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Article

On Structure of Infinite B^*-Matrices over Normed Fields

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Abstract: This article is devoted to the investigation of infinite B^*-matrices and linear operators over normed fields. Their structure is studied in the paper. Ideals and centers of the corresponding to them B^*-algebras are scrutinized.

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1. Introduction

Infinite matrices over the real field $\mathbf{R}$ and the complex field $\mathbf{C}$ and algebras of linear operators are indispensable in mathematics and its applications (see, for example [1–4] and references therein). In particular, C^*-algebras play a very important role. However, for infinite matrices and analogs of such algebras over other normed fields, comparatively little is known. This is caused by their specific features and additional obstacles arising from the structure of fields [5–12].

A lot of results in the classical case are based on the fact that the real field $\mathbf{R}$ has a linear ordering compatible with its additive and multiplicative structure. Moreover, the complex field $\mathbf{C}$ is algebraically closed, norm complete, locally compact, and is the quadratic extension of $\mathbf{R}$. Besides $\mathbf{R}$ and $\mathbf{C}$, there are not any other commutative fields with archimedean multiplicative norms and complete relative to their norms.

Notice, that in the ultrametric (non-archimedean) case the algebraic closure of the field $\mathbf{Q}_p$ is not locally compact. There is not any ordering on the infinite normed field such as $\mathbf{Q_p}$, $\mathbf{C_p}$ or $\mathbf{F_p}(t)$ compatible with its algebraic structure.

It is worth to mention that algebras over fields F other than $\mathbf{R}$ and $\mathbf{C}$, non-archimedean analysis, representations theory of groups and their applications develop fast in recent years [11,13–18]. Studies of matrices and linear algebras over fields with norms satisfying the strong triangle inequality are motivated not only by mathematical problems, but also by their applications in other sciences such as physics, quantum mechanics, quantum field theory, informatics, etc. (see, for example [19–25] and references therein).

This article is devoted to infinite B^*-matrices over normed fields. Their structure is studied in the paper. Ideals and centers of B^*-algebras are investigated in Theorems 1 and 2. Relations with operator theory and realizations of these algebras by algebras of infinite matrices are outlined. Theorems 3 and 4 about their embeddings into operator algebras are proven. This also provides tools for construction of wide families of such normed algebras. Realizations of elements of these algebras by infinite matrices are considered in Remark 3 and Example 8. Applications of obtained results are discussed in the conclusion.

All main results of this paper are obtained for the first time.

2. Embeddings of Normed*-Algebras

In this article, infinite matrices are considered over an infinite field F supplied with a multiplicative non-trivial norm denoted by $|\cdot|_F$, where $|\cdot|_F$ satisfies the strong triangle inequality:

$$|x+y|_F \le \max(|x|_F, |y|_F)$$

for each x and y in F. It is assumed that the field F is non-discrete and $\Gamma_F := \{|x|_F : x \in F \setminus \{0\}\} \subset (0, \infty) = \{r \in \mathbf{R} : 0 < r < \infty\}$.
Henceforth, the field F is supposed to be complete relative to its norm, if some other will not be specified.

Matrices with matrix elements belonging to F are naturally related with linear operators in normed spaces over the field F. Suppose that $V = c_0(\alpha, F)$ is a Banach space consisting of all vectors $x = (x_j : \forall j \in \alpha, \ x_j \in F)$ subjected to the condition

$card\{j \in \alpha : \ |x_j| > \epsilon\} < \aleph_0$ for each $\epsilon > 0$,

where V is supplied with the norm

$|x| = \sup_{j \in \alpha} |x_j|$,

where α is a set. For two normed spaces X and Y over the field F the linear space $L(X, Y)$ of all linear continuous operators $D : X \to Y$ is also normed:

$|D| := \sup_{x \in X \setminus \{0\}} |Dx|/|x|$.

Let $X = c_0(\alpha, F)$, $Y = c_0(\beta, F)$, $D \in L(X, Y)$, where α and β are sets. Then, to the operator D, a unique matrix $[D] = (d_{i,j} : i \in \beta, j \in \alpha)$ corresponds such that $Dx = y$ and $y_i = \sum_{j \in \alpha} d_{i,j} x_j$ with $d_{i,j} \in F$ for each $j \in \alpha$, $i \in \beta$, $x \in X$, $y \in Y$. The matrix $[D]$ is infinite, if $card(\alpha) \geq \aleph_0$ or $card(\beta) \geq \aleph_0$. Therefore, to any F-bimodule S contained in $L(X, Y)$, there corresponds a F-bimodule $[S]$ of matrices. In particular, $[S]$ is an algebra of matrices over F, if S is a subalgebra in $L(X, X)$.

Assume that A is a normed algebra over the field F such that a norm $|\cdot|_A$ on A satisfies the following conditions:

$|a|_A \in (\Gamma_F \cup \{0\})$ for each $a \in A$, also
$|a|_A = 0$ if and only if $a = 0$ in A,
$|ta|_A = |t|_F |a|_A$ for each $a \in A$ and $t \in F$,
$|a + b|_A \leq \max(|a|_A, |b|_A)$ and
$|ab|_A \leq |a|_A |b|_A$ for each a and b in A.

