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MATRIX TRANSFORMATIONS OF SOME GENERALIZED ANALYTIC SEQUENCE SPACES

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Abstract-The aim of this paper is to define and to investigate the generalized analytic sequence spaces $c_0^{\nu}(p,s)$, $l_\infty^{\nu}(p,s)$ and $l^{\nu}(p,s)$ and to determine the matrices of classes like, $(l_\infty^{\nu}(p,s), l_\infty)$, $(c_0^{\nu}(p,s), l_\infty)$, $(l^{\nu}(p,s), l_\infty)$, $(l_\infty^{\nu}(p,s), c)$, $(c_0^{\nu}(p,s), c)$, and $(l^{\nu}(p,s), c)$, where the sequence space and c are respectively the spaces of bounded and convergent complex sequences.
Key words-Generalized analytic sequence space, Matrix-Transformations

1. INTRODUCTION

Let $p=(p_n)$ be a bounded sequence of strictly positive real numbers and $v=(v_n)$ any fixed sequence of non-zero complex numbers such that

$$\lim_{n\to\infty} \inf |v_n|^{1/n} = r, \ (0<r<\infty)$$

We define Bilgin [2] the sequence spaces $c_0^{\nu}(p,s)$ and $l_\infty^{\nu}(p,s)$ and $l^{\nu}(p,s)$ as follows;

$$c_0^{\nu}(p,s) = \{x=(x_n) : n^{-s}|x_n v_n|^{p_n} \to 0 \text{ as } n\to\infty, \ s\geq 0\};$$

$$l_\infty^{\nu}(p,s) = \{x=(x_n) : \operatorname{Sup}_n n^{-s}|x_n v_n|^{p_n} < \infty, \ s\geq 0\};$$

$$l^{\nu}(p,s) = \{x=(x_n) : \sum_n n^{-s}|x_n v_n|^{p_n} < \infty, \ s\geq 0\}.$$

When $s=0$, $v_n=1$ and $p_n=1$ for every n the spaces $c_0^{\nu}(p,s)$, $l_\infty^{\nu}(p,s)$ and $l^{\nu}(p,s)$ turn out to be respectively the scalar sequence spaces c_0, l_∞ and l.

When $s=0$, $v_n=1$ for every n these spaces are respectively the well known spaces $c_0(p)$, $l_\infty(p)$, and $l(p)$ defined by Maddox [7] and Simons [11].

When , $v_n=1$ for every n these spaces are respectively the spaces $c_0(p,s)$, $l_\infty(p,s)$, and $l(p,s)$ defined by Başarır [1] and Çakar and Bulut [3].

When $s=0$ these spaces are respectively $D_0^\wedge(p)$, $D_\infty^\wedge(p)$ and $D^\wedge(p)$ defined by Ratha and Srivastava [10]. It may be noted here that the spaces $D_0^\wedge(p)$, $D_\infty^\wedge(p)$ and $D^\wedge(p)$ are the same as $(c_0(p))_v$, $(l_\infty(p))_v$, and $(l(p))_v$, (See [4])

Throughout the paper the following inequality will be used frequently. For any C>0 and any complex numbers a, b,

$$|a \cdot b| \leq C(C^{-q}|a|^q + |b|^p) \text{ where } 1<p<\infty \text{ and } p^{-1}+q^{-1}=1. \quad (1)$$

Using the same kind argument to that in [8], we get that the necessary and sufficient condition for above sequence spaces to be linear is $p\in l_\infty$. It is easy to see that $c_0^{\nu}(p,s)$ is paranormed space by

$$g(x) = \operatorname{Sup}_k (k^{-s}|x_k v_k|^{p_k})^{1/M}, \text{ where } H = \sup_k p_k \text{ and } M = \max(1,H).$$

Also $l_\infty^v(p,s)$ is paranormed by g(x) if and only if $\inf p_k > 0$.

The space $l^v(p,s)$ is paranormed by $h(x) = (\sum_k k^{-s}|x_k v_k|^{p_k})^{1/M}$.

All the spaces defined above are complete in their topologies.

