

## Article

# Intuition and Ingenuity: Gödel on Turing's "Philosophical Error"

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**Abstract:** Despite his unreserved appreciation of Turing's analysis for being a "precise and unquestionably adequate definition" of formal system or mechanical computability, Gödel nevertheless published a short note in 1972 claiming to have found a "philosophical error" in Turing's argument with regard to the finite nature of mental states and memory. A natural question arises: how could Gödel enjoy the generality conferred on his results by Turing's work, despite the error of its ways? Previous interpretative strategies by Feferman, Shagrir and others have mainly tried to resolve the disparity by distinguishing different types of arguments in Turing and taking Gödel to approve only some of them. By a more integral examination of their ideas, especially Turing's response to the "mathematical objection" based on Gödel's incompleteness theorem and Gödel's own conception of finite yet non-mechanical procedures, and taking some of the main ideas of current developments in machine learning into consideration, I will try to present a new explanation for the apparent disparity, arguing that there is no "error" on Turing's side and the seemingly conflicting views held by Turing and Gödel should best be seen as complementary, keeping intuition and ingenuity together.

**Keywords:** intuition; Gödel; Turing; mathematical objection



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## 1. Turing's "Philosophical Error"

Identifying the informal concept of 'effective calculability' with a rigorous mathematical notion of 'general recursiveness' or 'Turing computability' is called the "Church-Turing Thesis" and is usually considered one of the paradigm cases of conceptual clarification via logical methods. It is well known that Gödel, one of the founders of the theory of recursive functions, is dissatisfied with Church's analysis and thus Church's thesis, and is only totally convinced by Turing's analysis and Turing's Thesis. However, despite his unreserved appreciation of Turing's analysis for being a "precise and unquestionably adequate definition" [1] of formal system or mechanical computability, Gödel nevertheless published a short note in 1972 claiming to have found a "philosophical error" in Turing's argument:

Turing in his 1937 gives an argument which is supposed to show that mental procedures cannot go beyond mechanical procedures. However, this argument is inconclusive. What Turing disregards completely is the fact that mind, in its use, is not static, but constantly developing, i.e., that we understand abstract terms more and more precisely as we go on using them, and that more and more abstract terms enter the sphere of our understanding. There may exist systematic methods of actualizing this development, which could form part of the procedure. Therefore, although at each stage the number and precision of the abstract terms at our disposal may be finite, both (and, therefore, also Turing's number of distinguishable states of mind) may converge toward infinity in the course of the application of the procedure. Note that something like this indeed seems to happen in the process of forming stronger and stronger axioms of infinity in set theory. However, today this process is far from being sufficiently understood to form a well-defined procedure. It must be admitted that the construction of a well-defined procedure which could actually be carried out (and would yield a

non-recursive number-theoretic function) would require a substantial advance in our understanding of the basic concepts of mathematics. [2]

As indicated in a footnote by Gödel, this remark may be regarded as a further comment on his earlier claim in 1965 that the technical results, of the existence of undecidable arithmetical propositions and the non-demonstrability of the consistency of a system in the same system for every consistent formal system containing a certain amount of finitary number theory, do not “establish any bounds for the powers of human reason, but rather for the potentialities of pure formalism in mathematics” [1]. It is reasonable to suppose that here Gödel was responding to the claim made earlier by Emil Post that the generality of the incompleteness for all formal systems and the unsolvability for all methods of solvability required the Church-Turing Thesis to be seen as a “natural law”, which exhibits “a fundamental discovery in the limitations of the mathematicizing power of *Homo sapiens*” [3]. However, Gödel also realized that as much as it establishes a correct analysis of mechanical computability, Turing’s argument, *mutatis mutandis*, would also imply that mental procedures cannot go beyond mechanical procedures, thus placing the same kind of limitation on human reason as Post does. The puzzling problem for us is immediate: as Webb put it, how could Gödel “enjoy the generality conferred on his results by Turing’s work, despite the error of its ways” [4]?