Frequently, it is shortly written $|\cdot|$ instead of $|\cdot|_F$ or $|\cdot|_A$.

We remind necessary definitions and notations.

Definition 1. *Assume that the field F is of the characteristic $char(F) \neq 2$. Assume also that $B_2 = B_2(F)$ is the commutative associative algebra with one generator i_1 such that $i_1^2 = -1$ and furnished with the involution $(vi_1)^* = -vi_1$ for each $v \in F$. Suppose that A is a subalgebra in $L(X, X)$ such that A is also a B_2-bimodule, where $X = c_0(\alpha, F)$ is the Banach space over a field F, α is a set. Then, A is called a $*$-algebra if there is a continuous bijective (i.e., injective and surjective) F-linear operator*

(1) $\mathcal{I} : A \to A$ *such that*
(2) $\mathcal{I}(ab) = (\mathcal{I}b)(\mathcal{I}a)$ *and*
(3) $\mathcal{I}(ga) = (\mathcal{I}a)g^*$ *and* $\mathcal{I}(ag) = g^*(\mathcal{I}a)$
(4) $\mathcal{I}\mathcal{I}a = a$
(5) $(\theta(y))(ax) = (\theta((\mathcal{I}a)y))(x)$

for every a and b in A and $g \in B_2$ and x and y in X, where $\theta : X \hookrightarrow X'$ is the canonical embedding of X into the topological dual space X' such that $\theta(y)x = \sum_{j \in \alpha} y_j x_j$. For the sake of brevity, we can write a^* instead of $\mathcal{I}a$. The mapping $\mathcal{I}$ is called the involution.

For two normed $*$-algebras A and B over F a map $\phi : A \to B$ which is a continuous homomorphism of algebras and $\phi(a^*) = (\phi(a))^*$ for each $a \in A$ is called a $*$-homomorphism. If the $*$-homomorphism ϕ is bijective and $\phi^{-1} : B \to A$ is also a $*$-homomorphism, then ϕ is called a $*$-isomorphism, and the normed $*$-algebras A and B are called $*$-isomorphic.

For the normed $*$-algebra A and $a \in A$ (or $U \subset A$) by $alg^*(a)$ (or $alg^*(U)$, respectively) will be denoted a minimal normed subalgebra in A containing a (or U, respectively). By $\overline{alg^*(U)}$ will be denoted the closure of $alg^*(U)$ in A.

Remark 1. *In Definition 1, $\theta(y)x$ is a particular case of a bilinear functional (see [11] and Remark 2 in more details).*

Note that for matrices with entries in $\mathbf{C}$ *corresponding to operators on* $\mathbf{C}$*-linear spaces, their block form with entries in* $\mathbf{R}$ *is frequently used, because each complex number can be written as the* 2×2 *matrix with real entries. This is utilized for generating a complex* $*$*-algebra by the doubling procedure from a corresponding real algebra [1,3,5,26]. Using similar ideas, one can construct examples of Banach* $*$*-algebras over the field* F *other than* $\mathbf{R}$ *and* $\mathbf{C}$ *with* $char(F) \neq 2$*. This is evident for* $F = \mathbf{Q}_p$ *with* $\sqrt{-1} \notin \mathbf{Q}_p$ *for primes* p *such that* $p \neq 1 \pmod 4$ *by Corollary 6 in Ch. 1, Section 4 in [25].*

Example 1. *Let* $X = c_0(\alpha, F)$ *be the Banach space such that* $card(\alpha) \geq \aleph_0$*. In view of Theorem 5.13 in [11], the direct sum* $X \oplus X$ *is isomorphic with* X*. In more details, one can take for the set* α *any fixed partition* $\alpha = \alpha_1 \cup \alpha_2$ *with* $\alpha_1 \cap \alpha_2 = \emptyset$ *and* $card(\alpha_1) = card(\alpha)$, $card(\alpha_2) = card(\alpha)$*. Therefore,* $c_0(\alpha_j, F) = X_j$ *is isomorphic with* X *for each* $j \in \{1, 2\}$ *and* $X_1 \oplus X_2$ *is isomorphic with* X*. This also implies that* $L(X_j, X_j)$ *is isomorphic with* $L(X, X)$ *as the Banach algebra for each* $j \in \{1, 2\}$*.*

For row vectors $x \in X$ *and operators* $a \in L(X, X)$*, it is frequently written for convenience* xa *instead of* $x[a]$ *or* $a(x)$*, where* $[a]$ *is a matrix corresponding to an operator* a*. Assume that* $char(F) \neq 2$*. For* $char(F) \neq 2$ *the algebra* $B_2 = B_2(F)$ *can be embedded into* $L(X, X)$ *in the standard way, up to an automorphism of the Banach algebra* $L(X, X)$*, as induced by the formula* $(x_1, x_2)i_1 = (-x_2, x_1)$ *for each* $x = (x_1, x_2) \in X$*, where* $x_1 \in X_1$ *and* $x_2 \in X_2$*,* $i_1^2 = -I$*,* I *is the unit operator on* X*.*

