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Logical Problems in Analysis of Analogy

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Abstract: The paper discusses some logical problems concerning analogy. The traditional understanding of analogy as *proportion* (proportion) is inadequate, at least if proportionality is taken in mathematical sense. This situation is clear if we considered various special cases of analogy for instance *analogia legis* and *analogia juris.* Since analogy assumes a similarity of analogata (items being or investigated) as analogical, a general analysis of analogical relation must begin with the concept of similarity. It can be defined as possessing a common property. This idea is formalized by devices borrowed from logic and set theory.

Keywords: similarity; logic; set; reasoning; model; semantics; discovery

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1. Introduction

The term 'analogy' and the adjective 'analogical' (or 'analogous'; I will use both adjectives as synonyms, preferring the first one) occur in many contexts (according to a picturesque wording, "analogy is rather an "elastic" term"—see [1], p. 274; see also [2] for a list of cases in which analogy appears), for instance, we speak about reasoning by analogy (see [3]), discovering by analogy (for instance, in mathematics, see [4]; Stefan Banach used to say that average mathematicians see analogies between theorems, distinguished mathematicians—analogies between proofs, but great mathematicians—analogies between analogies), analogical transcendental concepts (see [5,6]), analogical usual (as kinds) concepts (see [1]), *analogia legis* and *analogia iuris* (see [7]), common sense analogies, metaphors as analogies, analogical models (see [8–10]), analogical computers, etc. (see [11] for a historical account of the concept of analogy). Clearly, this variety has something in common, namely the idea of being similar to some extent (see [11] for a brief historical account of the concept of analogy). This variety immediately calls for looking at a general characterization consisting in defining, or at least describing common features of particular cases of analogy. A preliminary intuition suggests that if a and b are analogous, they are similar. In particular, if a and b are similar in some respect, they are similar in other respects. However, this assertion does not explain of the degree in which a and b should be similar in order to satisfy the criteria of being qualified as analogous.

2. Analogy and Similarity

To see the problem of the nature of similarity in using analogy, let us consider *analogia legis* and *analogia juris.* The former consists in concluding about a not legally regulated case, on the base of a similar case explicitly regulated by law. Assume that a statute lists admissible sources of energy and a new one is discovered. Using *analogia legis,* we can conclude that the new source is admissible as well. Similarity consists here in the function, but it is presumed that, for instance, the new source of energy is not dangerous. *Analogia juris* does not assume that an explicit regulation exists. This procedure appeals to the spirit of law, for instance, to justice, legal security etc. Assume that law says that human rights are legally protected, but there is no list of them. One can argue that the right to (religious) unbelief is protected, because the opposite conclusion would not be proper from the point of view of justice-requirement. It is easy to see that the degree of similarity in the case of *analogia juris* is weaker than in the case of *analogia legis.* An additional feature of legal analogy is that its application in some situations can be restricted by additional constraints. For instance, penal codes usually prohibit analogy against interests of accused persons. The reason for this restriction is motivated by the principle *nullum cimen sine lege*.

One of the answers, coming back to ancient philosophy, points out that analogy consists in proportion. In other words, *a* and *b* are analogous if they are proportional, for instance, the relations 9:3 and 12:4 express proportionality; in fact, Greek ἀναλογία was translated as Latin *proportio.* Another example of an easy (mathematical) account of analogy is the definition of the similarity of two ordered sets as having the same order type, or saying that the sets of even integers, odd integers, natural numbers and integer numbers are analogous, because have the same cardinality. This description is precise, and we can even say that the idea of proportion is hidden in the phrase 'having the same cardinality'. Although we can say that all mentioned sets are analogical, this qualification is simply redundant, because it added nothing to the concept of collections having the same cardinality. Unfortunately, not all cases of analogy have a reasonable mathematical interpretation via the concept of proportionality. Consider two more advanced schemes of analogy, namely:

(A) If we assert similarity between n cases, the case $n + 1$ is also similar;
(B) If we assert a similarity in a sample of elements belonging to a class, all elements of this class have a given property.

