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# ON A GENERAL PROJECTION ALGORITHM FOR VARIATIONAL INEQUALITIES INVOLVING RELAXED LIPSCHITZ AND RELAXED MONOTONE MAPPINGS

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**Abstract-** In this paper, we consider the generalized nonlinear quasi-variational inequalities problem for set-valued mappings and construct an iterative algorithm for find the approximate solution of this problem by exploiting the projection method and prove the existence of the solution to our problem involving relaxed Lipschitz and relaxed monotone mappings and the convergence of the iterative sequences generated by this algorithm.

**Key words-** Variational inequality, Iterative sequences, Projection techniques, Relaxed Lipschitz and relaxed monotone mappings.

## 1. INTRODUCTION

Variational inequality theory has been extended and generalized in several directions using new, innovative and novel technique to study a wide class of problem arising in pure and applied sciences, see for example [2, 3, 4, 6] and references therein. Inspired and motivated by the recent research works [5, 9, 10], in this paper, we study the projection method which suggests a iterative algorithm for a generalized nonlinear quasi-variational inequality for set-valued mappings. We prove the existence for our inequality and the convergence of iterative sequences generated by the algorithms.

## 2. PRELIMINARIES

Let H be a real Hilbert space with norm and inner product are denoted by $||.||$ and $\langle .,. \rangle$, respectively. Let K be a nonempty closed convex subset of H. Let F, G be point-to-set mapping. Let A,T,S,g be nonlinear mappings from H into itself, and $K(.):H\to 2^H$ be a point-to-set mapping. Then, we consider the problem finding $u\in K$, $x\in F(u)$, $y\in G(u)$ such that $g(u)\in K(u)$ and the following relation is satisfied:

$$\langle A(g(u)),g(v)-g(u)\rangle \geq \langle A(u),g(v)-g(u)\rangle - \rho \langle T(x)-S(y),g(v)-g(u)\rangle, \quad (2.1)$$

for all $g(v)\in K(u)$, where $\rho > 0$ is a constant and set $K(u)$ is define as

$$K(u) = m(u) + K, \quad (2.2)$$

where m is a point-to-point mapping and K is a closed convex set. Then the problem (2.1) is called generalized nonlinear quasi-variational inequality problem.

To prove our main result, we need the following Lemmas and concepts.

**Definition 2.1-** A mapping $g:H\to H$ is said to be

(i) strongly monotone, if there exists a constant $\sigma > 0$, such that

$$\langle g(u_1) - g(u_2), u_1 - u_2\rangle \geq \sigma ||u_1 - u_2||^2, \text{ for all } u_i\in H, i=1,2;$$

(ii) Lipschitz continuous, if there exists a constant $\eta > 0$, such that

$$\|g(u_1) - g(u_2)\| \le \eta \|u_1 - u_2\|, \text{ for all } u_i \in H, i=1,2.$$

**Definition 2.2 [1,7]** - Let $F:H \to 2^H$ be a point-to-set mapping is said to be

(i)relaxed Lipschitz continuous with respect to a mapping $T:H \to H$, if there exists a constant $r \ge 0$, such that

$$\langle T(x_1) - T(x_2), u_1 - u_2 \rangle \le -r \|u_1 - u_2\|^2, \text{ for all } u_i \in H, x_i \in F(u_i), i=1,2;$$

(ii)relaxed monotone with respect to a mapping $S:H \to H$, if there exists a constant $s > 0$, such that

$$\langle S(x_1) - S(x_2), u_1 - u_2 \rangle \ge -s \|u_1 - u_2\|^2, \text{ for all } u_i \in H, x_i \in F(u_i), i=1,2;$$

(iii)Lipschitz continuous, if for $c \ge 1$,

$$\|x_1 - x_2\| \le c \|u_1 - u_2\|, \text{ for all } u_i \in H, x_i \in F(u_i), i=1,2.$$