## 2. Resolving the Disparity

An obvious and easy way to reconcile Gödel’s seemingly conflicting remarks is to distinguish different types of arguments in Turing’s claim. Indeed, Turing in his 1936 paper proposed three types of arguments to support the thesis that his machines could compute any functions which are calculable by finite and mechanical means. Type I is a direct analysis of the operations that an ideal human computer can perform and depends on the assumption, questioned by Gödel, of finitely many “states of mind”. Type II shows that the entire Hilbertian deductive apparatus of first order predicate logic can be simulated by one of his machines, i.e., a machine will produce the same theorems as the formal logic system. Type III is a “modification” of type I argument by replacing the notion of a state of mind by “a more physical and definite counterpart of it”, namely, a note of instructions explaining how the work is to be continued if the computer “breaks off from his work, to go away and forget all about it and later to come back and go on with it” [5]. Since any stage of the computation is “completely determined” by the instructions and the symbols on the tape of a previous stage, their relation is expressible in the functional calculus, and the entire computational history could be formalized in the calculus, therefore carried out by one of his machines. That is why Turing also regarded his type III argument as a “corollary” (*ibid.*) of type II. Thus, according to Webb, Feferman [6] maintains that “Gödel rejected only Turing’s type I argument, while accepting his ‘more physical’ type III argument” [4]. Under this interpretation Gödel rejects the type I argument because it associates the finite and mechanical nature of computational procedures in the assumption that human memory is necessarily limited, and in particular, the number of states of mind is bounded. He embraces the type III argument because it does not rest on this dubious assumption but on a “more physical and definite counterpart of it”. Webb also suggests that Gödel was of the opinion that “all Turing was really analyzing was the concept of ‘mechanical procedure,’ but in his arguments for the adequacy of his analysis he overstepped himself by dragging in the mental life of a human computer” (*ibid.* 302). As Gödel put it, reported by Wang, that “we had not perceived the sharp concept of mechanical procedures sharply before Turing, who brought us to the right perspective” [7]. The “memorial” role of mental states, that they depend on previous states and scanned symbols, is replaced in the more physical counterpart by instructions so numbered that they can refer to each other. While we may doubt the total number of states of mind in computing, we cannot doubt the finite feature of instructions in the outer symbol space.

A similar interpretative strategy is also adopted by Shagrir, whose conclusion to Gödel’s conflicting response to Turing’s analysis is that “Gödel praised Turing for his analy-

sis of an ideal human who calculates by means of finite and mechanical procedures. He was critical of what he deemed Turing's superfluous assumption that the finite and mechanical character of computation is somehow anchored in limitations on human cognitive capacities" [8]. According to Shagrir, the finite and mechanical nature of computation is rooted in its role of defining a formal system, which is the pivotal concept in foundational debates and discussions with which Gödel was directly involved. Being finite and mechanical is exactly the defining characteristic feature of a formal system and is neither open to question nor needs any justification. Turing's error is to anchor these two features in the human condition, especially in the number of states of mind, i.e., Turing's argument has implicitly made an extra physical or materialistic assumption about mind.

The explanations of Feferman and Shagrir have the merit of being neat and are supported by textual evidence too. In a conversation with Wang, published in Wang's 1974 book, Gödel made a very similar remark with his 1972 note, but with more details. He added "Turing's argument becomes valid under two additional assumptions, which today are generally accepted, namely: (1) There is no mind separate from matter; (2) The brain functions basically like a digital computer. (2 may be replaced by: 2' The physical laws, in their observable consequences, have a finite limit of precision.)". According to Wang, "while Gödel thinks that 2 is very likely and 2' practically certain, he believes that 1 is a prejudice of our time, which will be disproved scientifically (perhaps by the fact that there are not enough nerve cells to perform the observable operations of the mind)" [7]. Thus Gödel apparently holds that Turing's constraints, or a version of them, apply to the brain, but not the mind.

However, to distinguish three types of arguments for Turing and then argue that Gödel agrees with one but disagrees with another seems too artificial. When Gödel refers to Turing's "philosophical error" he does not point to one particular argument among many but Turing's arguments as a whole. Moreover, as Turing himself claims, his type I argument is "only an elaboration of the ideas" presented in Section 1 of his paper, and his type III argument can be regarded as a modification of type I or as a corollary of type II. So, after all there is basically one central argument based on the fact that "human memory is necessarily limited".