The scheme (A) can be illustrated by reasoning that if n cases of *opera seria* ends tragically, the same holds for the $n + 1$ case, but (B) by arguing that because some already investigated cases of *opera seria* are similar as having a tragic final, all other exemplars (or at least the majority of them) of this form of operatic work end tragically. However, it is hardly to say that (A) and (B) are based on proportionality in the strict meaning, even under interpretation of (B) in statistical terms. In order to maintain that this analogy is based on mathematical proportionality, one should assume that the world is mathematical, or at least completely mathematized. The Pythagoreans and their followers make this supposition, but it is a very risky metaphysical claim.

3. Analogy as a Relation

Yet, analogy of *a* and *b* is always a relation with *a* and *b* as so-called analogata. Formally speaking, we can write:

(1) $<a, b> \in$ **ANAL** if and only if *a* and *b* are analogical.

However, in order to avoid circularity, the adjective 'analogous' has to be defined in a way. In particular, the issue of a closer analysis of (1) is important for reasoning by analogy. Assume that we are interested in discovering properties of an object *a.* If we establish that *a* and *b* are analogous in some respects and that *b* has some properties $P_1, P_2, P_3, \ldots, P_n, \ldots$, we can conclude that *a* is P_n provided that *a* has properties $P_1, \ldots, P_{n-1}$ (this notation is only illustrative and temporary). Consider once again a legal example, namely the *argumentum a simile* (from similarity) used by lawyers. For instance, a person *O* distributes an amount of money as the inheritance over his or her two children. However, the third child *C*, unknown to *O*, claims to be an inheritor as well. The argument from similarity allows us to conclude that *C* should take part as a legal successor. Similarly, it would be difficult to say that *analogia legis* and *analogia juris* are grounded on proportions in the literal sense.

4. How to Define Analogy?

As we will see, a general definition explaining (1) is very problematic. To illustrate this point, I refer to the list in [5] (pp. 271–284), containing many cases of analogies but without providing a general definition—the author refers to a very vaguely characterized concept of analogical thinking. In fact, even special (concrete) cases of analogy are not easy to analyze and lead to several logical difficulties in their analysis. This situation can motivate a skeptical attitude toward the concept of analogy and take the route proposed by Wittgenstein (see [12], §66–67):

Consider for example the proceedings that we call a "game". I mean board-games, card games, ball-games, Olympic Games, and so on. What is common to them all? Do not say: "There *must* be something common, or they would not be called 'games'", but *look and see* whether here is anything common to all. For if you look at them, you will not see something that is common to *all*, but similarities, relationships, and a whole series of them at that. To repeat: do not think, but look! [. . .]. And the result of this examination is that we see a complicated network of similarities overlapping and criss-crossing; sometimes overall similarities, sometimes similarities of detail. [. . .]. I can think of no better expression to characterize these similarities than "family resemblances"; for the various resemblances between members of family: build, features, color of eyes, gait, temperament, etc. overlap and criss-cross in the same way; and I shall say: 'games' form a family.

If we substitute 'analogies' (or 'cases of analogy') for 'games' in the last quoted sentence, we obtain the assertion "analogies form a family".

Although many philosophers and logicians agree that attempts to form a general definition of analogy opens up the Pandora box, not all agree with skepticism illustrated by Wittgenstein's remarks on games and their supposed application to other concepts, for example, the notion analogy. In the history, we encounter many attempts to systematize particular cases falling under the label 'analogy' and trying to form some generalizations. In particular, we have the following distinctions (I follow [3,10,13]):

i. Analogy as proportion vs. analogy as attribution;
ii. Nominal analogy (*analogia nominalis*) vs. real analogy (*analogia realis*);
iii. Analogy (as reasoning) vs. deduction and induction;
iv. Categorical analogy vs. structural analogy;
v. Analogy as a method of discovery vs. analogy as a method of justification.

Ad (i). This distinction is crucial for scholasticism. Both kinds of analogy are used in predications about being. With some simplifications, the first kind of analogy occurs when we predicate on kinds of being comparing them with other kinds, for instance, saying that airplanes are similar to birds because they fly. This understanding functions as a generalization of the ancient *proportio* and can be understood as applicable to mathematical and empirical analogies. Attributive analogy is related to speaking about being as such (there are other uses, but I omit them for simplicity). The Schoolmen and their contemporary followers distinguish two kinds of general concepts, namely, universals (referring to kinds) and transcendental concepts (transendentalia). The latter are the most general notions. They include *ens* (being), *verum* (truth) and *bonum* (goodness), *unum* (one) and *res* (thing).

