Supplementary Material

☞ **Positivity and Boundedness**

Theorem 1: *The all possible solutions of the system (5) with the corresponding initial conditions always remains and bounded in the interior of* $\mathbb{R}^3_+$.

Proof: The system (5) can be written as:

$$\frac{dR(t)}{dt} = f_1(R(t), N(t), P(t))$$

$$\frac{dN(t)}{dt} = f_2(R(t), N(t), P(t))$$

$$\frac{dP(t)}{dt} = f_3(R(t), N(t), P(t))$$

So, from the above form of the system (5) with initial condition $R(0) > 0$, $N(0) > 0$ and $P(0) > 0$ we have

$$R(t) = R(0)e^{\int_0^t f_1(R(s),N(s),P(s))ds} > 0$$

$$N(t) = N(0)e^{\int_0^t f_2(R(s),N(s),P(s))ds} > 0$$

$$P(t) = P(0)e^{\int_0^t f_3(R(s),N(s),P(s))ds} > 0$$

Hence, from the above inequalities we may conclude that any possible that initiate in $\mathbb{R}^3_+$ remains in the interior of $\mathbb{R}^3_+$ for all future time.

To prove boundedness of the system (5), we proceed as follows.

From the first equation of the system (5) we have

$\frac{dR(t)}{dt} = rR(t)(1 - \frac{R(t)}{K}) - \frac{\alpha_1 R(t)N(t)}{\beta_1 + R(t)} - \frac{\alpha_2 R(t)(1-b)P(t)}{\beta_2 + R(t)}$

i.e., $\frac{dR(t)}{dt} \leq \frac{r}{K}R(t)(K - R(t))$

Now we compare $R(t)$ with well known Bernaulli equation

$$\frac{dx(t)}{dt} \leq ax(t)(1 - x(t)) \text{ with initial condition } x(0) \geq 0$$

we find $R(t) \leq \frac{R(0)}{R(0)+(K-R(0)e^{-\frac{r}{K}t})}$ with initial condition $R(0) \geq 0$

So, any solution of the system (5) satisfies $R(t) \leq K$ for $\forall t \geq 0$

Suppose, $(R(t), N(t), P(t))$ be any solution of (5) with some positive initial condition.

Let us define a function

$\eta = R(t) + \frac{1}{e_1}N(t) + \frac{1}{e_3}P(t)$

$\frac{d\eta}{dt} = \frac{dR(t)}{dt} + \frac{1}{e_1}\frac{dN(t)}{dt} + \frac{1}{e_3}\frac{dP(t)}{dt}$

So, $\frac{d\eta}{dt} \leq rK + \frac{e_2\alpha_4 a(1-b)}{e_1}NP + \frac{e_4\alpha_3 b}{e_3(\beta_3+N)}NP + \frac{d_1}{e_1}N + \frac{d_2}{e_3}P$

We chose a $\xi(> 0)$ such that

$\frac{d\eta}{dt} + \xi\eta \leq rK + \xi R + \{\frac{e_2\alpha_4 a(1-b)}{e_1} + \frac{e_4\alpha_3 b}{e_3(\beta_3+N)}\}NP + \frac{1}{e_1}N(d_1 - \xi) + \frac{1}{e_3}P(d_2 - \xi)P$

Let us define $\xi = \min\{d_1, d_2\}$, then $\frac{d\eta}{dt} + \xi\eta \leq rK + \xi R + \{\frac{e_2\alpha_4 a(1-b)}{e_1} + \frac{e_4\alpha_3 b}{e_3(\beta_3+N)}\}NP$

Suppose $U = rK + \xi R + \{\frac{e_2\alpha_4 a(1-b)}{e_1} + \frac{e_4\alpha_3 b}{e_3(\beta_3+N)}\}NP$

So, $\frac{d\eta}{dt} + \xi\eta \leq U$

Comparison Lemma: *If $x(t)$ is a absolutely continuous function which satisfies the differential inequality:*

$$\frac{dx(t)}{dt} + u_1 x(t) \leq u_2 \text{ such that } t \geq 0$$

where $(u_1,u_2)\in\mathbb{R}^2$ *and* $u_1\neq 0$. *Then, for all* $t\geq T\geq 0$, *we have* $x(t)\leq \frac{u_2}{u_1}-(\frac{u_2}{u_1}-x(T))e^{-u_1(t-T)}$
So, using above lemma, we have

$$\eta(t)\leq \frac{U}{\xi}-(\frac{U}{\xi}-\eta(T))e^{-\xi(t-T)}\ \ for\ \ t\geq T\geq 0$$

If $T=0$, then

$$\eta(t)\leq \frac{U}{\xi}-(\frac{U}{\xi}-\eta(0))e^{-\xi t}$$

Therefore, for $t\to\infty$ we have $0<\eta(t)\leq\frac{U}{\xi}$
Thus, all solution of (5) that initiate in $\mathbb{R}^3_+$ and restricted in the following region

$$\Omega=\left\{\Big(R(t),N(t),P(t)\Big)\in\mathbb{R}^3_+,\ \ 0<R(t)\leq K,\ \ 0<R(t)+\frac{N(t)}{e_1}+\frac{P(t)}{e_3}\leq\frac{U}{\xi}\right\}$$

Hence, all solutions of the system (5) are bounded.

☞ Persistence and Permanence

To establish the persistence for the model system (5), we need to recall the following lemma,

Lemma 1. *If* $m>0$, $n>0$ *and* $\frac{dz(t)}{dt}\leq(\geq)z(t)\Big(n-mz(t)\Big)$, $z(t_0)>0$, *then we have*

$$\limsup_{t\to\infty} z(t)\leq\frac{n}{m}\left(\liminf_{t\to\infty} z(t)\geq\frac{n}{m}\right)$$

Now, from the first equation of system (5) we have

$$\frac{dR(t)}{dt}=rR(t)\left(1-\frac{R(t)}{K}\right)-\frac{\alpha_1R(t)N(t)}{\beta_1+R(t)}-\frac{\alpha_2R(t)(1-b)P(t)}{\beta_2+R(t)}$$

$$\text{i.e., }\frac{dR(t)}{dt}\leq R(t)\Big(r-\frac{r}{K}R(t)\Big)$$

Using the Lemma 1, the above inequality becomes

$$\limsup_{t\to\infty} R(t)\leq K=M^*_R(say) \tag{1}$$

Also from the third equation of system (5) we have

$$\frac{dP(t)}{dt}=\frac{e_3\alpha_2R(t)(1-b)P(t)}{\beta_2+R(t)}+\frac{e_4\alpha_3\ N(t)bP(t)}{\beta_3+N(t)}-\gamma_2P^2(t)-\alpha_4aN(t)(1-b)P(t)-d_2P(t)$$