Example 2. *Let the condition of Example 1 be satisfied. Let* B *be a Banach subalgebra in* $L(X, X)$ *and let* $\psi : B \to B$ *be any continuous antiautomorphism of* B*. That is* ψ *is an* F*-linear map with continuous* ψ *and* ψ^{-1}*,* ψ *is a bijection (i.e., injection and surjection),* $\psi(ab) = \psi(b)\psi(a)$ *for each* a *and* b *in* B*. Such antiautomorphisms always exist, for example,* $\psi(ab) = ba$ *for each* a *and* b *in* B *[1,5,27]. Shortly, we denote* $\psi(a)$ *by* a^ψ*.*

There are natural embeddings η_j *of* $L(X_j, X_j)$ *into* $L(X, X)$ *as the Banach algebras such that* $(x_1, x_2)\eta_1(a) = (x_1 a, 0)$ *for each* $a \in L(X_1, X_1)$*,* $(x_1, x_2)\eta_2(b) = (0, x_2 b)$ *for each* $b \in L(X_2, X_2)$*, for each* $x = (x_1, x_2) \in X$ *with* $x_1 \in X_1$*,* $x_2 \in X_2$*, where* $j \in \{1, 2\}$*. Then, to each* $c \in L(X_1, X_2)$ *or* $d \in L(X_2, X_1)$*, one can pose operators* $\eta_3(c)$ *with* $(x_1, x_2)\eta_3(c) = (0, x_1 c)$ *and* $\eta_4(d)$ *with* $(x_1, x_2)\eta_4(d) = (x_2 d, 0)$ *for each* $x_1 \in X_1$ *and* $x_2 \in X_2$*. Shortly,* $\eta_1(a)$*,* $\eta_2(b)$*,* $\eta_3(c)$*,* $\eta_4(d)$ *will be denoted by* a, b, c, d*, respectively.*

Let ψ *be the antiautomorphism of* $L(X_1, X_1)$*. The Banach algebras* $L(X_1, X_1)$ *and* $L(X_2, X_2)$ *are isomorphic, hence* ψ *on* $L(X_1, X_1)$ *induces* ψ *on* $L(X_2, X_2)$*. For each* $a_1 \in L(X_1, X_1)$ *let* $\eta_5(a_1) \in L(X, X)$ *be such that* $(x_1, x_2)\eta_5(a_1) = (x_1 a_1, x_2 a_1^\psi)$ *for each* $x_1 \in X_1$ *and* $x_2 \in X_2$*. Notice that, with* $B_2 = B_2(F)$ *as in Example 1, evidently* $A_{L(X_1,X_1)} := \eta_5(L(X_1, X_1)) + \eta_5(L(X_1, X_1))i_1$ *is the* B_2*-bimodule, where* $U + W = \{x = u + w : u \in U, w \in W\}$ *for subsets* U *and* W *in a* F*-linear space* Y*.*

We take the antiautomorphism ψ *of* $L(X_1, X_1)$ *and extend it on* $A_{L(X_1,X_1)}$ *such that* $i_1^\psi = -i_1$ *and* $(a_1 + b_1 i_1)^\psi = a_1^\psi + i_1^\psi b_1^\psi$ *for each* a_1 *and* b_1 *in* $\eta_5(L(X_1, X_1))$*. For any* a_1 *and* b_1 *in* $A_{L(X_1,X_1)}$*, we put* $(a_1 i_1)^* = i_1^* a_1^*$ *and* $(a_1 + b_1 i_1)^* = a_1^* - b_1^\psi i_1$*. Hence,* $a_1^* = -i_1 a_1^\psi i_1$*,* $(a_1^*)^\psi = (a_1^\psi)^*$ *and* $(a_1 b_1 i_1)^* = -b_1^\psi a_1^\psi i_1$*, since* $b_1^\psi a_1^\psi = (a_1 b_1)^\psi$*. Therefore,* $(\alpha\beta)^* = \beta^* \alpha^*$ *for each* $\alpha = a_1 + b_1 i_1$ *and* $\beta = a_2 + b_2 i_1$ *with* a_1*,* a_2*,* b_1*,* b_2 *in* $\eta_5(L(X_1, X_1))$*, since* $(a_1 + a_2)^\psi = a_1^\psi + a_2^\psi$*,* $(a_1 a_2)^\psi = a_2^\psi a_1^\psi$*,* $(a_1^\psi)^\psi = a_1$*,* $(a_1^*)^* = a_1$*. This implies that* $A_{L(X_1,X_1)}$ *is the closed subalgebra in* $L(X, X)$ *and* $A_{L(X_1,X_1)}$ *is supplied with the* $*$*-algebra structure.*

