e., $h_1 = h_2 = h$. We denote $E_\infty(h, \tau) = \|U^N - u^N\|_\infty$ as the maximum error, where u^N and U^N are the numerical solution and the analytical solution at the N-th level, respectively.

For the convergence accuracy of the proposed difference schemes, we restrict the time step size and the space step size by $h^2 \approx \tau^{3-\alpha}$. To test the temporal convergence order, we set $h \approx \tau^{\frac{3-\alpha}{2}}$ in order that $E_\infty(h, \tau) \approx E_\infty(\tau)$. The temporal convergence order is obtained using $\text{Rate}_t = \log_2(E_\infty(2\tau)/E_\infty(\tau))$. To obtain the spatial convergence order, we set $\tau \approx h^{\frac{2}{3-\alpha}}$, so that $E_\infty(h,\tau) \approx E_\infty(h)$. The spatial convergence order is obtained using $\text{Rate}_x = \log_2(E_\infty(2h)/E_\infty(h))$.

Example 2. *In Equations (24)–(26), we choose the coefficients $\lambda_1 = 1, \lambda_2 = 1, D = 1$, and the initial values $\phi(x) = 0$ and $\psi(x) = 0$, together with the boundary conditions $\phi_l(t) = \phi_r(t) = 0$. The source term is given by $f(x,t) = \left(12t^2 + \frac{24t^{4-\alpha}}{\Gamma(5-\alpha)} + \pi^2 t^4\right)\sin(\pi x)$.*

The analytical solution of the above problem can be seen to be $u(x,t) = t^4\sin(\pi x)$. In the following simulation, we compute the numerical solution on domain $(x,t) \in [0,1] \times [0,1]$.

Tables 2 and 3 show the errors of numerical approximations and the corresponding convergence orders of the difference scheme (36)–(39) with various fractional orders $\alpha = 1.4$, $\alpha = 1.6$ and $\alpha = 1.8$, respectively. From Tables 2 and 3, one can see that, for all fractional orders considered, the numerical solutions obtained using the difference scheme (36)–(39) agree well with their corresponding analytical solutions. Moreover, it can be seen that the convergence rate of the difference scheme (36)–(39) confirms the accuracy of $\mathcal{O}(\tau^{3-\alpha} + h^2)$, which is consistent with the theoretical error estimate in Theorem 4.

Example 3. *In Equations (75)–(77), we choose coefficients $\lambda_1 = 1, \lambda_2 = 1, D = 1$, and the domain $\Omega = (0,1)^2$ and $T = 1$, and the initial conditions $\phi(x,y) = 0$ and $\psi(x,y) = 0$, and the boundary conditions $\varphi(x,0,t) = t^4\sin(\pi x)$, $\varphi(x,1,t) = -t^4\sin(\pi x)$, $\varphi(0,y,t) = t^4\sin(\pi y)$, $\varphi(1,y,t) = -t^4\sin(\pi y)$. The source term is given by $f(x,y,t) = \left(12t^2 + \frac{24t^{4-\alpha}}{\Gamma(5-\alpha)} + 2\pi^2 t^4\right)\sin(\pi(x+y))$.*

The analytical solution of the above problem can be seen to be $u(x,y,t) = t^4\sin(\pi(x+y))$. In our simulation, we use the difference scheme (78)–(79) and the ADI scheme (85)–(88) to numerically solve Example 3.

Tables 4 and 5 list the errors of the numerical approximations, the convergence orders, and the CPU times took by both schemes for three different fractional orders $\alpha = 1.4$, $\alpha = 1.6$ and $\alpha = 1.8$, respectively. From numerical results, one can see that both schemes effectively solve the governing problem, and the convergence rates of both schemes can achieve the accuracy of $\mathcal{O}(\tau^{3-\alpha} + h^2)$. Furthermore, apparently, the ADI scheme (85)–(88) costs less CPU time than the difference scheme (78)–(79) under the same time and space meshes. Numerical results confirm that the ADI scheme can significantly enhance the computational efficiency as compared with the non-ADI scheme for high-dimensional problems. This is because that the ADI technique factorizes the high-dimensional problem into several sets of independent one-dimensional cases, which can be computed using the Thomas algorithm and hence reduce storage expenses.