Now, we assume that interaction between intermediate predator and adult top predator is dominated by interaction between adult intermediate predator and juvenile top predator to avoid top predator population explosion and intermediate predator population extinction in future. So, we get

$$\frac{dP(t)}{dt}\leq\frac{e_3\alpha_2R(t)(1-b)P(t)}{\beta_2+R(t)}-\gamma_2P^2(t)-d_2P(t)$$

$$\text{i.e., }\frac{dP(t)}{dt}\leq P(t)\Big((e_3\alpha_2(1-b)K-d_2)-\gamma_2P\Big)$$

Now, with the help of that Lemma 1, we get

$$\limsup_{t\to\infty} P(t)\leq\frac{e_3\alpha_2(1-b)K-d_2}{\gamma_2}=M^*_P(say) \tag{2}$$

Again from the second equation of system (5) we have

$$\frac{dN(t)}{dt} = \frac{e_1\alpha_1 R(t)N(t)}{\beta_1+R(t)} - \frac{\alpha_3\ N(t)bP(t)}{\beta_3+N(t)} - \gamma_1 N^2(t) + e_2\alpha_4 aN(t)(1-b)P(t) - d_1N(t)$$

$$\text{i.e., } \frac{dN(t)}{dt} \leq \frac{e_1\alpha_1 R(t)N(t)}{\beta_1+R(t)} + e_2\alpha_4 aN(t)(1-b)P(t) - \gamma_1 N^2(t) - d_1N(t)$$

After some simple algebraic manipulation we have

$$\frac{dN(t)}{dt} \leq N(t)\Big((e_1\alpha_1 K + e_2\alpha_4 a(1-b)M_P^* - d_1) - \gamma_1 N(t)\Big)$$

Now, using the Lemma 1 we get

$$\limsup_{t\to\infty} N(t) \leq \frac{e_1\alpha_1 K + e_2\alpha_4 a(1-b)M_P^* - d_1}{\gamma_1} = M_N^*(say) \tag{3}$$

☞ <u>Persistence</u>

Theorem 2: *The proposed model system (5) is persistent if it satisfies the following conditions*
(i) $r - \alpha_1 M_N^* - \alpha_2(1-b)M_P^* > 0$,
(ii) $\frac{e_1\alpha_1 m_R}{\beta_1+K} - \alpha_3 bM_P^* - d_1 > 0$,
(iii) $\frac{e_3\alpha_2(1-b)m_R}{\beta_2+K} + \frac{e_4\alpha_3 bm_N}{\beta_3+M_N^*} - \alpha_4 a(1-b)M_N^* - d_2 > 0$

Proof. From the first equation of system (5) we have

$$\begin{aligned}
\frac{dR(t)}{dt} &= rR(t)\left(1-\frac{R(t)}{K}\right) - \frac{\alpha_1 R(t)N(t)}{\beta_1+R(t)} - \frac{\alpha_2 R(t)(1-b)P(t)}{\beta_2+R(t)}\\
\text{i.e., } \frac{dR(t)}{dt} &\geq rR(t)\left(1-\frac{R(t)}{K}\right) - \alpha_1 R(t)N(t) - \alpha_2(1-b)R(t)P(t)\\
\text{i.e., } \frac{dR(t)}{dt} &\geq R(t)\left(r - \alpha_1 N(t) - \alpha_2(1-b)P(t) - \frac{r}{K}R(t)\right)
\end{aligned}$$

Now using (2) and (3), the above inequality can be written as

$$\frac{dR(t)}{dt} \geq R(t)\left(r - \alpha_1 M_N^* - \alpha_2(1-b)M_P^* - \frac{r}{K}R(t)\right)$$

Using Lemma 1, from the above equation, we have

$$\liminf_{t\to\infty} R(t) \geq \frac{r - \alpha_1 M_N^* - \alpha_2(1-b)M_P^*}{\frac{r}{K}} = m_R(say) \tag{4}$$

From the second equation of system (5) we have

$$\begin{aligned}
\frac{dN(t)}{dt} &= \frac{e_1\alpha_1 R(t)N(t)}{\beta_1+R(t)} - \frac{\alpha_3 N(t)bP(t)}{\beta_3+N(t)} - \gamma_1 N^2(t) + e_2\alpha_4 aN(t)(1-b)P(t) - d_1N(t)\\
\frac{dN(t)}{dt} &\geq \frac{e_1\alpha_1 R(t)N(t)}{\beta_1+R(t)} - \alpha_3 bN(t)P(t) - \gamma_1 N^2(t) - d_1N(t)
\end{aligned}$$

Now using (2) and (4), the above inequality can be written as

$$\frac{dN(t)}{dt} \geq N(t)\left(\left(\frac{e_1\alpha_1 m_R}{\beta_1+K} - \alpha_3 bM_P^* - d_1\right) - \gamma_1 N(t)\right)$$

So, using lemma 1, from the above equation, we have

$$\liminf_{t\to\infty} N(t) \geq \frac{\frac{e_1\alpha_1 m_R}{\beta_1+K} - \alpha_3 b M_P^* - d_1}{\gamma_1} = m_N(say) \tag{5}$$

Also from the third equation of system (5) we have

$$\frac{dP(t)}{dt} = \frac{e_3\alpha_2 R(t)(1-b)P(t)}{\beta_2+R(t)} + \frac{e_4\alpha_3\ N(t)bP(t)}{\beta_3+N(t)} - \gamma_2 P^2(t) - \alpha_4 a N(t)(1-b)P(t) - d_2 P(t)$$

Now, with the help of (1), (3), (4) and (5), the above equation becomes

$$\frac{dP(t)}{dt} \geq P(t)\left\{\left(\frac{e_3\alpha_2(1-b)m_R}{\beta_2+K} + \frac{e_4\alpha_3 b m_N}{\beta_3+M_N^*} - \alpha_4 a(1-b)M_N^* - d_2\right) - \gamma_2 P(t)\right\}$$

So, using lemma 1, from the above equation, we have

$$\liminf_{t\to\infty} P(t) \geq \frac{\frac{e_3\alpha_2(1-b)m_R}{\beta_2+K} + \frac{e_4\alpha_3 b m_N}{\beta_3+M_N^*} - \alpha_4 a(1-b)M_N^* - d_2}{\gamma_2} = m_P(say) \tag{6}$$

Since, $R(t)$, $N(t)$, $P(t)$ and all other parameters associated to the system (5) are positive, so

$$\liminf_{t\to\infty} R(t) > 0,\ \ \liminf_{t\to\infty} N(t) > 0,\ \ \liminf_{t\to\infty} P(t) > 0 \ \text{ provides}$$

(i) $r - \alpha_1 M_N^* - \alpha_2(1-b)M_P^* > 0$,
(ii) $\frac{e_1\alpha_1 m_R}{\beta_1+K} - \alpha_3 b M_P^* - d_1 > 0$,
(iii) $\frac{e_3\alpha_2(1-b)m_R}{\beta_2+K} + \frac{e_4\alpha_3 b m_N}{\beta_3+M_N^*} - \alpha_4 a(1-b)M_N^* - d_2 > 0$ ∎

☞ Permanence

Theorem 3: *The system (5) is said to be permanent if ∃ positive constants m and M, with $0 < m \leq M$ such that*

$$\min\{\liminf_{t\to\infty} R(t), \liminf_{t\to\infty} N(t), \liminf_{t\to\infty} P(t)\} \geq m$$

and

$$\max\{\limsup_{t\to\infty} R(t), \limsup_{t\to\infty} N(t), \limsup_{t\to\infty} P(t)\} \leq M$$

for all solutions $\Big(R(t), N(t), P(t)\Big)$ of the model system (5) with positive initial values.