The concept of being plays a special role among all transcendental concepts, because other transcendentalia are compared to it. The fundamental principle proposed for transcendentals is captured by the formula: ff T and T' are transcendentals, both are mutually convertible (*conventuntur*), for instance, *ens et verum convertuntur.* The transcendentalia are co-extensional, but they have different senses. Since being the main transcendental, other express its different features. Yet being (as well as other transcendentalia) is not ambiguous (equivocal; I neglect the question whether concepts can be ambiguous—in order to simplify the issues, I will omit equivocal items)—it is an analogical concept. This theory distinguishes three sorts of concepts, namely univocal, equivocal and analogical. Clearly, univocal and equivocal notions can be analogical, so to speak, normally. Consequently, analogy in the context of the reported distinction can be normal (regular) and qualified (transcendental). The second type of analogy is not reducible to the first. Assume that Q is a genus (kind). Using so-called *infinitatio* (the operation of negation on concepts), we can form the kind non-Q. If we have Q and non-Q, there exists the genus R with Q and non-Q as its species. However, this construction cannot be performed with transcendentals, because they are the most general. Let me note, that some philosophers, following Duns Scotus, consider the concept of being as univocal. This view results in rejecting the idea of transcendental analogy. The distinction reported in this fragment is very interesting for ontology, but has no particular importance for analogy considered from the ordinary point of view.

Ad (ii). This distinction is related to (ii), if *analogia realis* is divided into *prima* (primary) and *secunda* (secondary). Goclenius (see in [14] (pp. 96–97)) says that *analogia realis prima* does not apply to species, but to being as such. This assertion implies that *analogia realis secunda* concerns particular beings and their species. If we omit *analogia realis prima,* all concepts are univocal, including the concept of being. *Analogia nominalis* can be identified with analogy holding between pairs (or more) names, for instance <'mathematical logic', 'philosophical logic'> or <'redness', 'correctness'>). The first case illustrates the situation of forming (inventing) the term 'philosophical logic' (in its contemporary sense) as patterned by 'mathematical logic', but the second example alludes to the fact that the ending 'ess' is characteristic for abstract nouns, expressing abstract properties. The label 'analogy of names' is ambiguous. In one sense, it might function as a device for analysis syntactic or grammatical facts, for instance, whether names are simple or complex, long or short, etc., but another usage suggests appealing to logical semantics. If we take this rule as a natural question of how the analogy of names is related to the analogy of things and reversely, that is, how the analogy of things is displayed by the analogy of names. If we return to (1), any analysis of the phrase 'a and b are analogical' should take its semantic properties into account.

Ad (iii). Define deduction as the inference in which the conclusion is logically entailed by some premises, and induction as the inference in which the general statement 'every a is P' is justified by the set of sentences of the form 'a_1 is P', 'a_2 is P', 'a_2 is P', ... and, additionally, (&) the conjunction of the premises is not equivalent to the conclusion, and (&&) for every n, the sentence 'a_n is P' is entailed by the sentence 'every a is P' (induction is a particular case of reduction). Let our reasoning proceed by passing (as above) from 'a is analogical with b', 'b has some common properties with a' and 'b has a property P which a is lacking' to the assertion 'a is P'. However, this reasoning (or inference) is neither deduction nor induction. What it is? According to Czeżowski (see [3]), analogy is an inference from the premise 'this a is P' to the conclusion 'other a is P' mediated by the general assertion of the form 'every a is P'. There are many problems related to understanding analogy as a kind of inference. Returning to the schemes (A) and (B) (see above), both can be also interpreted as species of induction, enumerative or statistical. In fact, the only mark (A) and (B) as instances of analogy consists in occurrence of the adjective 'similar', still not being explained. Furthermore, we cannot infer, via mathematical induction, that a_{n+1} is P, from the premises (#) a_0 is P, and (##) if a_{n-1} is P, then a_n is P, and (###) a is similar to some elements of the class being the reference of the predicate P, because we have no way to justify the premise (##).