However, not only is the above interpretation artificial, being ad hoc, but it may just be wrong. For in his argument Turing nowhere denied the possibility of the existence of an infinite number of mental states, his idea was rather that "if we admitted an infinity of states of mind, some of them will be 'arbitrarily close' and will be confused" [5]. The real assumption made by Turing under the above restriction is that "we will also suppose that the number of states of mind **which need to be taken into account** is finite" [ibid; my own emphasis]. Even Gödel admitted that this set is finite even for the mind in its current state of development but envisaged the possibility of "systematic methods" for so actualizing the development of our understanding of abstract terms that it would "converge to infinity". However, the idea of states of mind converging to infinity has not met with much enthusiasm among logicians. For example, Hao Wang thinks that it is not the number, but the way in which mind develops, computable or not, that constitutes the crucial point, for "it remains possible that the different stages of the mind's development are related in a computable manner, so that there is a sort of supercomputer which modified itself in such a way that, at each stage of the mind's development, the supercomputer functions like the computer which has the same states as the mind at that stage" [9]. The possibility mentioned by Gödel in his criticism of Turing was also severely criticized by Kleene, one of founders of the theory of computability. In his article "Reflection on Church's Thesis" he referred to Gödel's contemplation as "pie in the sky" and as far as he can predict, "the pie will remain stratospheric". The main objection of Kleene towards a potential infinity of states of mind is that " ... the idea of 'effective calculability' or an 'algorithm' involves a set of instructions that is fixed in advance. This condition is motivated by a publicity constraint, namely, that it must be possible to "convey a complete description of the effective procedure or algorithm by a finite communication, in advance of performing

computations in accordance with it. My version of the Church-Turing thesis is thus the ‘Public-Process Version’ [10].

As far as I can see, Kleene’s criticism is based on some misunderstanding on Gödel’s side. The potential infinite number of states is used by Gödel to show the possibility of an effective “mental procedure”, not to deny the correctness of Turing’s definition of an algorithm. It is to be noted that Gödel never used the word “effective” to describe the explicanda of Turing’s analysis, and on several occasions tried hard to make this distinction. In his 1965 Postscriptum he says explicitly that “finite procedure” is equivalent to general recursiveness only if “finite” is understood to be “mechanical procedure” and the “question of whether there exist finite non-mechanical procedures not equivalent with any algorithm, has nothing whatsoever to do with the adequacy of the definition of “formal system” and of “mechanical procedure” [1]. The real problem, I think, is Gödel’s challenge to Turing’s claim that allowing a finite number of states in computation does not make a serious difference, “since the use of more complicated states of mind can be avoided by writing more symbols on the tape” [5]. To see this point more clearly, we turn back to Turing’s basic fact about memory.

Webb, in his insightful introduction to this note in Gödel’s collected work, also claimed that “in fact, Turing has one basic argument, which is presented in Section I and whose central premise is “the fact that human memory is necessarily limited”. The heart of his argument was a novel abstract logical analysis of what it means to “effectively remember” things relevant to computation, such as symbols or how many times one has executed a subroutine: to do so one must be able to change from one distinguishable state to another, whether you are human or a machine. We presume indeed that states of mind may also carry memories beyond the wildest dreams of machines, but the only ones relevant to effective computation are those you are put into by symbols and processes arising in the course of computation. However, our memory is just as “necessarily limited” as a machine’s—in either case, to a finite number of recognizable state changes [4]. In other words, if we agree with Turing that what can be done effectively by the mind concerns memory, and since human memory is necessarily limited for otherwise some of them will be “arbitrary close” and confused, any effective procedure by the mind shall have the condition of finite number of states. However, a crucial difficulty of “second order” that might appear here is that, as the number of states increases, we can effectively number all the Turing machines, and as a result a human mind becomes a “universal” mind in that it can take any machine and imitate it. It is not obvious at all that any machine can exist that is as powerful as this “universal mind”. It is exactly this point that shows the enormous significance of Turing’s discovery of his universal machine which can take the code of any other arbitrary machine as input and imitate it on any other input. That is to say, the universal Turing machine, being infinite complex, can still exist with an internal memory of bounded complexity (finite of number of states), compensated by an external tape of symbols of unbounded size. This is exactly what Turing means when he writes that “the use of more complicated states of mind can be avoided by writing more symbols on the tape”. It is exactly here that Gödel expresses his doubt as to whether any effective state, no matter how complex, could always be compensated for in a purely symbolic way. Gödel would think that certain effective procedures involving understanding an abstract concept and its meaning would resist this symbolization, while all formal operations “whose essence is that reasoning is completely replaced by mechanical operations on formulas” [1] disregarding their meaning will do. That is to say, the real problem is whether in order for a state to be effectively distinguishable, it has to depend purely on remembering some symbol or it can also exist in a non-mechanical way by understanding an abstract concept. I think Webb was right in seeing Gödel’s criticism of Turing as based on the complexity of states rather than the potential number of states, and he is also right to point out that “it is really this kind of possibility [of a procedure involving a state exploiting meaning of abstract terms thus grasping infinitely complicated combinatorial relations] more than any convergence to an infinity of states that could undermine Turing’s argument”. However, I think he is too hasty