Proof. With the help of the equations (1)–(6) one can say that our proposed system (5) is also permanent which is summarized below.
From the equations (1)–(6), we have

$$m_R \leq \liminf_{t\to\infty} R(t) \leq \limsup_{t\to\infty} R(t) \leq M_R^*$$

$$m_N \leq \liminf_{t\to\infty} N(t) \leq \limsup_{t\to\infty} N(t) \leq M_N^*$$

$$m_P \leq \liminf_{t\to\infty} P(t) \leq \limsup_{t\to\infty} P(t) \leq M_P^*$$

Now, from the definition of limit inferior and limit superior, we have

$$\begin{aligned} R(t) &\geq m_R,\ \ \forall t \geq t_1, & R(t) &\leq M_R^*,\ \ \forall t \geq t_2 \\ N(t) &\geq m_N,\ \ \forall t \geq t_3, & N(t) &\leq M_N^*,\ \ \forall t \geq t_4 \\ P(t) &\geq m_P,\ \ \forall t \geq t_5, & P(t) &\leq M_P^*,\ \ \forall t \geq t_6 \end{aligned}$$

where t_j $(1 \leq j \leq 6,\ j \in \mathbb{N})$ are six positive constants. Now, if we define $m = \min\{m_R, m_N, m_P\}$, $M = \max\{M_R^*, M_N^*, M_P^*\}$ and $T = \max\{t_j :\ 1 \leq j \leq 6,\ j \in \mathbb{N}\}$, then the system (5) is permanent.
∎

☞ **Equilibrium points of the deterministic model**

(i) The trivial equilibrium point $E_0 = (0, 0, 0)$

(ii) The predators-free (axial) equilibrium point $E_1 = (K, 0, 0)$

(iii) The top predator-free (planar) equilibrium point $\bar{E} = (\bar{R}, \bar{N}, 0)$
where, $\bar{R} = \frac{(K-\beta_1)+\sqrt{(K-\beta_1)^2+4K(\beta_1-\frac{\alpha_1\bar{N}}{r})}}{2}$ and $\bar{N} = \frac{1}{\gamma_1}\left(\frac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}} - d_1\right)$.
Here, the top predator-free equilibrium point $\bar{E}$ is positively exists if
$\bar{N} < \frac{\beta_1 r}{\alpha_1}$ and $\bar{R} > \frac{d_1\beta_1}{e_1\alpha_1-d_1}$

(iv) The intermediate predator-free (planar) equilibrium point $\hat{E} = (\hat{R}, 0, \hat{P})$
where, $\hat{R} = \frac{(K-\beta_2)+\sqrt{(K-\beta_2)^2+4K(\beta_2-\frac{\alpha_2(1-b)\hat{P}}{r})}}{2}$ and $\hat{P} = \frac{1}{\gamma_2}\left(\frac{e_3\alpha_2(1-b)\hat{R}}{\beta_2+\hat{R}} - d_2\right)$.
Then, the intermediate predator-free equilibrium point $\hat{E}$ is positively exists if
$\hat{P} < \frac{\beta_2 r}{\alpha_2(1-b)}$ and $\hat{R} > \frac{d_2\beta_2}{e_3\alpha_2(1-b)-d_2}$

(v) The prey-free (planar) equilibrium point $\tilde{E} = (0, \tilde{N}, \tilde{P})$.
Suppose, $\psi_1 = \alpha_4 a(1-b)\beta_3 + (\gamma_2 P + d_2)a - e_3\alpha_3 ab$, ($< 0$ assume)
then $\tilde{N} = \frac{\psi_1}{2\alpha_4 a^2(1-b)}$ and $\tilde{P} = \frac{(\beta_3+a\tilde{N})(\gamma_1\tilde{N}+d_1)}{(\beta_3+a\tilde{N})e_2\alpha_4(1-b)-\alpha_3 ab}$
So, if $\psi_1^2 = 4\beta_3\alpha_4 a^2(1-b)(\gamma_2 P + d_2)$ and $\alpha_4 > \frac{\alpha_3 ab}{e_2(1-b)(\beta_3+aN)}$ then the prey-free equilibrium point $\tilde{E}$ is positively exist.

(vi) Suppose, $E^* = (R^*, N^*, P^*)$ be the coexisting equilibrium point where R^*, N^* and P^* is the positive root of the following system of equations

$$\left.\begin{aligned}
& rR^*(1-\tfrac{R^*}{K}) - \tfrac{\alpha_1 R^* N^*}{\beta_1+R^*} - \tfrac{\alpha_2 R^*(1-b)P^*}{\beta_2+R^*} = 0 \\
& \tfrac{e_1\alpha_1 R^* N^*}{\beta_1+R^*} - \tfrac{\alpha_3 a N^* b P^*}{\beta_3+aN^*} - \gamma_1 N^{*2} + e_2\alpha_4 a N^*(1-b)P^* - d_1 N^* = 0 \\
& \tfrac{e_3\alpha_2 R^*(1-b)P^*}{\beta_2+R^*} + \tfrac{e_4\alpha_3 a N^* b P^*}{\beta_3+aN^*} - \gamma_2 P^{*2} - \alpha_4 a N^*(1-b)P^* - d_2 P^* = 0
\end{aligned}\right\} \tag{7}$$

Now, solving the above system of equations we get

$$R^* = \frac{\beta_1 H}{e_1\alpha_1 - H}, \text{ where } H = \frac{\alpha_3 abP}{\beta_3 + aN} + \gamma_1 N - e_2\alpha_4 a(1-b)P + d_1$$
$$N^* = \frac{\beta_1 + R}{\alpha_1}\left\{r(1-\frac{R}{K}) - \frac{\alpha_2(1-b)P}{\beta_2 + R}\right\}$$
$$\text{and } P^* = \frac{1}{\gamma_1}\left\{\frac{e_3\alpha_2(1-b)R}{\beta_2 + R} + \frac{e_4\alpha_3 abN}{\beta_3 + aN} - \alpha_4 a(1-b)N - d_2\right\}$$