A peculiarity of analogy with respect to induction is that the relation of entailment does not hold in any direction (from premises to conclusion or back), because the sentence "every a is P" does not imply that a_n is similar to a_{n+1} for arbitrary n. Czeżowski's proposal is not quite adequate, because it omits the role of similarity in using the general premise in his treatment of analogy. Using similarity in analogical reasoning is responsible for this situation. On the other hand, contrary to induction, asserting some dissimilarity, considered as secondary, does not force, to reject the conclusion (see [15] (pp. 243–244)). Yet analogy is a fallible reasoning, that is, it can lead to false conclusions. And, as in the case of all non-deductive inferences, it is difficult to establish conditions of correctness of analogy. Even if we agree that premises of analogical reasoning should be true (material correctness), formal constraints (the formal relation) between premises and conclusion are controversial. To complete remarks in this fragment, let me note that every argument can be extended to deduction by adding some additional premises. However, such a step does not convert analogy into deduction.

Ad (iv). Roughly speaking, categorical analogy holds between objects, but structural between structures, in particular models. For instance, we can regard ordered sets as structures. Consider the set of even natural numbers and the set of all natural numbers. They are different, but analogical as having the same cardinality and the same order-type. Analogy can hold between semantic models, iconic (pictorial) models, theories and their mathematical models, etc. Sometimes, analogy is described via isomorphism (see [16]), sometimes via homomorphism (see [2]). At this point, I repeat that explaining analogy via well-defined mathematical concepts adds nothing to such locutions as two structures being isomorphic (homomorphic, have the same order-type, etc.). Eventually, searching for formal relations between structures and checking them can be involved in so-called analogical thinking (see next paragraph).

Ad (v) Most authors (for example, see [1,4,8,9]) seem to agree that analogy is rather a method of discovery (some authors speak about analogical thinking in this connection—this kind of thinking consists in looking for analogies using various methods) than a strategy of justification. I do not intend to enter into this issue. Thus, I restrict my remarks in this point to repeat well-known facts from the history of science on using analogies in zoological and botanical classifications, geological investigations, medicine, social comparisons, typologies of literary works, discoveries new theories, constructing models, etc. The role of analogy in historical episodes becomes obvious, it we look at labels 'the corpuscular theory of light', 'the wave theory of light', 'the planetary model of atom', 'the organic theory of state', 'logic of history, etc. On the other hand, such phrases have a clear metaphorical sense.

5. Formal Semantics and Analogy

In spite of the mentioned (and other omitted or overlooked) difficulties, I will try to apply formal semantics to the problem of analogy. My task does not consist in solving troubles of analogy, but to offer illuminative (I hope) remarks. First of all, semantics allows an analysis of *analogia nominalis* together with *analogia realis.* In order to that, the scheme (1) should be made more precise. I will focus on analogy as a relation between objects. Consequently, (1) concerns analogy between objects a and b (I consider the binary case only—generalizations for n-termed relations are straightforward). Since I intend to avoid analyzing analogy between structures, I assume that a and b are concrete individuals. The direction of analogy is important, and usually neglected. The notation $<a, b> \in \mathbf{ANAL}$ should be read 'a as the first analogatum is analogical with b as the second analogatum. In order to have a convenient terminology, let us introduce the terms 'analogandum' (the first analogatum) and analogans (the second analogatum); these labels are patterned by labels, like 'explicandum'—'explicans', 'explanandum—explanans' or 'definiendum—definiens'. According to this convention, analogical reasoning (thinking, etc.) consists in taking a as an object to be considered (analyzed, suggested, proposed, etc. as analogical to b. Analogy is always relativized to some properties. For instance, returning to a previous example, an unknown child of an inheritor is considered as having the same rights (via civil rights) as the children mentioned in the inheritor's last will.

We can say that analogical thinking consists in asking whether the analogandum has a property (natural or conventionally defined, for instance, by a legal system) possessed by the analogans. To simplify the further analysis, I assume that the analogy is related to just one property, let say $\mathbf{P_k}$ (relativizing to a collection of properties $\mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P_n}$ is immediate; we can suppose that analogy holds with respect to a finite collection of properties). If so, (1) can be rewritten as (I exchange both sides):

(2) a and b are analogical with respect to $\mathbf{P_k}$ if and only if $<a, b> \in \mathbf{ANAL}(\mathbf{P_k})$.

