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SOME FIXED POINT THEOREMS IN HILBERT SPACES

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Abstract. Some general fixed points theorems in Hilbert spaces are proved which generalize the results from [1].

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1. INTRODUCTION

Let R_+ be denote the set of all non-negative reals. Let ***H*** the set of all real function $g(t_1,\ ...,\ t_5): R_+^5 \to R_+$ satisfying the following conditions:

(H_1): g is non-decreasing in variables t_4 and t_5,

(H_2): $g(u,0,0,u,u) < u$, $\forall u > 0$,

(H_3): there exists $h \in (0,\ 1)$ such that for every $u, v \in R_+$ with

(H_a): $u \le g(v,v,u,u+v,0)$, or

(H_b): $u \le g(v,u,v,0,u+v)$,

we have $u \le h.v$.

Ex. 1. $g(t_1,\ ...,\ t_5) = q.\max\{t_1,\ t_2,\ t_3, \frac{1}{2}(t_4+t_5)\}$ where $q \in (0,\ 1)$ and $h = q$.

(H_1). Obviously.

(H_2). $g(u,0,0,u,u) = q.u < 0$, $\forall u > 0$.

(H_a). Let $u, v \ge 0$ be such that $u \le g(v,v,u,u+v,0)$ then $u \le q.\max\{v,v,u,\frac{1}{2}(u+v),0\}$ which implies $u \le q.v = h.v$.

(H_b). If $u \le g(v,u,v,0,u+v)$, similarly, we have $u \le h.v$.

Ex. 2. $g(t_1,\ ...,\ t_5) \le [at_1^k + bt_2^k + ct_3^k + d(t_4 t_5)^{k/2}]^{1/k}$ where $k \ge 1$; $a > 0$; $b,c,d \ge 0$ and $a+b+c+d < 1$.

(H_1). Obviously.

(H_2). $g(u,0,0,u,u) = [au^k + du^k]^{1/k} = (a+d)^{1/k}.u < u$, $\forall u > 0$.

(H_a). Let $u, v \in R_+$ be such that $u \le g(v, v, u, u+v, 0)$, then we have $u \le \left(\frac{a+b}{1-c}\right)^{1/k} . v = h_1 . v$ with $h_1 = \left(\frac{a+b}{1-c}\right)^{1/k} < 1$.

If $u \le g(v, u, v, 0, u+v)$, similarly, we have $u \le h_2 . v$ with $h_2 = \left(\frac{a+c}{1-b}\right)^{1/k} < 1$.

Thus g satisfies condition (H_3) with $h = \max\{h_1,\ h_2\}$.

Ex. 3. $g(t_1,\ \dots,\ t_5) = [at_1^2 + bt_3^2 \frac{1+t_2^2}{1+t_1^2} + ct_2^2 \frac{1+t_3^2}{1+t_1^2} + d\frac{t_4 t_5}{1+t_1^2}]^{1/2}$ where

$a > 0;\ b, c, d \ge 0$ and $a+b+c+d < 1$.

(H_1). Obviously.

(H_2). $g(u,0,0,u,u) = [au^2 + d\frac{u^2}{1+u^2}]^{1/2} \le (a+d)^{1/2} . u < u$, $\forall u > 0$.

(H_a). Let $u, v \in R_+$ be such that $u \le g(v, v, u, u+v, 0)$, then we have $u \le \left(\frac{a+c+av^2}{1-b+v^2(1-b-c)}\right)^{1/2} . v \le \left(\frac{a-b+av^2}{1-b+v^2(1-b-c)}\right)^{1/2} . v = h_1 . v$, where $h_1 \in (0,\ 1)$.

If $u \le g(v, u, v, 0, u+v)$, similarly, we have $u \le h_2 . v$ where $h_2 \in (0,\ 1)$.

Thus g satisfies condition (H_3) with $h = \max\{h_1,\ h_2\}$.

Remark 1. There exists the functions $g : R_+^5 \to R_+$ which satisfies conditions (H_1) - (H_3) and is decreasing in variable t_2 and t_3.

Ex. 4. $g(t_1,\ \dots,\ t_5) = [at_1^2 + \frac{bt_4 t_5}{t_2^2 + t_3^2 + 1}]^{1/2}$ where $a > 0,\ b \ge 0$ and $a+b < 1$.

(H_1). Obviously.

(H_2). $g(u,0,0,u,u) = (a+b)^{1/2} u < u$, $\forall u > 0$.

(H_3). Let $u, v \in R_+$ be such that $u \le g(v, v, u, u+v, 0)$, then we have $u \le a^{1/2} v = h.v$, where $h \in (0,\ 1)$. If $u \le g(v, u, v, o, u+v)$, then $u \le hv$ where $h = a^{1/2} < 1$.