If (X,g) is a paranormed space, with paranorm g, then we denote by X* the continuous dual of X, i.e., the set of all continuous linear functionals on X. If E is any set of complex sequences $x = (x_k)$ then E^α will denote the α-dual of E,

$E^\alpha = \{ a: \sum_k |a_k x_k| < \infty, \text{ for all } x \in E\}$

2. α-AND CONTINUOUS DUALS

In the following Lemmas we have the α-and continuous duals of $c_0^v(p,s)$, $l^v(p,s)$ and α-dual of $l_\infty^v(p,s)$ (see, Bilgin [2])

Lemma 2. 1. Let $0 < p_k \le \sup_k p_k < \infty$. Then

(i) $(c_0^v(p,s))^\alpha = M_0^v(p,s)$

$$\text{where } M_0^v(p,s) = \bigcup_{N>1} \left\{ a = (a_k) : \sum_k \left|\frac{a_k}{v_k}\right| k^{s/p_k} N^{-1/p_k} < \infty, s \ge 0 \right\}$$

(ii) $(c_0^v(p,s))^*$ is isomorphic to $M_0^v(p,s)$.

Lemma 2.2. i) If $1 < p_k \le \sup p_k < \infty$ and $p_k^{-1} + q_k^{-1} = 1$, $k = 0,1,2,....$ then

$(l^v(p,s))^\alpha = M^v(p,s)$ and $(l^v(p,s))^*$ is isomorphic to $M^v(p,s)$ where

$$M^v(p,s) = \left\{ a = (a_k) : \sum_k \left|\frac{a_k}{v_k}\right|^{q_k} k^{s(q_k-1)} N^{-q_k/p_k} < \infty, s \ge 0 \right\}$$

ii) If $0 < \inf p_k \le p_k \le 1$ then $(l^v(p,s))^\alpha = m^v(p,s)$ and $(l^v(p,s))^*$ is isomorphic to $m^v(p,s)$, where $m^v(p,s) = \left\{ a = (a_k) : \sup_k \left|\frac{a_k}{v_k}\right|^{p_k} k^s < \infty, s \ge 0 \right\}$

Lemma 2.3. $(l_\infty^v(p,s)) = M_\infty^v(p,s)$,where

$$M_\infty^v(p,s) = \bigcap_{N>1} \left\{ a = (a_k) : \sum_k \left|\frac{a_k}{v_k}\right| k^{s/p_k} N^{1/p_k} < \infty, s \ge 0 \right\}$$

3. MATRIX TRANSFORMATINOS

Let X and Y be any two nonempty subsets of s , the set of all sequences of real or complex numbers, and let $A = (a_{nk})$ be the infinite matrix of complex numbers a_{nk} (n, k= 1,2...). For every $x = (x_k) \in X$ and every integer n, we write

$$A_n(x) = \sum_k a_{nk} x_k \qquad (2)$$

The sum without limits in (2) is always taken from k =1 to k = ∞. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of $x = (x_k)$ by the matrix A. We write $A \in (X,Y)$ if and only $Ax \in Y$ whenever $x \in X$.

Necessary and sufficient conditions for a matrix $A = (a_{nk})$ to be in the calss (X, Y) for different sequence spaces X and Y are given by several authors ([1], [3], [5], etc.). Our results in this note characterize some of the classes like, $(l_\infty^\nu(p,s), l_\infty)$, $(c_0^\nu(p,s), l_\infty)$,

$(l^\nu(p,s), l_\infty)$, $(l_\infty^\nu(p,s), c)$, $(c_0^\nu(p,s), c)$,and $(l^\nu(p,s), c)$.

Theorem 3.1 . $A \in (l_\infty^\nu(p,s), l_\infty)$ if and only if there exists an integer N >1 such that

$$\sup_n \sum_k |a_{nk}/v_k| k^{s/pk} N^{1/pk} < \infty.$$

Proof. For sufficiency take an integer $N > \max(1, \operatorname{Sup}_k k^{-s}|v_k x_k|^{Pk})$. Then

$$\begin{aligned}\sup_n \left|\sum_k a_{nk}x_k\right| &\le \sup_n \sum_k |a_{nk}x_k| \\ &= \sup_n \sum_k |a_{nk}/v_k| k^{s/p_k}(k^{-s}|x_k v_k|^{p_k})^{1/p_k} \\ &< \sup_n \sum_k |a_{nk}/v_k| k^{s/pk} N^{1/pk} < \infty\end{aligned}$$

and therefore $A \in (l_\infty^\nu(p,s), l_\infty)$.

For the necessity suppose that $A \in (l_\infty^\nu(p,s), l_\infty)$ but there is an N >1 such that

$$\sup_n \sum_k |a_{nk}/v_k| k^{s/pk} N^{1/pk} = \infty.$$

Then the matrix $\mathbf{B} = (b_{nk}) = \left((a_{nk}/v_k)k^{s/p_k}N^{1/p_k}\right) \notin (\ell_\infty, \ell_\infty)$ for some integer N > 1. So there exists $x \in \ell_\infty$ with $\|x\| = 1$ such that $\mathbf{B}x \notin l_\infty$. Now $y = (y_k) = \left(k^{s/p_k}N^{1/p_k}x_k/v_k\right) \in l_\infty^\nu(p,s)$, but $Ay = \mathbf{B}x \notin l_\infty$ which contradicst the fact that $A \in (l_\infty^\nu(p,s), l_\infty)$ and this completes the proof.