☞ Local stability analysis

The stability of the a system is determined by the nature of the eigenvalues of the corresponding Jacobian matrix around the different possible equilibrium points of the system (5).
Suppose, $f_1 = rR(1-\frac{R}{K}) - \frac{\alpha_1 RN}{\beta_1+R} - \frac{\alpha_2 R(1-b)P}{\beta_2+R}$

$$f_2 = \frac{e_1\alpha_1 RN}{\beta_1 + R} - \frac{\alpha_3 aNbP}{\beta_3 + aN} - \gamma_1 N^2 + e_2\alpha_4 aN(1-b)P - d_1 N$$

$$f_3 = \frac{e_3\alpha_2 R(1-b)P}{\beta_2 + R} + \frac{e_4\alpha_3 aNbP}{\beta_3 + aN} - \gamma_2 P^2 - \alpha_4 aN(1-b)P - d_2 P$$

The Jacobian matrix of the system (5) is defined by

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial R} & \frac{\partial f_1}{\partial N} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial R} & \frac{\partial f_2}{\partial N} & \frac{\partial f_2}{\partial P} \\ \frac{\partial f_3}{\partial R} & \frac{\partial f_3}{\partial N} & \frac{\partial f_3}{\partial P} \end{bmatrix}$$

$\frac{\partial f_1}{\partial R} = r(1 - \frac{2R}{K}) - \frac{\alpha_1\beta_1 N}{(\beta_1+R)^2} - \frac{(1-b)\alpha_2\beta_2 P}{(\beta_2+R)^2}$

$\frac{\partial f_1}{\partial N} = -\frac{\alpha_1 R}{\beta_1+R}$

$\frac{\partial f_1}{\partial P} = -\frac{\alpha_2(1-b)R}{\beta_2+R}$

$\frac{\partial f_2}{\partial R} = \frac{e_1\alpha_1\beta_1 N}{(\beta_1+R)^2}$

$\frac{\partial f_2}{\partial N} = \frac{e_1\alpha_1 R}{\beta_1+R} - \frac{ab\alpha_3\beta_3 P}{(\beta_3+aN)^2} - 2\gamma_1 N + e_2\alpha_4 a(1-b)P - d_1$

$\frac{\partial f_2}{\partial P} = -\frac{\alpha_3 abN}{\beta_3+aN} + e_2\alpha_4 a(1-b)N$

$\frac{\partial f_3}{\partial R} = \frac{(1-b)e_3\alpha_2\beta_2 P}{(\beta_2+R)^2}$

$\frac{\partial f_3}{\partial N} = \frac{abe_4\alpha_3\beta_3 P}{(\beta_3+aN)^2} - \alpha_4 a(1-b)P$

$\frac{\partial f_3}{\partial P} = \frac{(1-b)e_3\alpha_2 R}{\beta_2+R} + \frac{e_4\alpha_3 abN}{\beta_3+aN} - 2\gamma_2 P - \alpha_4 a(1-b)N - d_2$

Theorem 4: *For the system (5),*

- *Trivial equilibrium E_0 is always unstable.*
- *Predators-extinction equilibrium E_1 is locally asymptotically stable if the following conditions hold:*
  $\alpha_1 < \frac{d_1}{e_1}\left(\frac{\beta_1}{K} + 1\right)$ *and* $\alpha_2 < \frac{d_2}{e_3(1-b)}\left(\frac{\beta_2}{K} + 1\right)$.
- *Top predator-extinction equilibrium $\bar{E}$ is locally asymptotically stable if* $\left(\frac{2r\bar{R}}{K} + \frac{\alpha_1\beta_1\bar{N}}{\beta_1+\bar{R}} + 2\gamma_1\bar{N} + d_1\right) > \left(r + \frac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}}\right)$, $\frac{e_1\alpha_1^2\beta_1\bar{R}\bar{N}}{(\beta_1+\bar{R})^3} > \left\{\frac{\alpha_1\beta_1\bar{N}}{\beta_1+\bar{R}} + r(\frac{2\bar{R}}{K} - 1)\right\}\left\{\frac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}} - (2\gamma_1\bar{N} + d_1)\right\}$ *and* $\alpha_4 > \alpha_4^*$ *(= threshold value of α_4).*
- *Intermediate predator-extinction equilibrium $\hat{E}$ is locally asymptotically stable if the following conditions hold:*
  $\alpha_3 > \frac{\beta_3}{b\hat{P}}\left\{\frac{e_1\alpha_1\hat{R}}{\beta_1+\hat{R}} + e_2\alpha_4 a(1-b)\hat{P} - d_1\right\} = \alpha_3^*$ *(= threshold value of α_3) and $tr(B) < 0$ and $det(B) > 0$.*
- *Prey-extinction equilibrium $\tilde{E}$ is locally asymptotically stable if $r < \frac{\alpha_1\tilde{N}}{\beta_1} + \frac{(1-b)\alpha_2\tilde{P}}{\beta_2}$ and $tr(C) = (D_1 + D_4) < 0$ and $det(C) = (D_1D_4 - D_2D_3) > 0$.*

Proof. • **For trivial equilibrium point** $E_0(0,0,0)$

The Jacobian at $E_0 = (0,0,0)$ is given by

$$[\mathbf{J}]_{E_0} = \begin{bmatrix} r & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$

The eigenvalues of $[\mathbf{J}]_{E_0}$ are r, $-d_1$ and $-d_2$.
So, the equilibrium point E_0 is unstable since $r(>0)$ is an eigenvalue.

- **For predators-free equilibrium point $E_1(K,0,0)$**

The Jacobian at $E_1 = (K,0,0)$ is given by

$$[\mathbf{J}]_{E_1} = \begin{bmatrix} -r & -\frac{\alpha_1 K}{\beta_1+K} & -\frac{\alpha_2(1-b)K}{\beta_2+K} \\ 0 & \frac{e_1\alpha_1 K}{\beta_1+K} - d_1 & 0 \\ 0 & 0 & \frac{(1-b)e_3\alpha_2 K}{\beta_2+K} - d_2 \end{bmatrix}$$

So, the eigenvalues of $[\mathbf{J}]_{E_1}$ are $-r$, $\frac{e_1\alpha_1 K}{\beta_1+K} - d_1$ and $\frac{(1-b)e_3\alpha_2 K}{\beta_2+K} - d_2$.
Hence, $E_1 = (K,0,0)$ is locally asymptotically stable if

$$\frac{e_1\alpha_1 K}{\beta_1+K} < d_1 ....(i)$$

and

$$\frac{(1-b)e_3\alpha_2 K}{\beta_2+K} < d_2 ....(ii)$$

But E_1 losses its stability when either the condition (i) or (ii) fail.
Hence we conclude that the predators free equilibrium point E_1 is locally asymptotically stable when both the boundary equilibria, i.e., top predator free equilibria $\bar{E}$ and intermediate predator free equilibria $\hat{E}$ do not exist.