Example 3. *Let the conditions of Example 2 be satisfied. We take any fixed antiautomorphism* ψ *on* $L(X_1, X_1)$ *extended on* $\eta_5(L(X_1, X_1)) + \eta_5(L(X_1, X_1))i_1$ *and inducing the involution as in Example 2. Certainly, for any given subset* V *in* $L(X_1, X_1)$*, there exists a minimal closed subalgebra* A_V *in* $(L(X, X), |\cdot|)$ *such that* $\eta_5(V) \subset A_V$*,* A_V *is the* B_2*-bimodule,* $A_V^* = A_V$*. This algebra* A_V *is the intersection of all closed* $*$*-subalgebras* W_k *in* $(L(X, X), |\cdot|)$ *such that* $\eta_5(V) \subset W_k$ *and* W_k *is the* B_2*-bimodule, where* W_k *is with the involution inherited from* $\eta_5(L(X_1, X_1)) + \eta_5(L(X_1, X_1))i_1$

and $W_k \subset \eta_5(L(X_1, X_1)) + \eta_5(L(X_1, X_1))i_1$. Evidently, A_V is the closure in $(L(X, X), |\cdot|)$ of a family $\mathcal{F}_V$ of all operators of the following form:

$$a = f_1\eta_5(v_1) + ... + f_n\eta_5(v_n) + f_{n+1}\eta_5(v_{n+1})i_1 + ... + f_{n+m}\eta_5(v_{n+m})$$

belonging to $L(X, X)$ with $f_1, ..., f_{n+m}$ in F, $v_1,...,v_{n+m}$ in $V \cup V^\psi$, where $V^\psi = \{v^\psi : v \in V\}$.

Take $X_{1,1} = c_0(\alpha_{1,1}, F)$, $X_{1,2} = c_0(\alpha_{1,2}, F)$ with $\alpha_1 = \alpha_{1,1} \cup \alpha_{1,2}$, $\alpha_{1,1} \cap \alpha_{1,2} = \emptyset$, $card(\alpha_{1,1}) = card(\alpha_1)$, $card(\alpha_{1,2}) = card(\alpha_1)$, where $card(\alpha_1) = card(\alpha)$, $card(\alpha) \geq \aleph_0$. Therefore, X_1 is isomorphic with $X_{1,1} \oplus X_{1,2}$ and with $X_{1,j}$ for each $j \in \{1, 2\}$. In particular, V can be taken contained in $L(X_{1,1}, X_{1,1})$, since there is the natural embedding $\eta_{1,1} : L(X_{1,1}, X_{1,1}) \hookrightarrow L(X_1, X_1)$ for $(X_{1,1}, X_1)$ analogously to η_1 for (X_1, X) described above. Therefore, there exists $V \neq IF$, $V \neq L(X_1, X_1)$ such that $A_V \neq A_{L(X_1,X_1)}$ and A_V is the nontrivial $*$-algebra.

Example 4. *Let the conditions of Examples 1 and 2 be satisfied. For Banach spaces B and Y over F by $B\hat{\otimes}_F Y$ is denoted the completion relative to the projective tensor product topology (see [11,28]) of the tensor product $B \otimes_F Y$ over the field F; or shortly $B\hat{\otimes}Y$ instead of $B\hat{\otimes}_F Y$, if F is specified. Let also $V \subset X_1\hat{\otimes}X_1$, where $X_1 = c_0(\alpha_1, F)$. In view of Theorems 4.33, 4.40, and 4.41 in [11], $X_1\hat{\otimes}X_1 \hookrightarrow L_c(X_1, X_1) \hookrightarrow L(X_1, X_1)$ and $X_1\hat{\otimes}X_1$ is the closed subalgebra in $L_c(X_1, X_1)$, where $L_c(X_1, X_1)$ denotes the algebra of all compact operators from X_1 into X_1. Notice that $L_c(X_1, X_1)$ is isomorphic with $X_1'\hat{\otimes}X_1$ by Theorem 4.41 [11], where X_1' is the topological dual space of X_1, $X_1' = L(X_1, F)$. We say that $X_1\hat{\otimes}X_1$ is the algebra of bicompact operators. In this case, it is possible to take $a^\psi = a^t$ for each $a \in X_1\hat{\otimes}X_1$, where $[a^t]_{k,j} = [a]_{j,k}$ for each j and k in α_1. This implies that $A_V \subset \eta_5(X_1\hat{\otimes}X_1) + \eta_5(X_1\hat{\otimes}X_1)i_1 \subset X\hat{\otimes}X$ (see Example 3). Hence, $A_V \neq A_{L(X_1,X_1)}$.*

Example 5. *Other examples of $*$-algebras which are proper subalgebras in $A_{L(X_1,X_1)}$ can be provided utilizing combinations of Examples 3 and 4. Note also that finite direct sums of $*$-algebras are $*$-algebras.*

Assume that Λ is an infinite set. For Banach spaces X_j over F for each $j \in \Lambda$, let $c_0(X_j : j \in \Lambda)$ denote a Banach space over F such that each $x \in c_0(X_j : j \in \Lambda)$ has the form $x = (x_j : \forall j \in \Lambda, x_j \in X_j)$ and $\forall \epsilon > 0$, $card\{j : |x_j| > \epsilon\} < \aleph_0$, $|x| = \sup_{j\in\Lambda} |x_j|$, $fx = (fx_j : \forall j \in \Lambda, x_j \in X_j)$ for each $f \in F$, $x + y = (x_j + y_j : \forall j \in \Lambda, x_j \in X_j, y_j \in X_j)$ for each x and y in $c_0(X_j : j \in \Lambda)$. Assume also that G_j is a Banach $*$-algebra over the field F for each $j \in \Lambda$. Then, $G := c_0(G_j : j \in \Lambda)$ is the Banach $*$-algebra with multiplication $xy = (x_j y_j : \forall j \in \Lambda, x_j \in G_j, y_j \in G_j)$ and inversion $x^* = (x_j^* : \forall j \in \Lambda, x_j \in G_j)$ for each x and y in G. We call G the c_0 direct sum of the Banach $*$-algebras G_j and denote it also by $G = \bigoplus_{j\in\Lambda}^{c_0} G_j$. Similarly, a l_∞ direct sum can be defined.