Theorem 3.2. $A \in (c_0^\nu(p,s), l_\infty)$ if and only if there exists an integer B >1 such that $D = \sup_n \sum_k |a_{nk}/v_k| k^{s/pk} B^{-1/pk} < \infty$

Proof. The proof is easy.

Theorem 3.3. i) If $1 < p_k \le \sup p_k < \infty$ then $A \in (l^\nu(p,s), l_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}/v_k|^{qk} R^{-qk} k^{s(qk-1)} < \infty \quad \text{for some integer } R > 1.$$

ii) If $0 < \inf p_k \le p_k \le \mathbf{1}$ then $A \in (l^\nu(p,s), l_\infty)$ if and only if

$$\sup_{n,k} k^s |a_{nk}/v_k|^{pk} < \infty$$

Proof. i) Sufficiency . By using the inequality (1) we get

$$|a_{nk}x_k| \le R\left(\left|\frac{a_{nk}}{v_k}\right|^{q_k} k^{s(q_k-1)} R^{-q_k} + k^{-s}|v_k x_k|^{p_k}\right)$$

for every n. Then , we obtain $(A_n(x)) \in l_\infty$,whenever $x \in l^{v}(p,s)$.

Necessity. Using the same kind of argument to that in [3],the necessity of the condition is obtained in a similar manner as done in Theorem 3(i), by choosing a sequence x

$\in l^{v}(p,s)$; $$x_k = \delta^{H/p_k} \left|\frac{a_{nk}}{v_k}\right|^{q_k - 1} \operatorname{sgn}\left(\frac{a_{nk}}{v_k}\right) k^{s(q_k - 1)} V^{-1} Q^{-q_k/p_k} \quad ;1 \le k \le k_0,$$

$$x_k = 0; k > k_0$$

for all n, where $V = \sum_{k=1}^{k_0} |a_{nk}/v_k|^{qk} Q^{-qk} k^{s(qk-1)}$ and an integer Q>1 such that

$Q\delta^H > L.(|A_n(x)| \le L)$

ii) The sufficiency and the necessity can be proved respectively by the same kind of argument used in Theorem 2 (ii)([3]), and by the uniform boundedness principle.

Theorem 3.4. $A \in (c_0^v(p,s), c)$ if and only if

i) there exists an integer B >1 such that

$D = \sup_n \sum_k |a_{nk}/v_k| k^{s/pk} B^{-1/pk} < \infty$

ii) $\lim_n (a_{nk}) = \alpha_k$, for each k.

Proof. Necessity. Let $A \in (c_0^v(p,s), c)$. Since $e_k = (0,0,0,...0,1,0,...)$ in $c_0^v(p,s)$ (ii) must hold. Put $y_n = \sum_k a_{nk}x_k$, (y_n) is a sequence of continuous linear functionals on $c_0^v(p,s)$ such that lim y_n exists. Therefore by uniform boundedness principle for $0 < \delta < 1$, there exists $S_\delta[0] \subset c_0^v(p,s)$ and constant K such that $|y_n| \le K$ for each n and $x \in c_0^v(p,s)$. Let us define $x^r = (x_k^r) \in c_0^v(p,s)$ by following

$$x_k^r = \begin{cases} \delta^{M/p_k} k^{s/p_k} \operatorname{sgn}(a_{n,k}/v_k)/v_k, & 0 \le k \le r \\ 0 & , \text{otherwise} \end{cases}$$

where $M = \max(1, \sup p_k)$. Now $x^r = (x_k^r) \in S_\delta[0]$ and

$$\sum_{k \le r} |a_{nk}/v_k| k^{s/pk} B^{-1/pk} \le K$$

for each n and r, where $B = \delta^{-M}$. Therefore (i) holds.