- **For top predator-free equilibrium point $\bar{E}(\bar{R},\bar{N},0)$**

The Jacobian evaluated at the top predator-free equilibrium point $\bar{E}$ is given by

$$[\mathbf{J}]_{\bar{E}} = \begin{bmatrix} r(1-\frac{2\bar{R}}{K}) - \frac{\alpha_1\beta_1\bar{N}}{\beta_1+\bar{R}} & -\frac{\alpha_1\bar{R}}{\beta_1+\bar{R}} & -\frac{\alpha_2(1-b)\bar{R}}{\beta_2+\bar{R}} \\ \frac{e_1\alpha_1\beta_1\bar{N}}{(\beta_1+\bar{R})^2} & \frac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}} - 2\gamma_1\bar{N} - d_1 & -\frac{\alpha_3 ab\bar{N}}{\beta_3+a\bar{N}} + e_2\alpha_4 a(1-b)\bar{N} \\ 0 & 0 & \frac{(1-b)e_3\alpha_2\bar{R}}{\beta_2+\bar{R}} + \frac{e_4\alpha_3 ab\bar{N}}{\beta_3+a\bar{N}} - \alpha_4 a(1-b)\bar{N} - d_2 \end{bmatrix}$$

One eigenvalue of the above Jacobian matrix $[\mathbf{J}]_{\bar{E}}$ is

$$\begin{aligned} \lambda_{\bar{E}} &= \frac{(1-b)e_3\alpha_2\bar{R}}{\beta_2+\bar{R}} + \frac{e_4\alpha_3 ab\bar{N}}{\beta_3+a\bar{N}} - \alpha_4 a(1-b)\bar{N} - d_2 \\ &= -\big(\alpha_4 a(1-b)\bar{N} + d_2\big) - \Big\{\frac{(1-b)e_3\alpha_2\bar{R}}{\beta_2+\bar{R}} + \frac{e_4\alpha_3 ab\bar{N}}{\beta_3+a\bar{N}}\Big\} \end{aligned}$$

and the other two eigenvalues are the roots of the characteristic polynomial of the following matrix

$$A = \begin{bmatrix} r(1-\frac{2\bar{R}}{K}) - \frac{\alpha_1\beta_1\bar{N}}{\beta_1+\bar{R}} & -\frac{\alpha_1\bar{R}}{\beta_1+\bar{R}} \\ \frac{e_1\alpha_1\beta_1\bar{N}}{(\beta_1+\bar{R})^2} & \frac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}} - 2\gamma_1\bar{N} - d_1 \end{bmatrix}$$

So, eigenvalues of A will be negative or have negative real part if $tr(A) < 0$ and $det(A) > 0$.

$$\begin{aligned} \text{Now, } tr(A) &= r - \frac{2r\bar{R}}{K} - \frac{\alpha_1\beta_1\bar{N}}{\beta_1+\bar{R}} + \frac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}} - 2\gamma_1\bar{N} - d_1 \\ &= -\Big[\Big(\frac{2r\bar{R}}{K} + \frac{\alpha_1\beta_1\bar{N}}{\beta_1+\bar{R}} + 2\gamma_1\bar{N} + d_1\Big) - \Big(r + \frac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}}\Big)\Big] \end{aligned}$$

and $det(A) = \dfrac{e_1\alpha_1^2\beta_1\bar{R}\bar{N}}{(\beta_1+\bar{R})^3} - \Big\{\dfrac{\alpha_1\beta_1\bar{N}}{\beta_1+\bar{R}} + r(\dfrac{2\bar{R}}{K}-1)\Big\}\Big\{\dfrac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}} - (2\gamma_1\bar{N}+d_1)\Big\}$

Hence, $tr(A) < 0$ and $det(A) > 0$ if $\big(\frac{2r\bar{R}}{K} + \frac{\alpha_1\beta_1\bar{N}}{\beta_1+\bar{R}} + 2\gamma_1\bar{N} + d_1\big) > \big(r + \frac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}}\big)$ and $\frac{e_1\alpha_1^2\beta_1\bar{R}\bar{N}}{(\beta_1+\bar{R})^3} > \big\{\frac{\alpha_1\beta_1\bar{N}}{\beta_1+\bar{R}} + r(\frac{2\bar{R}}{K}-1)\big\}\big\{\frac{e_1\alpha_1\bar{R}}{\beta_1+\bar{R}} - (2\gamma_1\bar{N}+d_1)\big\}$ respectively.
Also, we have $\lambda_{\bar{E}} < 0$ when $\big(\alpha_4 a(1-b)\bar{N} + d_2\big) > \frac{(1-b)e_3\alpha_2\bar{R}}{\beta_2+\bar{R}} + \frac{e_4\alpha_3 ab\bar{N}}{\beta_3+a\bar{N}}$

$$\text{i.e., } \alpha_4 > \frac{1}{a(1-b)\bar{N}}\Big[\frac{(1-b)e_3\alpha_2\bar{R}}{\beta_2+\bar{R}} + \frac{e_4\alpha_3 ab\bar{N}}{\beta_3+a\bar{N}} - d_2\Big] = \alpha_4^* \text{ (= threshold value of } \alpha_4)$$

Hence, stability of the system (5) around $\bar{R}$ changes when α_4 crosses it threshold value of α_4^*

- **For intermediate predator-free equilibrium point** $\hat{E}(\hat{R}, 0, \hat{P})$

The Jacobian evaluated at the top predator-free equilibrium point $\hat{E}$ is given by

$$[\mathbf{J}]_{\hat{E}} = \begin{bmatrix} r(1-\frac{2\hat{R}}{K}) - \frac{(1-b)\alpha_2\beta_2\hat{P}}{\beta_2+\hat{R}} & -\frac{\alpha_1\hat{R}}{\beta_1+\hat{R}} & -\frac{\alpha_2(1-b)\hat{R}}{\beta_2+\hat{R}} \\ 0 & \frac{e_1\alpha_1\hat{R}}{\beta_1+\hat{R}} - \frac{ab\alpha_3\beta_3\hat{P}}{\beta_3^2} + e_2\alpha_4 a(1-b)\hat{P} - d_1 & 0 \\ \frac{(1-b)e_3\alpha_2\beta_2\hat{P}}{(\beta_2+\hat{R})^2} & \frac{abe_4\alpha_3\beta_3\hat{P}}{\beta_3^2} - \alpha_4 a(1-b)\hat{P} & \frac{(1-b)e_3\alpha_2\bar{R}}{\beta_2+\hat{R}} - 2\gamma_2\hat{P} - d_2 \end{bmatrix}$$