**Lemma 2.1-** Let $K(u)$ be of type (2.2). Then $u \in K$ is a solution to the problem (2.1) if and only if $u \in K$, $x \in F(u)$, $y \in G(u)$ satisfies $g(u) \in K(u)$ and

$$\langle u - \phi(u), g(v) - g(u) \rangle \ge 0, \text{ for all } g(v) \in K(u), \tag{2.3}$$

where $\phi(u):H \to H$ and $F,G:H \to 2^H$ is point-to-set mapping, for some constant $\rho > 0$,

$$\langle \phi(u), g(v) \rangle = \langle u, g(v) \rangle - \rho \langle T(x) - S(y), g(v) \rangle + \langle [Ao(I-g)](u), g(v) \rangle, \tag{2.4}$$

for all $g(v) \in K(u)$, where the operator $Ao(I-g)$ is defined as

$$[Ao(I-g)](u) = A(u) - A(g(u)), \text{ for all } u \in H.$$

**Lemma 2.2-** Let $K(u)$ be defined as (2.2). Then, $u \in K$, $x \in F(u)$ and $y \in G(u)$ is a solution to the problem (2.1) if and only if $u \in K$, $x \in F(u)$ $y \in G(u)$ satisfies $g(u) \in K(u)$ and

$$g(u) = m(u) + \mathrm{Proj}_K H[g(u) - u + \phi(u) - m(u)], \tag{2.5}$$

where $m:H \to H$, $\phi(u)$ is defined as (2.4), and $\mathrm{Proj}_K H$ denotes the projection of H onto K.

**Lemma 2.3-** Let $A,g:H \to H$ be Lipschitz continuous with Lipschitz constants $\xi$ and $\eta$, respectively; and $T,S:H \to H$ be Lipschitz continuous with constants $\beta > 0$ and $\alpha > 0$, respectively. Let $F:H \to 2^H$ be a relaxed Lipschitz continuous with respect to T and Lipschitz continuous with corresponding constants $r \le 0$ and
$\varepsilon \ge 1$. Let $G:H \to 2^H$ be a relaxed monotone with respect to S and Lipschitz continuous with corresponding constants $s > 0$ and $c \ge 1$, respectively. Then, for any constant $\rho > 0$, there exists $\theta > 0$, such that

$$\|\phi(u_1) - \phi(u_2)\| \le \theta \|u_1 - u_2\|, \text{ for all } u_1, u_2 \in H,$$

where $\phi(u)$ is defined as (2.4). It turn out that

$$\theta = [1 + 2\rho(r - s) + \rho^2(\beta\varepsilon + \alpha c)^2]^{1\backslash 2} + \xi(1 + \eta). \tag{2.6}$$

**Proof-** For each $u_1, u_2 \in H$, by (2.4) we have

$$\begin{aligned} |\langle \phi(u_1) - \phi(u_2), g(v) \rangle| &\le |\langle u_1 - u_2 - \rho((T(x_1) - S(y_1)) - (T(u_2) - S(y_2))), g(v) \rangle| \\ &\quad + |\langle A(u_1) - A(u_2) - (A(g(u_1)) - A(g(u_2))), g(v) \rangle| \\ &\le \|u_1 - u_2 - \rho(T(x_1) - T(x_2)) + \rho(S(y_1) - S(y_2))\| \, \|g(v)\| \\ &\quad + \|A(u_1) - A(u_2) - (A(g(u_1)) - A(g(u_2)))\| \, \|g(v)\|. \end{aligned} \tag{2.7}$$

Further, since F is a relaxed Lipschitz and G is a relaxed monotone, we have

$$\begin{aligned} \|u_1 - u_2 - \rho(T(x_1) - T(x_2)) + \rho(S(y_1) - S(y_2))\|^2 &= \|u_1 - u_2\|^2 \\ &\quad - 2\rho \langle T(x_1) - T(x_2), u_1 - u_2 \rangle + 2\rho \langle S(y_1) - S(y_2), u_1 - u_2 \rangle \\ &\quad + \rho^2 \|(T(x_1) - T(x_2)) - (S(y_1) - S(y_2))\|^2 \end{aligned}$$

$$\leq \|u_1-u_2\|^2+2\rho r\|u_1-u_2\|^2-2\rho s\|u_1-u_2\|^2+\rho^2(\beta\varepsilon+\alpha c)^2\|u_1-u_2\|^2$$
$$\leq [1-2\rho(s-r)+\rho^2(\beta\varepsilon+\alpha c)^2]\|u_1-u_2\|^2. \quad (2.8)$$

And, A and g are Lipschitz continuous, we have

$$\begin{aligned}\|A(u_1)-A(u_2)-(A(g(u_1))-A(g(u_2)))\| &\leq \|A(u_1)-A(u_2)\| \\ &\quad +\|A(g(u_1))-A(g(u_2))\| \\ &\leq \xi\|u_1-u_2\|+\eta\xi\|u_1-u_2\| \\ &\leq \xi(1+\eta)\|u_1-u_2\|. \quad (2.9)\end{aligned}$$