Example 4. *We investigate the dynamic of the following 2D wave equation with fractional damping*

$$\begin{cases} \dfrac{\partial^2 u(\mathbf{x},t)}{\partial t^2} + {}_0^C\mathcal{D}_t^\alpha u(\mathbf{x},t) = \Delta u(\mathbf{x},t), \ \ \mathbf{x} \in \Omega, \ \ t \in (0,T], \\ u(\mathbf{x},0) = \text{sech}(V1) + \text{sech}(V2), \quad \dfrac{\partial u(\mathbf{x},0)}{\partial t} = \text{sech}(V3), \ \ \mathbf{x} \in \bar{\Omega}, \\ u(\mathbf{x},t) = 0, \ \mathbf{x} \in \partial\Omega, \ \ t \in [0,T], \end{cases} \tag{118}$$

where $V1 = (x+2)^2 + y^2$, $V2 = (x-2)^2 + y^2$, $V3 = x^2 + y^2$, and the computational domain $\Omega = (-16,16)^2$.

We use the ADI scheme (85)–(88) to obtain the solution of the problem (118). In our simulation, we divided Ω into meshes with the space step size of $h = 1/8$ and compute the time up to $T = 5$ with the time step size $\tau = 1/20$.

Table 2. Numerical error and temporal convergence accuracy in Example 2 with $h \approx \tau^{\frac{3-\alpha}{2}}$.

α	τ	$E_\infty(\tau)$	Rate$_t$
1.4	1/2000	3.3961×10^{-6}	-
	1/4000	1.0747×10^{-6}	1.6599
	1/8000	3.4298×10^{-7}	1.6477
	1/16,000	1.1031×10^{-7}	1.6366
1.6	1/2000	1.6207×10^{-5}	-
	1/4000	6.0741×10^{-6}	1.4159
	1/8000	2.2812×10^{-6}	1.4129
	1/16,000	8.5974×10^{-7}	1.4078
1.8	1/2000	8.5707×10^{-5}	-
	1/4000	3.7238×10^{-5}	1.2026
	1/8000	1.6175×10^{-5}	1.2030
	1/16,000	7.0371×10^{-6}	1.2007

Table 3. Numerical error and spatial convergence accuracy in Example 2 with $\tau \approx h^{\frac{2}{3-\alpha}}$.

α	h	$E_\infty(h)$	Rate$_x$
1.4	1/64	2.0137×10^{-4}	-
	1/128	4.5200×10^{-5}	2.1555
	1/256	1.0417×10^{-5}	2.1174
	1/512	2.4460×10^{-6}	2.0904
	1/1024	5.8316×10^{-7}	2.0685
1.6	1/64	1.7620×10^{-4}	-
	1/128	4.2168×10^{-5}	2.0630
	1/256	1.0291×10^{-5}	2.0348
	1/512	2.5373×10^{-6}	2.0200
	1/1024	6.2953×10^{-7}	2.0110
1.8	1/64	1.9261×10^{-4}	-
	1/128	4.7801×10^{-5}	2.0106
	1/256	1.1916×10^{-5}	2.0041
	1/512	2.9753×10^{-6}	2.0018
	1/1024	7.4337×10^{-7}	2.0009

Table 4. Numerical error, temporal convergence accuracy, and CPU time in Example 3 with $h \approx \tau^{\frac{3-\alpha}{2}}$.

α	τ	ADI Scheme (85)–(88)			Difference Scheme (78)–(79)		
		$E_\infty(\tau)$	Rate$_t$	CPU(s)	$E_\infty(\tau)$	Rate$_t$	CPU(s)
1.4	1/100	2.1905×10^{-4}	-	2.5415	2.5769×10^{-4}	-	10.7989
	1/200	7.4343×10^{-5}	1.5590	9.1474	8.3892×10^{-5}	1.6190	235.2666
	1/400	2.4530×10^{-5}	1.5996	84.1986	2.6911×10^{-5}	1.6403	8814.8094
	1/800	8.1856×10^{-6}	1.5834	624.8911	-	-	-
1.6	1/100	6.4844×10^{-4}	-	1.2507	6.8376×10^{-4}	-	1.3788
	1/200	2.4367×10^{-4}	1.4120	4.5492	2.5242×10^{-4}	1.4377	19.2717
	1/400	9.3717×10^{-5}	1.3785	27.1594	9.5907×10^{-5}	1.3961	363.7173
	1/800	3.5293×10^{-5}	1.4089	207.1005	3.5844×10^{-5}	1.4199	10,007.9790
1.8	1/100	1.8465×10^{-3}	-	0.9686	1.9000×10^{-3}	-	1.0571
	1/200	8.1979×10^{-4}	1.1715	2.3394	8.2721×10^{-4}	1.1997	2.6146
	1/400	3.6132×10^{-4}	1.1820	13.4669	3.6318×10^{-4}	1.1876	18.8149
	1/800	1.5601×10^{-4}	1.2116	83.4417	1.5648×10^{-4}	1.2147	281.8330