2. MAIN RESULTS

Theorem 1. Let T_1 and T_2 be two mappings from Hilbert space X into itself such inequality

(1) $\|T_1 x - T_2 y\| \le g(\|x-y\|, \|x - T_1 x\|, \|y - T_2 y\|, \|x - T_2 y\|, \|y - T_1 x\|)$ holds for all $x, y \in X$

where $g \in \boldsymbol{H}$, then $F_{T_1} = F_{T_2}$, where $F_T = \{x \in X:\ \ x = Tx\}$.

Proof. Let $u \in F_{T_1}$ be, then

$\|u - T_2u\| = \|T_1u - T_2u\| \le g(\|u-u\|, \|u - T_1u\|, \|u - T_2u\|, \|u - T_2u\|, \|u - T_1u\|) =$

$= g(0,0,\|u - T_2u\|, \|u - T_2u\|, 0)$. By ($H_a$) we have $\|u - T_2u\| \le 0$ which implies $u = T_2u$ thus $u \in F_{T_2}$ and $F_{T_1} \subset F_{T_2}$. Similarly, by (H_b), we have $F_{T_2} \subset F_{T_1}$.

Theorem 2. Let T_1 and T_2 be two mappings from Hilbert space X into itself such that inequality (1) holds for all $x, y \in X$ where g satisfies (H_2). If T_1 and T_2 have a common fixed point z, then z is a unique common fixed point for T_1 and T_2.

Proof. Suppose that T_1 and T_2 have a second common fixed point $z' \neq z$. Then

$\|z - z'\| = \|T_1z - T_2z'\| \le g(\|z - z'\|, \|z - T_1z\|, \|z' - T_2z'\|, \|z - T_2z'\|, \|z' - T_1z\|) =$

$= g(\|z - z'\|, 0, 0, \|z - z'\|, \|z - z'\|) < \|z - z'\|$, a contradiction.

In [1] is proved following theorem.

Theorem 3. Let X be a closed subset of a Hilbert space and T_1 and T_2 be mappings of X into itself satisfying

$$(2)\ \|T_1x - T_2y\|^2 \le a.\|x-y\|^2 + b.\|y - T_2y\|^2 \frac{1 + \|x - T_1x\|^2}{1 + \|x - y\|^2} + c\|x - T_1x\|^2 \frac{1 + \|y - T_2y\|^2}{1 + \|x - y\|^2}$$

for all x, y in X, where a,b,c are non-negative reals with $a + b + c < 1$. Then T_1 and T_2 have a unique common fixed point in X.

The purpose of this paper is to extend Theorem 3 and others results from [1] for the functions $g \in \boldsymbol{H}$.

Theorem 4. Let X be a closed subset of a Hilbert space and T_1 and T_2 be mappings of X into itself satisfying inequality (1) for all x, y in X, where $g \in \boldsymbol{H}$. Then T_1 and T_2 have a unique common fixed point in X.

Proof. For arbitrary $x_0 \in X$, define the sequence $\{x_n\}$ as

$$x_1 = T_1x_0,\ \ x_2 = T_2x_1,\ \ \ldots,\ \ x_{2n} = T_1x_{2n}, x_{2n+2} = T_2x_{2n+1} \ldots$$

Then we have

$\|x_{2n+1} - x_{2n}\| = \|T_1x_{2n} - T_2x_{2n-1}\| \le g(\|x_{2n} - x_{2n-1}\|, \|x_{2n} - T_1x_{2n}\|, \|x_{2n-1} - T_2x_{2n-1}\|,$

$\|x_{2n} - T_2x_{2n-1}\|, \|x_{2n-1} - T_1x_{2n}\|) = g(\|x_{2n} - x_{2n-1}\|, \|x_{2n} - x_{2n+1}\|, \|x_{2n-1} - x_{2n}\|, \|x_{2n} - x_{2n}\|,$