in saying that “once he [Turing] discovered the universal machine he saw that it could indeed compensate symbolically for a surprisingly wide class of increasingly complicated machine states” [4]. As we will see, both Gödel and Turing will have more to say about what stands above Turing machines, i.e., above the mechanic method.

### 3. Gödel’s Conception of Finite Effective yet Non-Mechanical Procedure

Gödel’s idea that mental procedure, or mind infinitely surpassing the power of any finite machine is explored in detail in his own comprehensive reflection about the philosophical significance of his own incompleteness theorems, namely, the Gibbs lecture in 1951. There is the famous disjunctive conclusion, which for him is an inevitable mathematically established fact, first appears: “Either mathematics is incomplete in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable Diophantine problems of the type specified” [11]. Although not specified in this lecture, Gödel was definitely certain of the falsity of the second disjunct. In a lecture given around 1938, when discussing Hilbert’s famous idea that every precisely formulated mathematical question has a unique answer, i.e., a proof or disproof can always be found, Gödel points out that proof in Hilbert’s sense should be understood as “something which starts from evident axioms and proceeds by evident inferences” [12]. If, however, we formalize the notion of proof as given in the usual logical sense as a sequence of formulas then Hilbert’s idea becomes amenable to mathematical treatment and the answer turns out to be negative, due to his incompleteness theorem. However, this result only shows that (1) there may exist absolutely unsolvable problems or (2) in the transition from evidence to formalism (about the notion of proof) something is lost. Additionally, Gödel was quick to point out that “it is easily seen that actually the second is the case, since the number-theoretic questions which are undecidable in a given formalism are always decidable by evident inferences not expressible in the given formalism” (ibid.) and which has the same evident character as those in the old formalism. The right conclusion to draw here is that it is not possible to formalize mathematical evidence or mechanize mathematical reasoning even in the domain of number theory, and Hilbert’s conviction remains entirely untouched. So, the natural question is which aspects of mathematical reasoning or evidence defy formalization? In his 1974 discussion with Wang, Gödel mentioned two “vaguely defined” effective but non-mechanical procedures: the process of defining recursive well-orderings of integers representing larger and larger ordinals and the process of forming stronger and stronger axioms of infinity in set theory. The second procedure, of forming axioms of large cardinal numbers in set theory was Gödel’s favorite example and he reiterated it again in the same 1972 note about “another version of” his first completeness theorem where he invoked as an example against the idea of absolutely unsolvable problems for the human mind: “There do exist unexplored series of axioms which are analytic in the sense that they only explicate the concepts occurring in them”, just like the axioms of infinity, which “only explicates the content of the general concept of set” [2]. This emphasis on meaning and explication of abstract concepts, rather than the mere combinatorial properties of concrete symbols, lies at the heart of Gödel’s conception of an effective non-mechanical procedure, which is unique for the human mind. As we noted earlier Gödel praised Turing’s analysis for mechanical procedures because “this meaning [mechanical], however, is required by the concept of formal system, whose essence it is that reasoning is completely replaced by mechanical operations on formulas”. Note that the question of whether there exist finite non-mechanical procedures not equivalent with any algorithm, has nothing whatsoever to do with the adequacy of [Turing’s] definition of ‘formal system’ and of ‘mechanical procedure’ . . . (For theories and procedures in this more general sense [those involving the use of abstract notions] the situation [with respect to complete formal systems and deciding procedures for arithmetic relations] may be different.) Note that the results do not establish any bounds for the powers of human reason, but rather for the potentialities of pure formalism in mathematics” [1]. The example Gödel gave