3. Bilinear Functionals on Algebras

Definition 2. *For a topological algebra A over a field F and a subset S of A, the left annihilator of S is defined by $\mathsf{L}(A, S) := \{x \in A : xS = 0\}$ and the right annihilator of S by $\mathsf{R}(A, S) := \{x \in A : Sx = 0\}$. Shortly, they also will be denoted by $A_l(S) := \mathsf{L}(A, S)$ and $A_r(S) := \mathsf{R}(A, S)$ correspondingly.*

The algebra A is called an annihilator algebra if it satisfies conditions (6)–(8):

(6) $A_l(A) = A_r(A) = 0$ *and*

(7) $A_l(J_r) \neq 0$ *and*

(8) $A_r(J_l) \neq 0$

for all proper closed right J_r and left J_l ideals in A.

Definition 3. *Let A and B be two Banach algebras over the normed field F. Let $A\hat{\otimes}_F B$ be the completion relative to the projective tensor product topology (see [11,28]) of the tensor product $A \otimes_F B$ over the field F.*

Suppose that B is a Banach algebra over the normed field F, and x is an element in B. It will be said that x has a left core quasi-inverse y if for any complete normed (valued) field extension H of F an element $y \in B_H$ exists satisfying the equality $x + y + yx = 0$, where $B_H = B\hat{\otimes}_F H$, where H is such that $|b|_H = |b|_F$ for each $b \in F$. In particular, if only the field $H = F$ is considered, it is called a left quasi-inverse.

Assume that A is a unital Banach algebra over F. Suppose also that an element $x \in A$ has the property: for any complete normed (valued) field extension G of F the left inverse $(1+yx)_l^{-1}$ exists in A_G for each $y \in A_G$, where G is normed such that $|b|_G = |b|_F$ for each $b \in F$. Then, we call x a generalized core nil-degree element. The family of all generalized core nil-degree elements of A is called a core radical and it is denoted by $R_c(A)$. A radical of the algebra A is denoted by $R(A)$.

Remark 2. *We recall known necessary facts about bilinear functionals. Let E and Y be Banach spaces over the field F. A space $Bil(E,Y;F)$ of all continuous bilinear functionals $T : E \times Y \to F$ is normed and isomorphic with $L(E, L(Y,F))$ and with $L(Y, L(E,F))$ (Ch. 4 in [11]).*

Let $X = c_0(\mathbf{N}, F)$ be a Banach space over the field F (F is complete relative to its multiplicative norm, as it was supposed above). Let $S : X \to X$ be a compact operator. Then, for each x and y in $L(X,X)$, the trace $Tr(xSy)$ exists and

(9) $|Tr(xSy)| \le |x||S||y|$.

Indeed, let S be a marked compact operator, $S \in L_c(X,X)$. A series $\sum_{j=1}^{\infty} b_j$ with $b_j \in F$ for each $j \in \mathbf{N}$ converges in F if and only if $\lim_{j\to\infty} b_j = 0$ by Proposition 23.1 in [16], since the field F is complete relative to its norm. Notice that the trace $Tr(C) = \sum_j C_{j,j}$ is defined for each compact operator $C \in L_c(X,X)$ by Theorem 4.40 in [11] (see also Definition 2.2 in [9]). By virtue of Theorems 4.37 and 4.40 in [11] $L_c(X,X)$ is the closed two-sided ideal in $L(X,X)$. This implies that $Tr(xSy)$ exists for each x, y in $L(X,X)$ (see also Theorem 2 in [9]). Since $|Tr(C)| \le |C|$ and $|xSy| \le |x||S||y|$, then $|Tr(xSy)| \le |x||S||y|$ for each x, y in $L(X,X)$.

Notice that for a subalgebra A in the algebra $L(X,X)$ the trace $Tr(xSy)$ exists for each x, y in A as the restriction of this bilinear continuous functional from $L(X,X)$ on A. In this article bilinear functionals are considered on Banach spaces.

Lemma 1. *Let G be a normed $*$-algebra over F of $char(F) \ne 2$ such that $G \subset L(X,X)$ and $G = G_1 \oplus G_1 i_1$ with G_1 being a subalgebra in $L(X_1,X_1)$ (see Example 3). Let also $(\cdot,\cdot)_1 : G_1^2 \to F$ be a bilinear functional on G_1 such that for each nonzero $a_1 \in G_1 \setminus \{0\}$ there exists $a_2 \in G_1$ with $(a_1,a_2)_1 \ne 0$. Then, the bilinear functional $(\cdot,\cdot)_1$ has a bilinear extension $(\cdot,\cdot) : G^2 \to F$ such that for each nonzero $a \in G \setminus \{0\}$ there exists $b \in G$ with $(a,b) \ne 0$.*