$\|x_{2n-1} - x_{2n+1}\|) \le g(\|x_{2n} - x_{2n-1}\|, \|x_{2n} - x_{2n+1}\|, \|x_{2n-1} - x_{2n}\|, 0, \|x_{2n-1} - x_{2n}\| + \|x_{2n} - x_{2n+1}\|)$

which implies, by condition (H_b), that

$$\|x_{2n+1} - x_{2n}\| \le h.\|x_{2n-1} - x_{2n}\|.$$

Similarly, by condition (H_b), we have

$$\|x_{2n} - x_{2n-1}\| \le h.\|x_{2n-1} - x_{2n-2}\|.$$

Hence we get

$$\|x_{n+1} - x_n\| \le h^n.\|x_1 - x_0\| \text{ for all } n \in N^*.$$

Hence $\{x_n\}$ is a Cauchy sequence. Since X is closed, there exists $u \in X$ which is the limit of x_n, i.e. $\lim\ x_n = u$. Since $x_{2n+1} = T_1 x_{2n}$ and $x_{2n+2} = T_2 x_{2n+1}$ are subsequence of $\{x_n\}$, $\{T_1 x_{2n}\}$ and $\{T_2 x_{2n+1}\}$ also converge to the same limit u. We now prove that u is a common fixed point of T_1 and T_2. Consider

$$\|u - T_2 u\|^2 = \|(u - x_{2n+1}) + (x_{2n-1} - T_2 u)\|^2 \le \|u - x_{2n+1}\|^2 + 2\operatorname{Re} < u - x_{2n+1}, x_{2n+1} - T_2 u > +$$
$$+ \|(x_{2n+1} - T_2 u)\|^2 = \|u - x_{2n+1}\|^2 + \|T_1 x_{2n} - T_2 u\|^2 + 2\operatorname{Re} < u - x_{2n+1}, x_{2n+1} - T_2 u > \le$$
$$\le \|u - x_{2n+1}\|^2 + g^2(\|x_{2n} - u\|, \|T_1 x_{2n} - x_{2n}\|, \|T_2 u - u\|, \|x_{2n} - T_2 u\|, \|u - T_1 x_{2n}\|) +$$
$$+ 2\operatorname{Re} < u - x_{2n+1}, x_{2n+1} - T_2 u > .$$

Letting $n \to \infty$, so that $x_{2n}, x_{2n+1} \to u$ and $\operatorname{Re} < u - x_{2n+1}, x_{2n+1}, T_2 u > \to 0$ we get
$\|u - T_2 u\| \le g(0, 0, \|T_2 u - u\|, \|T_2 u - u\|, 0)$.
By condition (H_a) follows that $\|u - T_2 u\| \le 0$, which implies $T_2 u = u$. By Theorems 1 and 2 follows that u is unique common fixed point for T_1 and T_2.

Corollary 1. Let X be a closed subset of a Hilbert space and T_1 and T_2 be mappings on X into itself such that

a) $\|T_1 x - T_2 y\| \le k.\max\{\|x - y\|, \|x - T_1 x\|, \|y - T_2 y\|, \frac{1}{2}(\|x - T_2 y\| + \|y - T_1 x\|)\}$

where $k \in (0,\ 1)$, or

b) $\|T_1 x - T_2 y\|^k \le a\|x - y\|^k + b\|x - T_1 x\|^k + c\|y - T_2 y\|^k + d[\|x - T_2 y\| \cdot \|y - T_1 x\|]^{k/2}$

where $k \ge 1,\ a > 0,\ b, c, d \ge 0$ and $a + b + c + d < 1$, or

c)
$$\|T_1 x - T_2 y\|^2 \le a\|x - y\|^2 + b\|y - T_2 y\|^2 \frac{1 + \|x - T_1 x\|^2}{1 + \|x - y\|^2} +$$
$$+ c\|x - T_1 x\|^2 \frac{1 + \|y - T_2 y\|^2}{1 + \|x - y\|^2} + d\frac{\|x - T_2 y\| \cdot \|y - T_1 x\|}{1 + \|x - y\|^2}$$

where $a > 0,\ b, c, d \ge 0$ and $a + b + c + d < 1$,
holds for all x, y in X. Then T_1 and T_2 have a unique common fixed point in X.

Remark 2. From Corollary 1(c) for $d = 0$ follows Theorem 3.

Theorem 5. Let X be a closed subset of a Hilbert space and $\{T_n\}_{n\in N}$ a sequence of mapping on X into itself satisfying inequality

(3). $\|T_n x - T_{n+1y}\| \le g(\|x - y\|, \|x - T_n x\|, \|y - T_{n+1} y\|, \|x - T_{n+1} y\|, \|y - T_n x\|)$ for all $x, y \in X$, where $g \in \boldsymbol{H}$. Then the sequence $\{T_n\}_{n\in N}$ has unique common point in X.

Proof. By Theorem 4, T_1 and T_2 have a unique common fixed point. By Theorem 1, z is unique fixed point for the sequence $\{T_n\}_{n\in N}$.