One eigenvalue of the above Jacobian matrix $[\mathbf{J}]_{\hat{E}}$ is

$$\begin{aligned} \lambda_{\hat{E}} &= \frac{e_1\alpha_1\hat{R}}{\beta_1+\hat{R}} + e_2\alpha_4 a(1-b)\hat{P} - \Big\{\frac{ab\alpha_3\beta_3\hat{P}}{\beta_3^2} + d_1\Big\} \\ &= -\Big\{\Big(\frac{ab\alpha_3\beta_3\hat{P}}{\beta_3^2} + d_1\Big) - \Big(\frac{e_1\alpha_1\hat{R}}{\beta_1+\hat{R}} + e_2\alpha_4 a(1-b)\hat{P}\Big)\Big\} \end{aligned}$$

Now, $\lambda_{\hat{E}} < 0$ when $\frac{ab\alpha_3\beta_3\hat{P}}{\beta_3^2} + d_1 > \frac{e_1\alpha_1\hat{R}}{\beta_1+\hat{R}} + e_2\alpha_4 a(1-b)\hat{P}$
i.e., $\alpha_3 > \frac{\beta_3}{ab\hat{P}}\Big\{\frac{e_1\alpha_1\hat{R}}{\beta_1+\hat{R}} + e_2\alpha_4 a(1-b)\hat{P} - d_1\Big\} = \alpha_3^*$ (= threshold value)
The other two eigenvalues of the matrix $[\mathbf{J}]_{\hat{E}}$ are the roots of characteristic polynomial of the following matrix:

$$B = \begin{bmatrix} r(1-\frac{2\hat{R}}{K}) - \frac{(1-b)\alpha_2\beta_2\hat{P}}{\beta_2+\hat{R}} & -\frac{\alpha_2(1-b)\hat{R}}{\beta_2+\hat{R}} \\ \frac{(1-b)e_3\alpha_2\beta_2\hat{P}}{(\beta_2+\hat{R})^2} & \frac{(1-b)e_3\alpha_2\bar{R}}{\beta_2+\hat{R}} - 2\gamma_2\hat{P} - d_2 \end{bmatrix}$$

Now, eigenvalues of B will be negative or have negative real part if $tr(B) < 0$ and $det(B) > 0$.

- **For prey-free equilibrium point** $\tilde{E}(0, \tilde{N}, \tilde{P})$

The Jacobian evaluated at the prey-free equilibrium point $\tilde{E}$ is given by

$$[\mathbf{J}]_{\tilde{E}} = \begin{bmatrix} r - \frac{\alpha_1\beta_1\tilde{N}}{\beta_1^2} - \frac{(1-b)\alpha_2\beta_2\tilde{P}}{\beta_2^2} & 0 & 0 \\ \frac{e_1\alpha_1\beta_1\tilde{N}}{\beta_1^2} & -\frac{ab\alpha_3\beta_3\tilde{P}}{(\beta_3+a\tilde{N})^2} - 2\gamma_1\tilde{N} + e_2\alpha_4 a(1-b)\tilde{P} - d_1 & -\frac{\alpha_3 ab\tilde{N}}{\beta_3+a\tilde{N}} + e_2\alpha_4 a(1-b)\tilde{N} \\ \frac{(1-b)e_3\alpha_2\beta_2\tilde{P}}{\beta_2^2} & \frac{abe_4\alpha_3\beta_3\tilde{P}}{(\beta_3+a\tilde{N})^2} - \alpha_4 a(1-b)\tilde{P} & \frac{abe_4\alpha_3\tilde{N}}{\beta_3+a\tilde{N}} - 2\gamma_2\tilde{P} - \alpha_4 a(1-b)\tilde{N} - d_2 \end{bmatrix}$$

One eigenvalue of the above matrix $[\mathbf{J}]_{\tilde{E}}$ is

$$\lambda_{\tilde{E}} = r - \frac{\alpha_1\beta_1\tilde{N}}{\beta_1^2} - \frac{(1-b)\alpha_2\beta_2\tilde{P}}{\beta_2^2}$$

Now, $\lambda_{\tilde{E}} < 0$ when $r < \frac{\alpha_1\beta_1\tilde{N}}{\beta_1^2} + \frac{(1-b)\alpha_2\beta_2\tilde{P}}{\beta_2^2}$.
And the other eigenvalues of $[\mathbf{J}]_{\tilde{E}}$ are the roots of characteristic polynomial of the following matrix:

$$C = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

where, $D_1 = -\frac{ab\alpha_3\beta_3\tilde{P}}{(\beta_3+a\tilde{N})^2} - 2\gamma_1\tilde{N} - d_1 + e_2\alpha_4 a(1-b)\tilde{P}$
$D_2 = -\frac{\alpha_3 ab\tilde{N}}{\beta_3+a\tilde{N}} + e_2\alpha_4 a(1-b)\tilde{N}$
$D_3 = \frac{abe_4\alpha_3\beta_3\tilde{P}}{(\beta_3+a\tilde{N})^2} - \alpha_4 a(1-b)\tilde{P}$
$D_4 = \frac{abe_4\alpha_3\tilde{N}}{\beta_3+a\tilde{N}} - 2\gamma_2\tilde{P} - \alpha_4 a(1-b)\tilde{N} - d_2$
Hence, the roots of the characteristic polynomial of the matrix C will be negative or negative real part if $tr(C) = (D_1 + D_4) < 0$ and $det(C) = (D_1D_4 - D_2D_3) > 0$. ∎

Theorem 5: *The coexisting equilibrium point (E^*) is locally asymptotically stable if $\xi_1 > 0$, $\xi_3 > 0$ and $\xi_1\xi_2 - \xi_3 > 0$ where ξ_1, ξ_2 and ξ_3 are the coefficients of the characteristic equation of the Jacobian matrix $[J]_{E^*}$ which is $\lambda^3 + \xi_1\lambda^2 + \xi_2\lambda + \xi_3 = 0$.*

Proof. The corresponding Jacobian matrix for the interior equilibrium point E^* is given by

$$[\mathbf{J}]_{E^*} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

Now, the characteristic equation of the Jacobian matrix $[J]_{E^*}$ is given by
$\lambda^3 + \xi_1\lambda^2 + \xi_2\lambda + \xi_3 = 0$ where, the parametric values of ξ_1, ξ_2, ξ_3 and all other associated parameters are expressed in the following table:

Table 1: Expressions for ξ_1, ξ_2, and ξ_3.