here is his *Dialectica* Interpretation [13,14], where he used finite types of primitive recursive functionals to prove the consistency of Peano arithmetic. Gödel considered this method as “an extension of finitary method” that involved the use of abstract terms on the basis of their meaning. Abstract notions, for Gödel are “those that are essentially of second or higher order, that is, notions that do not involve properties or relations of concrete objects (for example, of combinations of signs), but that relate to mental constructs (for example proofs, meaningful statements and so on); and in the proofs we make use of insights, into these mental constructs, that spring not from the combinatorial (spatiotemporal) properties of the sign combinations representing the proofs, but only from their meaning”. (Ibid.) It is exactly this characteristic ability of the human mind, the process of forming indefinitely new axioms based on the understanding and deepening of the meaning of abstract concepts that resist mechanization and formalization and make possible the existence of effective yet not mechanical procedures<sup>1</sup>.

#### 4. Turing and the “Mathematical Objection”

The problem of whether mental procedures can go beyond mechanical procedures, or whether mathematical reasoning can be fully mechanized takes a more concrete form in the question “Can machines be intelligent?” in the case of Turing. In his post-war program of constructing an “intelligent machine” Turing had already anticipated an objection based on the limitative logical result, which he called “the Mathematical Objection”. In a 1948 report “Intelligent Machinery”, Turing said that

The theorem of Gödel and related results have shown that if one tries to use machines for such purposes as determining the truth or falsity of mathematical theorems and one is not willing to tolerate an occasional wrong result, then any given machine will in some cases be unable to give an answer at all. On the other hand, the human intelligence seems to be able to find methods of ever-increasing power for dealing with such problems “transcending” the methods available to machines. [16]

Take the theorem of the undecidability of first order logic as an example: Turing and Church had shown that there is no mechanical uniform method, i.e., no machine that can tell whether an arbitrary formula of the system is a theorem or not, in some cases (when given an unprovable formulae) it must fail to give an answer. On the other hand, if an able mathematician is confronted with such a problem he can search around and find whatever new methods of proof available so that he ought eventually to be able to reach a correct decision about any given formula. This possibility, even if not conclusively, should still make Gödel’s claim very plausible that the human mind infinitely surpasses any machine and at the same time, make Turing’s conception of “intelligent machinery” doubtful. Before we turn to Turing’s reply to this seemingly very powerful mathematical objection, we will have a brief look at some of Turing’s earlier considerations, especially in his conception of “ordinal logic”, which gives us the impression that Turing did consider the possibility of a humanly effective yet non-mechanical procedure.

Already in his 1936 paper where he argued for the synonymous of a “machine process” and a “rule of thumb or mechanical” process, he also hinted at the possibility of the existence of a non-machine-computable but humanly computable number  $\delta$ :

It is (so far as we know at present) possible that any assigned number of figures of  $\delta$  can be calculated, but not by a uniform process. When sufficiently many figures of  $\delta$  have been calculated, an essentially new method is necessary in order to obtain more figures. [5]

This raises the problem whether there are essential differences between the power of human effective procedures, where new mathematical methods can always be created and found, and power of formal mechanical system, which can only embody a fixed set of rules and methods.

In his PhD dissertation “Systems of Logic Based on Ordinals”, written under the supervision of Church during between 1936 and 1938 in Princeton, Turing took on the arduous task of overcoming Gödel’s incompleteness through an infinite sequence of logical systems based on ordinals. Gödel had already proved his famous incompleteness results in 1931 that in any single logical system containing a certain amount of arithmetic like *Principia Mathematica* or other related systems there exists true but unprovable propositions. Turing’s basic idea was to form stronger and stronger logical systems by adding the true yet provable proposition in the previous system to the next new ones as an axiom. By iterating this process infinitely many times he achieved a certain completeness result for the resulting sequence of logical systems, i.e., any true arithmetic sentences could be derived within one or another system of the sequence<sup>2</sup>. However, the whole sequence of logical systems was non-constructive in the sense that there exists no uniform method (or a Turing machine) that could generate all of them. Turing was fully aware that his ordinal logic was not a formal system in the proper sense, since the axioms and thus proofs in his sequence of systems are not decidable. In Section 11 of his dissertation where he explained “the purpose of ordinal logics”, he introduced the interesting comparison of “intuition” and “ingenuity” in mathematical reasoning:

Mathematical reasoning may be regarded rather schematically as the exercise of combination of two faculties<sup>3</sup>, which we may call intuition and ingenuity. The activity of the intuition consists in making spontaneous judgments which are not the result of conscious trains of reasoning. These judgments are often but by no means invariably correct (leaving aside the question what is meant by “correct”). Often it is possible to find some other way of verifying the correctness of an intuitive judgment. We may, for instance, judge that all positive integers are uniquely factorizable into primes; a detailed mathematical argument leads to the same result. This argument will also involve intuitive judgments, but they will be less open to criticism than the original judgment about factorization. [17]

According to Turing, the exercise of ingenuity in mathematics was to verify the intuitive results by “suitable arrangements of propositions, and perhaps geometrical figures or drawing”, i.e., to find the necessary proof for the intuitive result and to turn the intuitive judgment into a theorem by eliminating the doubt of every inferential step. Although not an absolute distinction, we can nearly always tell in particular cases the different roles played by these two faculties. Through the introduction of a formal system for the corresponding mathematical field, these two faculties take an even more definite shape: intuition is reduced by setting down the axioms and formal inferential rules which are always considered to be intuitively valid while ingenuity determines which steps, among the considerable variety of possible steps in any stage of a proof, are the more profitable for the purpose of proving the desired proposition. Under this distinction, the significance of Gödel’s incompleteness is even more striking:

In pre-Gödel times it was thought by some that it would probably be possible to carry this program to such a point that all the intuitive judgments of mathematics could be replaced by a finite number of these rules. The necessity for intuition would then be entirely eliminated. (ibid. 209)

What Turing was saying, probably with Hilbert in mind, was that a single formal complete system for mathematics, where for each proposition either itself or its negation was provable, was just not possible. For the existence of such a system would eliminate intuition once and for all, leaving only the task for searching the proof, i.e., ingenuity behind. What Turing was proposing was rather the opposite by eliminating ingenuity and seeing “how far it is possible to eliminate intuition, and leave only ingenuity”. Of course, the possibility of not considering ingenuity is only conceivable in a formal system in a theoretical sense in that we can always, due to the recursive nature of a formal system, effectively enumerate all the possible proofs and the corresponding theorems, thus replacing ingenuity with, so to speak, brutal patience. Since it is impossible to find a

formal logic that wholly eliminates the necessity of using intuition, “we naturally turn to ‘non-constructive’ systems of logic with which not all the steps in a proof are mechanical, some being intuitive” [17]. What this amounts to is that in Turing’s (partial) complete system of ordinal logic some steps are non-mechanical in the sense that their validity is equivalent to the verification that some particular formulae are ordinal formulae, which again is equivalent to the truth of some number-theoretic statement. Its non-mechanical nature lies in the fact that, although being true, their truth cannot be established in a fixed system with limited methods of proof, i.e., they have to be invoked sometimes as a “oracle”. The wording here might suggest to some [18] that Turing was endorsing, at least for this period of his life, an anti-mechanist view and hence, agreeing with Gödel that mental procedures surpass the machine. However, what Turing was really stressing was only that “we want it to show quite clearly when a step makes use of intuition, and when it is purely formal. The strain put on the intuition should be a minimum” [17]. That is to say, even though any single formal system would not suffice for solving arithmetic problems, a system of them might do; by the same token, any single machine might not decide some propositions, many of them together might do. This fits well into the idea that although it is impossible to formalize and mechanize all mathematical intuition and its methods of proof in one single formal system, nonetheless it is possible that every mathematical intuition or method of proof is formalizable, thus allowing more and more powerful machines approximating truth by provability as well as anyone desires.

The same strategy, by inventing stronger and stronger formal systems/machines rather than creating a machine with a faculty for intuition, is adopted again in Turing’s conception of “intelligent machinery” and lies at the heart of his reply to the mathematical objection. The key assumption in the mathematical objection is that “the machine must not make mistakes. However, this is not a requirement for intelligence” [16]. If indeed a machine is expected to be infallible, it cannot also be intelligent, for discipline alone can never produce intelligence. Turing made the observation that:

If the untrained infant’s mind is to become an intelligent one, it must acquire both discipline and initiative . . . But discipline is certainly not enough in itself to produce intelligence. That which is required in addition we call initiative. This statement will have to serve as a definition. Our task is to discover the nature of this residue as it occurs in man, and to try and copy it in machines. (ibid.)