Sufficiency. Suppose (i)- (ii) hold and $x \in c_0^v(p,s)$. Hence for $0 < \varepsilon < 1$, there exists r ;

$$\forall k > r \qquad \left|k^{-s/pk} x_k v_k\right|^{pk/M} \le \frac{\varepsilon}{B(2D+1)} < 1$$

and therefore $k > r$

$$B^{1/pk}\left|k^{-s/pk} x_k v_k\right| < B^{M/pk}\left|k^{-s/pk} x_k v_k\right| < \left(\frac{\varepsilon}{2D+1}\right)^{M/p_k} < \frac{\varepsilon}{2D+1} < B.$$

By (i) and (ii) we have $\sum_k |\alpha_k / v_k| k^{s/p_k} B^{-1/p_k} < D$ and

$$\sum_k |(a_{nk} - \alpha_k)(x_k)| \leq \sum_k (|a_{nk} x_k| + |\alpha_k x_k|)$$
$$= \sum_k (|(a_{nk} / v_k) v_k x_k| + |(\alpha_k / v_k) v_k x_k|)$$
$$\leq B(\sum_k |a_{nk} / v_k| k^{s/p_k} B^{-1/p_k}$$
$$+ \sum_k |\alpha_k / v_k| k^{s/p_k} B^{-1/p_k}) < 2BD < \infty$$

for each n. Hence $\sum_{k>r} |(a_{nk} - \alpha_k)(x_k)| < \varepsilon$ for each n.

Therefore, we have $\lim_n \sum_k a_{nk} x_k = \sum_k \alpha_k x_k$. This proves that $A \in (c_0^v(p,s), c)$.

Theorem 3.5. $A \in (l_\infty^v(p,s), c)$ if and only if

i) $\sum_k |a_{nk}/v_k| k^{s/pk} N^{1/pk}$ convergens uniformly in n for all integers $N > 1$,

ii) $\lim_n (a_{nk}) = \alpha_k$, for each k

Proof. Sufficiency. By (i) $\sum_k a_{nk} x_k$ converges uniformly in n for each $x \in l_\infty^v(p,s)$. Therefore $\lim_n \sum_k a_{nk} x_k = \sum_k \alpha_k x_k$ and hence sufficiency holds.

Necessity. Suppose that $A \in (l_\infty^v(p,s), c)$. Since $e_k \in l_\infty^v(p,s)$, (ii) must hold. If (i) does not hold, then $((a_{nk} / v_k) k^{s/p_k} N^{1/p_k}) \notin (\ell_\infty, c)$ for some integer $N > 1$, whence as in Theorem 3.1 there is $x \in l_\infty^v(p,s)$ such that $(\sum_k a_{nk} x_k) \notin c$. This completes the proof of the Theorem.

Theorem 3.6. $A \in (l^v(p,s), c)$ if and only if

i) $C(R) = \sup_n \sum_k |a_{nk}/v_k|^{qk} R^{-qk} k^{s(qk-1)} < \infty$ for some integer $R > 1$ $(1 < p_k \leq \sup p_k < \infty)$ and $\sup_{n,k} k^s |a_{nk} / v_k|^{p_k} < \infty$ $(0 < \inf p_k \leq k \leq 1)$

ii) $\lim_n (a_{nk}) = \alpha_k$, for each k,

Proof. We consider only the case $1 < p_k \leq \sup p_k < \infty$. Necessity. Let $A \in (l^v(p,s), c)$. Since $e_k \in l^v(p,s)$, (ii) must hold. Now, y_n exists for each n and $x \in l^v(p,s)$ ($y_n = \sum_k a_{nk} x_k$). If we put $A_n = (y_n)$, then (A_n) is a sequence of continuous real functionals on $l^v(p,s)$ and further $\sup_n |A_n| < \infty$ on $l^v(p,s)$. By uniform boundedness principle desired result (i) follows.

Sufficiency. Suppose that the conditions (i) and (ii) hold. Then the series $\sum_k a_{nk} x_k$ converges for each n and $x \in l^v(p,s)$. We have

$$\lim_r \lim_n \sum_{k=1}^{r} |a_{nk} / v_k|^{q_k} R^{-q_k} k^{s(q_k-1)} \leq C(R) \quad \text{that is,}$$

$$\sum_k |\alpha_k/v_k|^{q_k} R^{-q_k} k^{s(q_k-1)} < \sup_n \sum_k |a_{nk}/v_k|^{qk} R^{-qk} k^{s(qk-1)} < \infty$$

Thus, $\sum_k \alpha_k x_k$ converges for each $x \in l^v(p,s)$. For each $x \in l^v(p,s)$ we can choose
$r \geq 1$such that

$$\sum_{k>r} k^{-s} |x_k v_k|^{p_k} < 1.$$

By using the inequality (1) it is easy to check that

$$\sum_{k>r} |(a_{nk} - \alpha_k)x_k| < 2R(2\,C(R)+1)(\sum_{k>r} k^{-s}|x_k v_k|^{p_k})^{1/H} \text{ (H=Sup } p_k).$$

Therefore $\lim_n \sum_k a_{nk} x_k = \sum_k \alpha_k x_k$

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