$\xi_1 = -(J_{11} + J_{22} + J_{33})$.	
$\xi_2 = J_{11}J_{22} + J_{22}J_{33} + J_{33}J_{11} - J_{12}J_{21} - J_{23}J_{32} - J_{13}J_{31}$.	
$\xi_3 = J_{11}J_{23}J_{32} + J_{22}J_{13}J_{31} + J_{33}J_{12}J_{21} - J_{11}J_{22}J_{33} - J_{12}J_{23}J_{31} - J_{21}J_{32}J_{13}$.	
$J_{11} = r(1 - \frac{2R}{K}) - \frac{\alpha_1\beta_1 N}{(\beta_1+R)^2} - \frac{(1-b)\alpha_2\beta_2 P}{(\beta_2+R)^2}$	$J_{22} = \frac{e_1\alpha_1 R}{\beta_1+R} - \frac{ab\alpha_3\beta_3 P}{(\beta_3+aN)^2} - 2\gamma_1 N + e_2\alpha_4 a(1-b)P - d_1$
$J_{12} = -\frac{\alpha_1 R}{\beta_1+R}$	$J_{23} = -\frac{\alpha_3 abN}{\beta_3+aN} + e_2\alpha_4 a(1-b)N$
$J_{13} = -\frac{\alpha_2(1-b)R}{\beta_2+R}$	$J_{31} = \frac{(1-b)e_3\alpha_2\beta_2 P}{(\beta_2+R)^2}$.
$J_{21} = \frac{e_1\alpha_1\beta_1 N}{(\beta_1+R)^2}$	$J_{32} = \frac{abe_4\alpha_3\beta_3 P}{(\beta_3+aN)^2} - \alpha_4 a(1-b)P$
$J_{33} = \frac{(1-b)e_3\alpha_2 R}{\beta_2+R} + \frac{e_4\alpha_3 abN}{\beta_3+aN} - 2\gamma_2 P - \alpha_4 a(1-b)N - d_2$	

Therefore, using Routh-Hurwitz criterion, E^* will be locally asymptotically stable, i.e., the eigenvalues of the characteristic equation will be negative or roots have negative real part if $\xi_1 > 0$, $\xi_3 > 0$ and $\xi_1\xi_2 - \xi_3 > 0$; otherwise the system will be unstable around $E^* = (R^*, N^*, P^*)$. ∎

☞ Global stability analysis

Theorem 6: *The positive coexisting equilibrium $E^*(R^*, N^*, P^*)$ is globally asymptotically stable with respect to all the solutions initiating in the interior of $\mathbb{R}^3_+$ if the following condition hold:*
(i) $e_2 > 1$,
(ii) $e_1 > 1$ and $A' < \beta_1(e_1 - 1)$,
(iii) $\frac{A'}{K} > \frac{\alpha_1 B}{(\beta_1+A')(\beta_1+R^*)}$

Proof. We show the global stability of coexisting equilibrium $E^*(R^*, N^*, P^*)$ by by constructing a suitable Lyapunav function $V(R, N, P) : \mathbb{R}^3_+ \to \mathbb{R}$ such that

$$V(R, N, P) = V_1(R, N, P) + V_2(R, N, P) + V_3(R, N, P)$$

where, $V_1(R, N, P) = \big(R - R^* - R^* \ln(\frac{R}{R^*})\big)$
$V_2(R, N, P) = \big(N - N^* - N^* \ln(\frac{N}{N^*})\big)$
$V_3(R, N, P) = \big(P - P^* - P^* \ln(\frac{P}{P^*})\big)$

Apparently, the function $V(R, N, P)$ is defined and continuous on the interior of $\mathbb{R}^3_+$. Now, it can be easily seen that the function $V(R, N, P)$ is zero at (R^*, N^*, P^*) and is positive in the interior of $\mathbb{R}^3_+$. Thus $V(R, N, P)$ takes minimum value at (R^*, N^*, P^*). We evaluate the time derivative of this scalar valued function $V(R, N, P)$ along the solutions of the model (5).

So, the time derivative of $V(R, N, P)$ along the solutions of (5) is

$$\frac{dV}{dt} = \frac{dV_1}{dt} + \frac{dV_2}{dt} + \frac{dV_3}{dt}$$

Thus, $\frac{dV_1}{dt} = \frac{1}{R}(R - R^*)\frac{dR}{dt}$
$\frac{dV_2}{dt} = \frac{1}{N}(N - N^*)\frac{dN}{dt}$
$\frac{dV_3}{dt} = \frac{1}{P}(P - P^*)\frac{dP}{dt}$

Now, $\frac{dV_1}{dt} = (R - R^*)\Big[r\big(1 - \frac{R}{K}\big) - \frac{\alpha_1 N}{\beta_1 + R} - \frac{\alpha_2(1 - b)P}{\beta_2 + R}\Big]$
Since (R^*, N^*, P^*) satisfies (5), we have $\frac{\alpha_2(1 - b)P^*}{\beta_2 + R^*} = r\big(1 - \frac{R^*}{K}\big) - \frac{\alpha_1 N^*}{\beta_1 + R^*}$.
After some algebraic simplification and with the help of above equation, the time derivative of V_1 takes the following form

$$\frac{dV_1}{dt} = -\Big\{\frac{R}{K} - \frac{\alpha_1 N}{(\beta_1 + R)(\beta_1 + R^*)}\Big\}(R - R^*)^2 - \frac{\alpha_1}{\beta_1 + R^*}(R - R^*)(N - N^*)$$

Similarly, we have

$$\frac{dV_2}{dt} = -\gamma_1(N - N^*)^2 + e_2\alpha_4 a(1 - b)(N - N^*)(P - P^*) + \frac{e_1\alpha_1\beta_1}{(\beta_1 + R)(\beta_1 + R^*)}(R - R^*)(N - N^*)$$

$$\frac{dV_3}{dt} = -\gamma_2(P - P^*)^2 - \alpha_4 a(1 - b)(N - N^*)(P - P^*) + \frac{e_3\alpha_2\beta_2(1 - b)}{(\beta_2 + R)(\beta_2 + R^*)}(R - R^*)(P - P^*)$$

Now, putting the value of $\frac{dV_1}{dt}$, $\frac{dV_2}{dt}$ and $\frac{dV_3}{dt}$ we get

$$\begin{aligned}\frac{dV}{dt} \;=\; & -\Big\{\frac{R}{K} - \frac{\alpha_1 N}{(\beta_1 + R)(\beta_1 + R^*)}\Big\}(R - R^*)^2 - \gamma_1(N - N^*)^2 - \gamma_2(P - P^*)^2 \\ & + \Big[\frac{e_1\alpha_1\beta_1}{(\beta_1 + R)(\beta_1 + R^*)} - \frac{\alpha_1}{\beta_1 + R^*}\Big](R - R^*)(N - N^*) \\ & + (e_2 - 1)\alpha_4 a(1 - b)(N - N^*)(P - P^*) + \frac{e_3\alpha_2\beta_2(1 - b)}{(\beta_2 + R)(\beta_2 + R^*)}(R - R^*)(P - P^*)\end{aligned}$$

If we consider,

$$\limsup_{t\to\infty} R(t) = A \qquad \liminf_{t\to\infty} R(t) = A'$$
$$\limsup_{t\to\infty} N(t) = B \qquad \liminf_{t\to\infty} N(t) = B'$$
$$\limsup_{t\to\infty} P(t) = C \qquad \liminf_{t\to\infty} P(t) = C'$$