What Turing wrote in the rest of that paper concerned the possible ways to “copy” the initiative in machine, notably by “learning” through various methods of “educating” or “teaching” or “searching”, or by placing a random element in machines. The result would be a machine able to alter its own instruction tables through experience and learning, transcending the original methods available to it and thus displaying intelligence. As Copeland points out correctly: “In his post-war writing on mind and intelligence . . . the term “intuition” drops from view and what comes to the fore is the closely related idea of learning—in the sense of devising and discovering new methods of proof” [19].

## 5. Mechanizing Mathematical Intuition: Gödel and Turing Reconciled?

Whether Turing really meant to argue for the claim that “mental procedures cannot go beyond mechanical procedures”, either in 1936 or later when he was more concerned with the possibility of “learning machines”, and whether Gödel’s criticism and his opposing position was right or not, do not seem, from our discussion above, to be as urgent a problem as it looks. What is certain is that both realized there is a “residue” part in human thinking which transcends the limitations of any particular formalism or machine, be it “intuition”, “abstract understanding”, or “initiative”, although the way they would develop it was different. Gödel was certain about the extent of mechanical procedures with Turing machines and explored the possibility of the existence of humanly finite effective, yet non-mechanical procedures based on an understanding of abstract concepts with the help of mathematical intuition, and which also provide sufficient evidence for his conviction that human minds infinitely surpasses any finite machine. Turing, on the



other hand, being more practical rather than speculative, tried to embody the “initiative” in machines too, thus making them intelligent. Notwithstanding the differences, Gödel’s more philosophical conception of number and precision of states converging toward infinity and Turing’s more practical idea of devising stronger and stronger machines endowed with more and more powerful capacities for approximating truth and capturing intuition may just be seen as complementary ways to realizing the “residue” part in human intelligence, thus there is an ultimate agreement between Gödel and Turing. The great successes we recognize of Turing’s insights and ideas today definitely prove them to be a very fruitful philosophical program. The correctness of Gödel’s own philosophical position will depend, on the one hand, on the extent of the success of Turing’s program, and, on the other hand the power of Gödel’s own arguments and the developments of mathematics itself <sup>4</sup>. Latest developments in Artificial Intelligence, using models more sophisticated than Turing machines by combining the idea of making mistakes and learning with the more mechanical part, proves to be extremely powerful in helping mathematicians solve old problems. Relying on its unparalleled ability in generating large datasets of the representations of objects and searching for detectable patterns, AI can help mathematicians see the connection that would normally go unnoticed and nearly impossible to understand, as the recent two examples in mathematical theories of knots and symmetries show <sup>5</sup>. As the authors indicate, their main motivations are not to “use machine learning to directly generate conjectures”, but “focus on helping guide the highly tuned intuition of expert mathematicians, yielding results that are both interesting and deep” [20]. After all, rather than contrasting intuition with mechanical methods, they might benefit each other in a reciprocal way by development of “an artificial style of intuition”. We cannot conclude our discussion better than by quoting Turing: “We can only see a short distance ahead, but we can see plenty there that needs to be done” [22].

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## Notes

- <sup>1</sup> We cannot but hint at several central points of Gödel’s discussion here. For more see [15].
- <sup>2</sup> Turing’s partial completeness results only apply to sentences of the same logical form as Gödel’s original undecidable sentence, i.e., formulas with only universal quantifiers in front of the them.
- <sup>3</sup> In a footnote Turing said that he didn’t consider the most important faculty “which distinguishes topics of interest from others”, since he was here regarding the function of the mathematician as simply to determine the truth or falsity of propositions.
- <sup>4</sup> Such as the possibility of deciding CH based on large cardinals in set theory, and the already successful example of proving the consistency of Peano arithmetic using the abstract notion of primitive recursive functionals in Gödel’s famous “*Dialectica* interpretation”.
- <sup>5</sup> For more details, see [20,21].

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