From (2.7)-(2.9), we have

$$|\langle\phi(u_1)-\phi(u_2),g(v)\rangle|\leq\theta\|u_1-u_2\|\;\|g(v)\|,$$

where $\theta$ is given by (2.6), it follows that

$$\|\phi(u_1)-\phi(u_2)\|\leq \sup_{g(v)\in H}|\langle\phi(u_1)-\phi(u_2),g(v)\rangle|/\|g(v)\|\leq\theta\|u_1-u_2\|.$$

## 3. MAIN RESULTS

In this section, we consider those conditions under which the solution of the generalized nonlinear quasivariational inequalities problem (2.1) exists and the sequences of approximate solution which is defined in Algorithm 3.1, converges to the exact solution of the generalized nonlinear quasivariational inequalities problem (2.1).

**Algorithm 3.1-** Given $u_0\in H$, compute $u_{n+1}$ by the rule

$$u_{n+1}=u_n-(g-m)(u_n)+\mathrm{Proj}_K H[(g-m)(u_n)-u_n+\phi(u_n)], \quad (3.1)$$

where $(g-m)$ is defined by

$$(g-m)(u)=g(u)-m(u),$$

and $\rho>0$ is a constant, and $n=0,1,2,\ldots$.

**Theorem 3.1-** Let $g:H\to H$ be a Lipschitz continuous with Lipschitz constant $\eta>0$ and strongly monotone with constant $\sigma>0$ ; and let $A,T,S:H\to H$ be Lipschitz continuous with Lipschitz constants $\xi>0$, $\beta>0$ and $\alpha>0$, respectively. Let $F:H\to 2^H$ be a relaxed Lipschitz continuous with respect to T and Lipschitz continuous with corresponding constants $r\leq 0$ and $\varepsilon\geq 1$; and $G:H\to 2^H$ be a relaxed monotone with respect to S and Lipschitz continuous with corresponding constants $s>0$ and $c\geq 1$. Let $m:H\to H$ be a Lipschitz continuous with respect to constant $\upsilon>0$.

Assume that

$$\langle m(u_1)-m(u_2),u_1-u_2-(g(u_1)-g(u_2))\rangle\leq\lambda\|u_1-u_2\|^2, \text{ for all } u_1,u_2\in H, \quad (3.2)$$

for some constant $\lambda$ such that $\lambda_0\leq\lambda\leq\upsilon(1-2\sigma+\eta^2)$, where

$\lambda_0=\inf\{M:\langle m(u_1)-m(u_2),u_1-u_2-(g(u_1)-g(u_2))\rangle\leq M\|u_1-u_2\|^2, \text{for all } u_1,u_2\in H\}$.

Further assume that

$$q=[(1-2\sigma+\eta^2)+\upsilon^2+2\lambda]^{1\backslash 2}+\xi(1-\eta)2^{-1}<2^{-1}, \quad (3.3)$$
$$(s-r)>2(\beta\,\varepsilon+c\,\alpha)[q(1-q)]^{1/2}.$$

Then, the problem (1.1) has a solution $u^*\in H$, $x^*\in F(u^*)$, $y^*\in G(u^*)$, and for all $\rho>0$, such that

$$|\rho-(s-r)(\beta\varepsilon+c\alpha)^{-2}|<[(s-r)^2-4q(1-q)(\beta\varepsilon+c\alpha)^2]^{1\backslash 2}(\beta\varepsilon+c\alpha)^{-2}, \quad (3.4)$$

the approximate solution of sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ generated by the Algorithm 3.1, converges strongly to $u^*$, $x^*$ and $y^*$, respectively.

**Proof-** By Lemma 2.2, $(u^*,x^*,y^*)$ is a solution of the problem (2.1) if and only if $u^*\in H, x^*\in F(u^*)$, $y^*\in G(u^*)$ and satisfies (2.5).

Let us introduce the mapping $N:H\to H$ defined by

$$N(u) = u - g(u) + m(u) + \mathrm{Proj}_K H[g(u) - u + \phi(u) - m(u)], \text{ for } u\in H,$$

where $\phi(u)$ is given by (2.4). For all $u_1,u_2\in H$, we have

$$\|N(u_1) - N(u_2)\| \le 2\|u_1 - u_2 - [g(u_1) - g(u_2)] + m(u_1) - m(u_2)\| + \|\phi(u_1) - \phi(u_2)\|. \tag{3.5}$$