Table 5. Numerical error, spatial convergence accuracy, and CPU time in Example 3 with $\tau \approx h^{\frac{2}{3-\alpha}}$.

α	h	ADI Scheme (85)–(88)			Difference Scheme (78)–(79)		
		$E_\infty(h)$	Rate$_x$	CPU(s)	$E_\infty(h)$	Rate$_x$	CPU(s)
1.4	1/16	1.2811×10^{-3}	-	0.2432	1.7000×10^{-3}	-	0.2673
	1/32	3.3922×10^{-4}	1.9171	1.0039	4.0663×10^{-4}	2.0638	2.4947
	1/64	8.6549×10^{-5}	1.9706	6.9007	9.8228×10^{-5}	2.0495	134.1365
	1/128	2.1895×10^{-5}	1.9829	90.4336	2.3946×10^{-5}	2.0364	14,028.9227
1.6	1/16	1.5517×10^{-3}	-	0.5325	1.7000×10^{-3}	-	0.5844
	1/32	3.9889×10^{-4}	1.9598	2.6282	4.1654×10^{-4}	2.0290	4.2683
	1/64	1.0014×10^{-4}	1.9940	27.9728	1.0257×10^{-4}	2.0219	279.9205
	1/128	2.5086×10^{-5}	1.9971	466.9703	2.5422×10^{-5}	2.0125	32,390.5134
1.8	1/16	1.8236×10^{-3}	-	0.8777	1.9000×10^{-3}	-	0.9089
	1/32	4.6245×10^{-4}	1.9794	8.7622	4.6530×10^{-4}	2.0298	9.2940
	1/64	1.1576×10^{-4}	1.9982	197.7385	1.1605×10^{-4}	2.0034	797.0400
	1/128	2.8946×10^{-5}	1.9997	4326.2059	2.8975×10^{-5}	2.0019	105,978.3261

Figure 1 displays numerical solutions obtained from the ADI scheme (85)–(88) at different moments $t = 0$(s), $t = 2.5$(s) and $t = 5$(s) with various fractional orders $\alpha = 1$, $\alpha = 1.2$, $\alpha = 1.5$, and $\alpha = 1.8$. The graphs of initial values for all considered fractional orders are showed to be two spike type in first column of Figure 1. It can be seen that, with time increases, the amplitude of solutions decreases for each fractional order. Moreover, the larger the fractional order α is, the smaller the amplitude of solutions becomes. Numerical results indicate that the fractional order α could play an important role in regulating dynamic of damping.