Proof. We put

(10) $(a,b) = (a_1,a_2)_1 - (b_1,b_2)_1$

for each $a = a_1 + b_1 i_1$ and $b = a_2 + b_2 i_2$ in G with a_1, a_2, b_1 and b_2 in G_1. If $a \ne 0$, then either $a_1 \ne 0$ or $b_1 \ne 0$. For $a_1 \ne 0$ one can take a_2 in G_1 with $(a_1,a_2)_1 \ne 0$ and put $b_1 = 0$. For $b_1 \ne 0$ one can take $b_2 \in G_1$ with $(b_1,b_2)_1 \ne 0$ and put $a_2 = 0$. □

4. B^*-Matrices and Algebras

Definition 4. *Let A be an normed algebra over the field F, (see Introduction), satisfying the following conditions:*

(11) *A is a Banach $*$-algebra and*

(12) *there exists a bilinear functional $(\cdot,\cdot) : A^2 \to F$ such that $|(x,y)| \le q|x||y|$ for all x and y in A, where $0 < q < \infty$ is a constant independent of x and y,*

(13) *$(x,y) = (y,x)$ and $(x,y) = (x^*,y^*)$ for each x and y in A,*

(14) *if $(x,y) = 0$ for each $y \in A$, then $x = 0$;*

(15) *$(xy,z) = (x,zy^*)$ for every x, y and z in A,*

(16) *$xx^* \ne 0$ for each nonzero element $x \in A \setminus (0)$.*

Then we call A a B^*-algebra. If an operator D belongs to the B^*-algebra A, $A \subset L(X,X)$, $X = c_0(\alpha,F)$, then the corresponding matrix $[D]$ is called a B^*-matrix.

Lemma 2. *For a $*$-subalgebra A of $L(X,X)$ with $X = c_0(\mathbf{N},F)$, a bilinear functional $(\cdot,\cdot)$ satisfying conditions (12), (13), and (15) exists.*

Proof. We put $(x,y) = Tr(x^*Sy)$, where S is a marked compact operator such that $S^* = S$, $S \in L_c(X,X)$, $X = c_0(\mathbf{N},F)$ (see Remark 2). From the inequality (9) it follows that Condition (12) is valid. From $Tr(C^*) = (Tr(C))^* = Tr(C)$ for each $C \in L_c(X,X)$ and

$(x^*Sy)^* = y^*Sx$ property (13) follows, since $b^* = b$ for each $b \in F$. Then, using the identity $Tr(CD) = Tr(DC) = \sum_{k,j} C_{k,j}D_{j,k}$ for each $C \in L_c(X,X)$ and $D \in L(X,X)$ we deduce that $(xy,z) = Tr(y^*x^*Sz) = Tr(x^*Szy^*) = (x,zy^*)$ for every x, y and z in A, since $(xy)^* = y^*x^*$. $\square$

Lemma 3. *Let $X = c_0(\mathbf{N},F)$ and let A be a Banach $*$-algebra over F such that $A \subset L(X,X)$. Let also either (1) or (2) be satisfied:*

(i) *If $char(F) \neq 2$ and for each $a \in A \setminus \{0\}$ there exists a normed extension K_a of the field F such that there exists a $*$-homomorphism ϕ_a from $K_a\hat{\otimes}_F\overline{alg^*(a)}$ into a $*$-algebra B_a, such that $B_a \subset L(X_{K_a},X_{K_a})$ with $X_{K_a} = c_0(\mathbf{N},K_a)$ and $\phi_a(aa^*) = J^{-1}DJ$, where $J \in L(X_{K_a},X_{K_a})$ is an invertible operator and a matrix $[D]$ of an operator $D \in L(X_{K_a},X_{K_a})$ is diagonal and nonzero, $[D] \neq 0$; or*

(ii) *if $char(F) = 0$ and for each $a \in A \setminus \{0\}$ there exists a $*$-homomorphism $\psi_a : alg^*(a) \to A_{L(X_1,X_1)}$ with nonzero image $\psi_a(a)$, $\psi_a(a) \neq 0$ (see Examples 2 and 3),*

then conditions (12)–(16) are also valid. Moreover,

(iii) *if A_j is a B^*-algebra over F for each $j \in \mathbf{N}$,*

then $A := \bigoplus_{j\in\mathbf{N}}^{c_0} A_j$ and $B := \bigoplus_{j\in\mathbf{N}}^{l_\infty} A_j$ are B^*-algebras.

Proof. In cases (i) and (ii), in view of Lemmas 2, 1, and Formula (10), Conditions (12), (13) and (15) are satisfied. Indeed, using injective $*$-homomorphism it is possible to choose $S \in L_c(X,X)$ for which the decomposition $S = T^{-1}YT$ is such that $T : X \to X$ is an automorphism of the Banach space X and $S^* = S$, also $Ye_j = Y_{j,j}e_j$ with $Y_{j,j} \neq 0$ for each j, while $Y_{i,j} = 0$ for each $i \neq j$, where $\{e_k : k \in \mathbf{N}\}$ is the standard basis of X, since $f^* = f$ for each $f \in F$. Then, we get property (14), since $Tr(x^*Sy) \in F$.