Using the above consideration, we have

$$\begin{aligned}\frac{dV}{dt} &< -\Big[\frac{A'}{K} - \frac{\alpha_1 B}{(\beta_1 + A')(\beta_1 + R^*)}\Big](R-R^*)^2 - \gamma_1(N-N^*)^2 - \gamma_2(P-P^*)^2 \\ &\quad + \Big[\frac{e_1\alpha_1\beta_1}{(\beta_1 + A')(\beta_1 + R^*)} - \frac{\alpha_1}{\beta_1 + R^*}\Big](R-R^*)(N-N^*) \\ &\quad + (e_2-1)\alpha_4 a(1-b)(N-N^*)(P-P^*) + \frac{e_3\alpha_2\beta_2(1-b)}{(\beta_2 + A')(\beta_2 + R^*)}(R-R^*)(P-P^*)\end{aligned}$$

If $e_2 > 1$, $e_1 > 1$ and $A' < \beta_1(e_1 - 1)$, then the above equation reduces to the following form

$$\frac{dV}{dt} < -\Big[\frac{A'}{K} - \frac{\alpha_1 B}{(\beta_1 + A')(\beta_1 + R^*)}\Big](R-R^*)^2 - \gamma_1(N-N^*)^2 - \gamma_2(P-P^*)^2$$

Hence, we conclude that if the hypotheses of Theorem are satisfied then $\frac{dV}{dt} < 0$ along all the trajectories in $\mathbb{R}^3_+$ except (R^*, N^*, P^*).

Therefore, $E^*(R^*, N^*, P^*)$ is globally asymptotically stable. ∎

☞ Bifurcation analysis

Theorem 7: *The necessary and sufficient condition for occurrence of Hopf bifurcation of the system (5) at $b = b^*$ are*

(i) $\xi_i(b^*) > 0$ for $i = 1, 2, 3$ and $\xi_1(b^*)\xi_2(b^*) - \xi_3(b^*) = 0$

(ii) $Re[\frac{d\lambda_i}{db}]_{b=b^*} \neq 0$ for $i = 1, 2, 3$

where λ_i are the roots of the characteristic equation corresponding to the coexisting equilibrium point E^*.

Proof. The characteristic equation of the Jacobian matrix around interior equilibrium point $E^* = (R^*, N^*, P^*)$ is

$$\lambda^3 + \xi_1\lambda^2 + \xi_2\lambda + \xi_3 = 0 \tag{8}$$

So, the interior equilibrium point $E^* = (R^*, N^*, P^*)$ is locally asymptotically stable when $\xi_1 > 0$, $\xi_3 > 0$ and $\xi_1\xi_2 - \xi_3 > 0$ where ξ_1, ξ_2, and ξ_3 are given in Table 1.

When $b = b^*$, the characteristic equation (8) becomes

$$(\lambda^2 + \xi_2)(\lambda + \xi_1) = 0 \tag{9}$$

Then, the roots of the above equation (9) are given by $\lambda_1 = +i\sqrt{\xi_2}$, $\lambda_2 = -i\sqrt{\xi_2}$ and $\lambda_3 = -\xi_1$. Due to the condition $\xi_1\xi_2 - \xi_3 = 0$ at $b = b^*$, there exist an open interval $(b^* - \epsilon, b^* - \epsilon)$ for some positive ϵ. Thus for $b^* \in (b^* - \epsilon, b^* - \epsilon)$, the characteristic equation (8) has no roots containing negative real parts.

Now, for all b, the roots of (9) are in the following general form

$$\lambda_1(b) = \mu_1(b) + i\mu_2(b)$$
$$\lambda_2(b) = \mu_1(b) - i\mu_2(b)$$

$\lambda_3(b) = -\xi_1(b)$

Now, we shall verify the transversality condition

$$\frac{d}{db}\Big[Re(\lambda_i(b))\Big]_{b=b^*} \neq 0 \ \text{ for } \quad 1 = 1, 2$$

Now, substituting $\lambda_1(b) = \mu_1(b) + i\mu_2(b)$ into the characteristic equation (9), separating the real and imaginary part after calculating the derivatives, we have

$A(b)\mu_1'(b) - B(b)\mu_2'(b) + C(b) = 0$

$B(b)\mu_1'(b) + A(b)\mu_2'(b) + D(b) = 0$

where,

$A(b) = 3\mu_1^2(b) - 3\mu_2^2(b) + 2\mu_1(b)\xi_1(b) + \xi_2(b)$

$B(b) = 6\mu_1(b)\mu_2(b) + 2\mu_2(b)\xi_1(b)$

$C(b) = (\mu_1^2(b) - \mu_2^2(b))\xi_1'(b) + \mu_1(b)\xi_2'(b) + \xi_3'(b)$

$D(b) = 2\mu_1(b)\mu_2(b)\xi_1'(b) + \mu_2(b)\xi_2'(b)$

Noticing that $\mu_1(b^*) = 0$ and $\mu_2(b^*) = \sqrt{\xi_2(b^*)}$, we have

$A(b^*) = -2\xi_2(b^*)$, $B(b^*) = 2\xi_1(b^*)\sqrt{\xi_2(b^*)}$, $C(b^*) = \xi_3'(b^*) - \xi_2(b^*)\xi_1'(b^*)$ and $D(b^*) = \sqrt{\xi_2(b^*)}\xi_2'(b^*)$

$$\begin{aligned}
\text{Now, } \frac{d}{db}\Big[Re(\lambda_i(b))\Big]_{b=b^*} &= -\frac{B(b^*)D(b^*) + A(b^*)C(b^*)}{A(b^*)^2 + B(b^*)^2} \\
&= -\frac{(2\xi_1(b^*)\sqrt{\xi_2(b^*)}) \times (\sqrt{\xi_2(b^*)}\xi_2'(b^*)) + (-2\xi_2(b^*)) \times (\xi_3'(b^*) - \xi_2(b^*)\xi_1'(b^*))}{(-2\xi_2(b^*))^2 + (2\xi_1(b^*)\sqrt{\xi_2(b^*)})^2} \\
&= -\frac{\xi_1(b^*)\xi_2'(b^*) + \xi_2(b^*)\xi_1'(b^*) - \xi_3'(b^*)}{2\{\xi_2(b^*) + \xi_1^2(b^*)\}}
\end{aligned}$$

Therefore, $\frac{d}{db}\Big[Re(\lambda_i(b))\Big]_{b=b^*} \neq 0$ if $\xi_1(b^*)\xi_2'(b^*) + \xi_2(b^*)\xi_1'(b^*) - \xi_3'(b^*) \neq 0$ and $\lambda_3(b^*) = -\xi_1(b^*) \neq 0$

Hence, the transversality conditions hold and Hopf bifurcation occurs for system (5) at a critical value $b = b^*$.

Henceforth the theorem. ∎