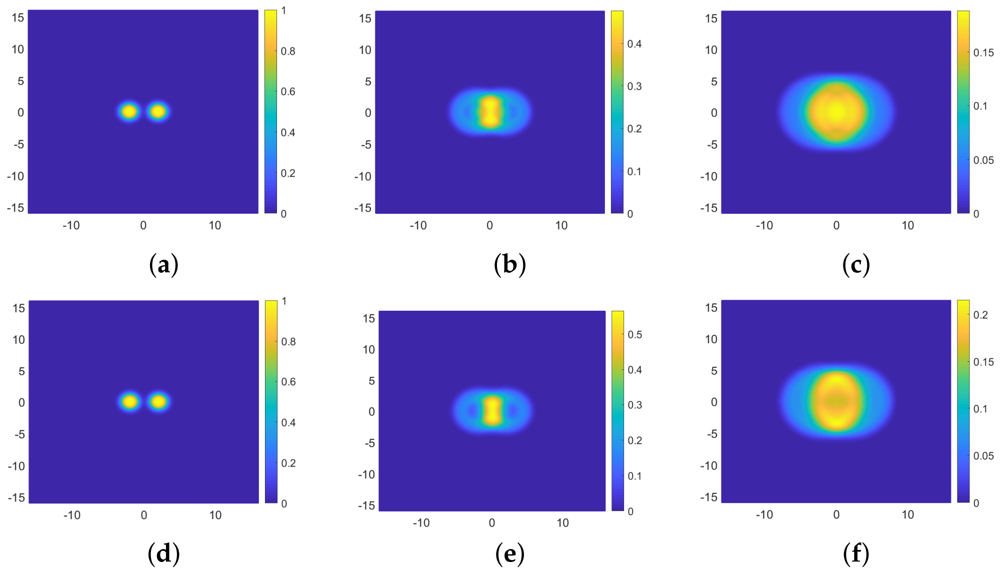

Figure 1. *Cont.*

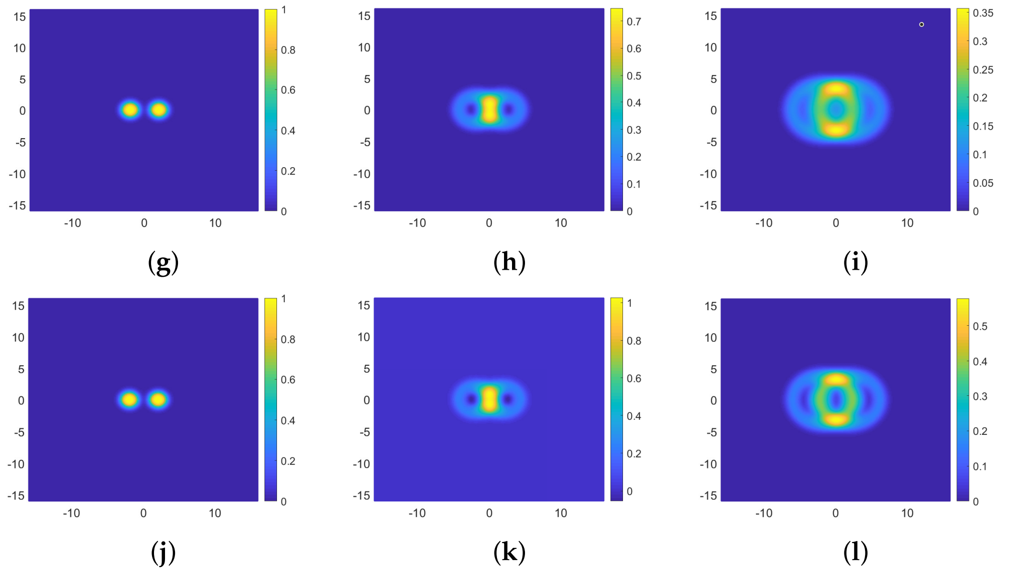

Figure 1. Effect of damping for different derivatives on peak value at different times. (**a**) $\alpha = 1,\ t = 0$, (**b**) $\alpha = 1,\ t = 2.5$, (**c**) $\alpha = 1,\ t = 5$, (**d**) $\alpha = 1.2,\ t = 0$, (**e**) $\alpha = 1.2,\ t = 2.5$, (**f**) $\alpha = 1.2,\ t = 5$, (**g**) $\alpha = 1.5,\ t = 0$, (**h**) $\alpha = 1.5,\ t = 2.5$, (**i**) $\alpha = 1.5,\ t = 5$, (**j**) $\alpha = 1.8,\ t = 0$, (**k**) $\alpha = 1.8,\ t = 2.5$, (**l**) $\alpha = 1.8,\ t = 5$.

6. Conclusions

We have developed efficient difference schemes for solving the one- and two-dimensional wave equations with fractional damping. In addition, we have proved the unconditional stability and convergence of the proposed difference schemes. The accuracy of the present difference schemes and the application of the wave equation with fractional damping have been tested in several numerical examples. Numerical simulations show that the values of fractional orders of the Caputo derivative in the damping model have a significant effect on the wave process.

It should be pointed out that the time discretization technique used in this paper is based on the uniform time mesh, which requires strong regularity assumptions. Recent works [18,25,38,39] have discussed the lack of the smoothness near the initial time of the solution of the time-fractional PDEs. Future study will consider the numerical method in a graded time mesh in order to overcome initial time singularity and provide a theoretical basis for the subsequent numerical scheme.

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