In case (i), we deduce that $\phi_a(a)(\phi_a(a))^* \neq 0$, hence $aa^* \neq 0$, since ϕ_a is the $*$-homomorphism and $\phi_a(aa^*) = \phi_a(a)(\phi(a))^*$. This implies (16).

In case (ii), let $x = \psi_a(a)$. By the imposed conditions in (2) x is nonzero, $x \neq 0$. On the other hand, $(bx)(bx)^* = b(xx^*)b^*$ and $(xx^*)^* = xx^*$ for each $b \in A_{L(X_1,X_1)}$. Let $E_{j,k} = e'_j \otimes e_k$, $e'_j = \theta(e_j)$ for each j and k in α_1 (see also Definition 1 and Examples 2, 3). Therefore, considering $b \in A$ of the form $b = b_1 + b_2i_1$ with $b_l = \sum_{k,j} f_{l;j,k}E_{j,k}$ with $f_{l;j,k} \in F$ for every $l \in \{1,2\}$, j and k in $\mathbf{N}$, one finds coefficients $f_{l;j,k}$ such that $(bx)(bx)^* \neq 0$, since $E_{j,k} \in X_1\hat{\bigotimes}_F X_1 \hookrightarrow L(X_1,X_1)$ for each j,k in α_1, $x \neq 0$. Note that $(bx)(bx)^* \neq 0$ implies that $xx^* \neq 0$ and consequently, $aa^* \neq 0$, since the algebra A is associative and ψ_a is the $*$-homomorphism. Thus, property (16) also is fulfilled.

(iii). For each $x \in A$ (or $x \in B$), there is the decomposition

(17) $x = \bigoplus_{j\in\mathbf{N}}^{c_0} x_j$ (or $x = \bigoplus_{j\in\mathbf{N}}^{l_\infty} x_j$, respectively) with $x_j \in A_j$ for each $j \in \mathbf{N}$. Therefore, $xx^* = \bigoplus_{j\in\mathbf{N}}^{c_0} x_jx_j^*$ (or $xx^* = \bigoplus_{j\in\mathbf{N}}^{l_\infty} x_jx_j^*$, respectively). Hence if $x \neq 0$, then $xx^* \neq 0$.

For each $j \in \mathbf{N}$ there exists a constant $w_j > 0$ such that $|(x_j,y_j)_j| \leq w_j|x_j||y_j|$ for each x_j, y_j in A_j, where $(x_j,y_j)_j$ denotes the bilinear functional on A_j satisfying the conditions of Definition 4. We choose $\pi \in F$ such that $0 < |\pi| < 1$, because the field F is infinite non-discrete. For each $j \in \mathbf{N}$ there exists $l(j) \in \mathbf{N}$ such that $|\pi|^j < w_j|\pi|^{l(j)} \leq |\pi|^{j-1}$. Let

(18) $(x,y) = \sum_{j=1}^{\infty}(x_j,y_j)_j\pi^{l(j)}$,

then $|(x,y)| \leq |x||y|/(1-|\pi|)$ for each x and y in A (or B, respectively). This implies that the bilinear functional given by Formula (18) satisfies conditions of Definition 4. $\square$

Lemma 4. *If J_r and J_l are proper or improper right and left ideals in a B^*-algebra A, then $\mathsf{L}(A,J_r)$ and $\mathsf{R}(A,J_l)$ are orthogonal relative to the family of bilinear functionals $\{(\cdot,\cdot)_a : a \in A\}$ complements of the sets J_r^* and J_l^* in the Banach space A, where $(x,y)_a = (ax,ay)$ for every a, x and y in A.*

Proof. If $x \in \mathsf{L}(A, J_r)$ (see Definition 2), then $xJ_r = (0)$, hence $(axJ_r, aA) = 0$ for each $a \in A$ and consequently, $(ax, aAJ_r^*) = 0$ by identity (15) and inevitably $(ax, aJ_r^*) = 0$. This means that $x \in A \ominus J_r^*$ relative to $\{(\cdot,\cdot)_a : a \in A\}$, that is $\mathsf{L}(A, J_r)$ is the orthogonal complement of J_r^*. Similarly, $\mathsf{R}(A, J_l)$ is the orthogonal complement of J_l^* in A as the Banach space relative to the family $\{(\cdot,\cdot)_a : a \in A\}$ of bilinear functionals. □

Proposition 1. *Any B^*-algebra A is dual.*

Proof. If J_r and J_l are right and left ideals in A, then by Lemma 4 $\mathsf{R}(A, \mathsf{L}(A, J_r)) = \mathsf{R}(A, A \ominus J_r^*) = A \ominus (A \ominus J_r) = J_r$ and analogously $\mathsf{L}(A, \mathsf{R}(A, J_l)) = J_l$, since $A^* = A$ and $(J_r^*)^* = J_r$. □

Theorem 1. *Any B^*-algebra A over the spherically complete field F with $R_c(A) = R(A)$ is representable as the direct sum of its two-sided minimal closed ideals, which are simple B^*-algebras and pairwise orthogonal relative to the family of bilinear functionals $\{(\cdot,\cdot)_a : a \in A\}$.*

Proof. By virtue of Theorem 8 in [9] and Proposition 1, the algebra A is the completion (relative to the norm) of the direct sum of its minimal closed two-sided ideals which are simple dual subalgebras (see also Definition 3). Consider a two-sided minimal closed non null ideal J in A. The involution mapping $x \mapsto \mathcal{I}x = x^*$ provides from it the minimal closed two-sided ideal J^* due to Condition (1).