Corollary 2. Let X be a closed subset of a Hilbert space and $\{T_n\}_{n\in N}$ a sequence of mappings on X into itself such that

a) $\|T_n x - T_{n+1}y\| \le k.\max\{\|x-y\|, \|x-T_n x\|, \|y-T_{n+1}y\|, \frac{1}{2}(\|x-T_{n+1}y\| + \|y-T_n x\|)\}$

where $k\in(0,\ 1)$, or

b) $\|T_n x - T_{n+1}y\|^k \le a\|x-y\|^k + b\|x-T_n x\|^k + c\|y-T_{n+1}y\|^k + d[\|x-T_{n+1}y\|\cdot\|y-T_n x\|]^{k/2}$

where $k\ge 1$, $a>0$, $b,c,d\ge 0$ and $a+b+c+d<1$, or

c)
$$\|T_n x - T_{n+1}y\|^2 \le a\|x-y\|^2 + b\|y-T_{n+1}y\|^2 \frac{1+\|x-T_n x\|^2}{1+\|x-y\|^2} +$$
$$+c\|x-T_n x\|^2 \frac{1+\|y-T_{n+1}y\|^2}{1+\|x-y\|^2} + d\frac{\|x-T_{n+1}y\|\cdot\|y-T_n x\|}{1+\|x-y\|^2}$$

where $a>0$, $b,c,d\ge 0$ and $a+b+c+d<1$,
holds for all x,y in X. Then the sequence $\{T_n\}_{n\in N}$ have a unique common fixed point.

Theorem 6. Let X be a closed subset of a Hilbert space and T_1 and T_2 be mapping on X into itself satisfying inequality

(4). $\|T_1^p x - T_2^q y\| \le g(\|x-y\|, \|x-T_1^p x\|, \|y-T_2^q y\|, \|x-T_2^q y\|, \|y-T_1^p x\|)$ for all $x,y\in X$,

where $g\in \boldsymbol{H}$, and p,q are some positive integers. Then T_1 and T_2 have a unique common point in X.

Proof. T_1^p and T_2^q satisfy all conditions of the Theorem 4. Hence they have a unique common fixed point, say u, so that $T_1^p u = u$, $T_2^q u = u$.
Now, $T_1^p u = u$ implies $T_1(T_1^p u) = T_1 u$ and $T_1^p(T_1 u) = T_1 u$. Hence $T_1 u$ is a fixed point of T_1^p. Similarly, $T_2 u$ is a fixed point of T_2^q. Now if $u \ne T_2 u$, we have

$$\|u-T_2u\| = \|T_1^p u - T_2^q(T_2u)\| \le g(\|u-T_2u\|, \|u-T_1^p u\|, \|T_2u - T_2^q(T_2u)\|, \|u-T_2^q(T_2u)\|,$$
$$\|T_2u - T_1^q u\|) = g(\|u-T_2u\|, 0, 0, \|u-T_2u\|, \|u-T_2u\|) < \|u-T_2u\|$$

which is a contradiction. Thus $u=T_2u$. Similarly we get $u=T_1u$. If v is another common fixed point of T_1 and T_2 then clearly v is also a common fixed point of T_1^p and T_2^q. By Theorem 4, T_1^p and T_2^q have a unique common fixed point.

Corollary 3. Let X be a closed subset of a Hilbert space and T_1 and T_2 be mappings on X into itself such that

a) $\|T_1^p x - T_2^q y\| \le k.\max\{\|x-y\|, \|x-T_1^p x\|, \|y-T_2^q y\|, \frac{1}{2}(\|x-T_2^q y\| + \|y-T_1^p x\|)\}$

where $k\in(0,\ 1)$, or

b) $\|T_1^p x - T_2^q y\|^k \le a\|x-y\|^k + b\|x - T_1^p x\|^k + c\|y - T_2^q y\|^k + d[\|x - T_2^q y\| \cdot \|y - T_1^p x\|]^{k/2}$

where $k \ge 1$, $a > 0$, $b,c,d \ge 0$ and $a+b+c+d<1$, or

c)

$$\|T_1^p x - T_2^q y\|^2 \le a\|x-y\|^2 + b\|y - T_2^q y\|^2 \frac{1+\|x - T_1^p x\|^2}{1+\|x-y\|^2} +$$

$$+ c\|x - T_1^p x\|^2 \frac{1+\|y - T_2^q y\|^2}{1+\|x-y\|^2} + d\frac{\|x - T_2^q y\| \cdot \|y - T_1^p x\|}{1+\|x-y\|^2}$$

where $a > 0$, $b,c,d \ge 0$ and $a+b+c+d<1$,
holds for all x,y in X and p,q positive integers. Then T_1 and T_2 have a unique common fixed point.

Remark 3. Corollary 3(c) for $d = 0$ is Corollary 2 of [1].

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