Since g is strongly monotone and Lipschitz continuous with constants $\sigma$ and $\eta$, respectively, and m is Lipschitz continuous with constant $\upsilon$, it can be obtained that using (3.2)

$$\begin{aligned}\|u_1 - u_2 - (g(u_1)-g(u_2)) + m(u_1) - m(u_2)\|^2 &= \|u_1 - u_2 - (g(u_1)-g(u_2))\|^2 + \|m(u_1) - m(u_2)\|^2 + 2\langle m(u_1) - m(u_2), u_1 - u_2 - (g(u_1) - g(u_2))\rangle \\ &\le [(1 - 2\sigma + \eta^2) + \upsilon^2 + 2\lambda]\,\|u_1 - u_2\|^2, \text{ for all } u_1,u_2\in H.\end{aligned} \tag{3.6}$$

From (3.5) - (3.6) and Lemma 2.3, we obtain

$$\|N(u_1) - N(u_2)\| \le (2q + \theta_1)\,\|u_1 - u_2\|, \tag{3.7}$$

where

$$\theta_1 = [1 + 2\rho(r - s) + \rho^2(\beta\varepsilon + \alpha c)^2]^{1/2},$$
$$q = [(1 - 2\sigma + \eta^2) + \upsilon^2 + 2\lambda]^{1\backslash 2} + \xi(1 + \eta)\, 2^{-1},$$

since $r < s$, $q < 1\backslash 2$ and $(s - r) > 2(\beta\varepsilon + \alpha c)[q(1 - \rho)]^{1\backslash 2}$,

for all $\rho > 0$, such that

$$|\rho - (s - r)(\beta\varepsilon + c\alpha)^{-2}| < [(s - r)^2 - 4q(1 - q)(\beta\varepsilon + c\alpha)^2]^{1\backslash 2}(\beta\varepsilon + c\alpha)^{-2},$$

we have

$$2q + \theta_1 = 2q + [1 + 2\rho(r - s) + \rho^2(\beta\varepsilon + \alpha c)^2]^{1\backslash 2} < 1.$$

Hence N is a contraction mapping. Then it follows that N has a unique fixed point $u^*\in H$. By Lemma 2.2, $u^*\in H$, $x^*\in F(u^*)$, $y^*\in G(u^*)$ is solution of the problem (2.1). Since $u^*\in H$, $x^*\in F(u^*)$ and $y^*\in G(u^*)$ satisfies (2.5), then from (3.1), we deduce that

$$\begin{aligned}\|u_{n+1} - u^*\| &\le 2\|u_n - u^* - (g(u_n) - g(u^*)) + m(u_n) - m(u^*)\| + \|\phi(u_n) - \phi(u^*)\| \\ &\le (2q + \theta_1)\|u_n - u^*\| \\ &\le (2q + \theta_1)^n\|u_1 - u^*\|.\end{aligned}$$

Noting that $2q + \theta_1 < 1$, we obtain that $\{u_n\}$ converges strongly to $u^*$. Similarly , we show that $\{x_n\}$ and $\{y_n\}$ strongly converges to $x^*$ and $y^*$, respectively. This completes the proof.

## REFERENCES

1.M. R. Bai, Y. Y. Tang and Y. P. Liu, Generalized implicit quasi-variational inequalities with relaxed Lipschitz and relaxed monotone mappings, Appl. Math. Lett. **12**, 69-76, 1999.

2.C. Baiocchia and A. Capelo, Variational and quasivariational inequalities, J. Wiley and sons, New York, 1984.

3.R. Glowinski, J. L. Lions and R. Tremolieres, Numerical analysis of variational inequalities, North-Holland, Amsterdam, Holland, 1981.

4.D. Kinderlehrer and G. Stampacchia, An introduction to variational inequalities and their applications, Academic Press, New York, 1980.
5.Z. Luchuan, Iterative algorithms for finding approximate solutions for general strongly nonlinear variational inequalities, J. Math. Anal. Appl. **187**, 352-360, 1994.
6.U. Mosco, Implicit variational problems and quasivariational inequalities, Lecture notes in Math. Springer-Verlag, Berlin, Germany, **543**, 1976.
7.Salahuddin, Generalized multivalued nonlinear variational inequalities,(to appear).
8.L. C. Zeng, Completely generalized strongly nonlinear quasivariational problems in Hilbert spaces, J. Math. Anal. Appl. **193**, 706-714, 1995.
9.L. C. Zeng, Iterative algorithms for finding approximate solution to completely generalized strongly nonlinear quasivariational inequalities, J. Math. Anal. Appl. **201**, 180-194, 1996.
10.L. C. Zeng, On a general projection algorithm for variational inequalities, J. Optim. Theo. Appl. **97**, 229-235, 1999.