If we assert the left (right) side of (2), we initially assume that the sentence (*) 'b is P_k' (P_k is a predicated letter referring to the property $\mathbf{P_k}$) is true; analogy is asserted in such sentences. If the sentence (**) 'a is P_k' is true as well, a and b are analogical with respect to $\mathbf{P_k}$, otherwise these objects are not analogous with respect to $\mathbf{P_k}$. Clearly, they can be analogical in another respect.

Both analogata have many other properties than $\mathbf{P_k}$. Denote the class of a-properties as $\mathbf{P^a}$ and the set of b-properties by $\mathbf{P^b}$. We have

(3) If $<a, b> \in \mathbf{ANAL}(\mathbf{P_k})$, then $\mathbf{P^a} \cap \mathbf{P^b} \neq \emptyset$.

The condition that the intersection of set of properties possessed by both analogata is not empty (in the considered case, this intersection has one element only) can be considered as a general condition that the relation of analogy holds. Of course, one can claim that additional constraints should be added, for example, that properties grounding analogy should be relevant, essential, interesting, etc., but such additions are extralogical.

Assume that $\mathbf{P^a}$ and $\mathbf{P^b}$ include all properties of *a* and *b*. If $\mathbf{P^a} = \mathbf{P^b}$¸ both analogata are identical (by the Leibniz law); this situation is not excluded by the condition $\mathbf{P^a} \cap \mathbf{P^b} \neq \emptyset$. Thus, we can say that identity is a special case of analogy. Let $\mathbf{X^a}$ and $\mathbf{X^b}$ be sets of true sentences and *a* and *b* modulo are the identity relation between both objects. This assumption leads to the conclusion that $\mathbf{X^a}$ and $\mathbf{X^b}$ have the same semantic models. They are isomorphic by definition. If $\mathbf{P^a} \subseteq \mathbf{P^b}$, homomorphism can be taken into account. Another formal possibility is related to the categoricity of models. However, at least in my opinion, the most typical situation appears when *a* and *b* are different (as in our example) and, in the consequence, the models of (*) and (**) are different as well. On the other hand, the formula 'x is P_k' (x is a free variable) is satisfied in both semantic structures. Call models **M′** and semantically analogical if the same open formulas of the type 'x is P' are satisfied in these models. In our example, **M*** and M** are semantically analogous, because the formulas (*) and (**) are satisfied in both models. Perhaps this intuition motivates the consideration of isomorphism as a generally grounding analogy. However, this intuition is not correct. In fact, being semantically analogical should be interpreted as a weaker property than isomorphism, for it does require neither the same cardinality of analogata nor the same properties of relations(s) on which analogy is grounded. Yet, this analysis shows how to combine a linguistic and objectual approach to analogy—it is roughly related to *analogia nominalis* and *analogia realis*.

The above analysis illuminates formal properties of the analogy relation. Clearly, every object is analogical with itself—this means that analogy is reflexive. It is also symmetric, because properties relevant for analogy are possessed by the analogandum and the analogans. However, analogy is not transitive in general, that is, if *a* is analogical with *b* and *b* is analogical with *c*, it does not entail, that *a* is analogical with *c*. Consequently, analogy does usually not produce congruence, abstraction classes or ordering, but only in the cases in which mathematical proportions are available (identity, isomorphism, homomorphism, categoricity). This circumstance is responsible for the fact that logical analysis of analogy is so difficult. Finally, let me remark that analogy is coherent with some amount of vagueness. Thereby, fuzzy logic can provide new tools to analyze the analogy-relation, but it is a separate topic.

6. A Final Remark

I am fully aware of the fact that the above analysis is formal and ignores pragmatic factors of analogy as a procedure performing in concrete scientific or ordinary situations. In particular, I disregard a huge amount of studies on analogy in contemporary cognitive science. On the other hand, I hope that a formal treatment of analogical reasoning has some importance for a better understanding of analogy as related to human activities. I am indebted to anonymous referees for their fruitful comments.

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