Suppose that $J^* \neq J$, then $JJ^* = (0)$, since the ideal J is minimal. From $aJ \subset J$ and $Ja \subset J$ for each $a \in A$ we deduce that $AJJ^*A = (0)$. Together with condition (16) imposed on the B^*-algebra, this would imply that $x = 0$ for each $x \in J$ contradicting $J \neq (0)$. Thus, $J^* = J$.

Notice that properties (11)–(13) and (15) for J are inherited from that of A. Then, condition (16) on A implies that $J^2 \neq (0)$, since $J^* = J$ and $AJ \subseteq J$, also $JA \subseteq J$. However, J is minimal, hence $J^2 = J$. Therefore, property (14) on J follows from that of on A and (15) and $J^2 = J$, since for each $u \in J$ there exists x and y in J with $u = xy$ and $(u, z) = (xy, z) = (x, zy^*)$ for all $z \in A$, also since $zy^* \in J$. Then, for each $y \in J \setminus (0)$ an element $x \in J \setminus (0)$ exists such that $xy \neq 0$, hence $u = xy \in J \setminus (0)$. Then, we have that $uu^* \neq 0$ by (16) on A. Hence, $(xy)(xy)^* \neq 0$, consequently, $yy^* \neq 0$, since the algebra A is associative and $x(yy^*)x^* \neq 0$. Therefore, property (16) on J is valid. Thus, J is the B^*-algebra.

If J and S are two distinct minimal closed two-sided ideals in A, then $JS = (0)$. From Lemma 4, it follows that $S \subset \mathsf{R}(A, J) = A \ominus J^* = A \ominus J$. Thus, these ideals J and S are orthogonal relative to the family $\{(\cdot,\cdot)_a : a \in A\}$ of bilinear functionals.

Using condition (14) and Lemma 4, we infer that A is the direct sum of its two-sided minimal closed ideals. □

Theorem 2. *Let A be a simple unital B^*-algebra over the spherically complete field F with $R_c(A) = R(A)$ and let a division algebra G be provided by Theorem 2 in [10]. Then, the following conditions are equivalent:*

(i) *A_G is finite dimensional over G;*
(ii) *A_G is unital;*
(iii) *the center $Z(A_G)$ of A_G is non-null.*

Proof. Let $\{w_j : j \in \Lambda\}$ be a maximal system of irreducible idempotents provided by Theorem 2 in [10].

$(i) \Rightarrow (ii)$. If A_G is finite dimensional over G, then according to Theorem 1, a maximal system $\{w_j : j \in \Lambda\}$ of irreducible idempotents is finite, that is $card(\Lambda) < \aleph_0$. Then, their sum $w = \sum_{j \in \Lambda} w_j$ is the idempotent fulfilling the condition $x = \sum_{j \in \Lambda} xw_j = xw$ and $x = \sum_{j \in \Lambda} w_j x = wx$. Thus, w is the unit in A_G.

$(ii) \Rightarrow (iii)$. If A_G contains a unit w, then $Z(A_G)$ contains w, consequently, $Z(A_G)$ is non-null.

$(iii) \Rightarrow (i)$. Let $Z(A_G) \neq (0)$ and x be a non-zero element of $Z(A_G)$, $x \neq 0$. In view of Theorem 2 in [10] $xw_j = (xw_j)w_j = w_jxw_j = w_j^2xw_j$, hence $xw_j = b_jw_j = w_jb_jw_j$, where $b_j \in G$. Thus $(b_jw_j)w_j = w_j(b_jw_j)$. Therefore, $x = \sum_j xw_j = \sum_j b_jw_j$ and hence $b_jw_{j,k} = b_jw_jw_{j,k} = xw_jw_{j,k} = xw_{j,k} = w_{j,k}x = w_{j,k}w_kx = w_{j,k}xw_k = w_{j,k}b_kw_k$. Similarly, $b_kw_{k,j} = w_{k,j}b_jw_j$, consequently, $b_jw_{j,k}w_{k,j} = b_jw_j = w_{j,k}b_kw_kw_{k,j} = w_{j,k}b_kw_{k,j}$ and hence $\sum_j b_jw_j = b_kw_k + \sum_{j,j\neq k} w_{j,k}b_kw_{k,j} = \sum_j w_{j,k}b_kw_{k,j}$.

Note that $w_jA_Gw_j = Gw_j$ for each j, where w_j plays the role of the unit in Gw_j. Then,

$$Gw_j \supseteq w_j(w_{j,k}A_Gw_{k,j})w_j = w_{j,k}A_Gw_{k,j}$$
$$= w_{j,k}(w_kA_Gw_k)w_{k,j} \supseteq w_{j,k}(w_{k,j}A_Gw_{j,k})w_{k,j} = w_jA_Gw_j = Gw_j$$

for each j and k, hence $Gw_k \ni b \mapsto w_{j,k}bw_{k,j} \in Gw_j$ is the isomorphism of normed algebras Gw_j with Gw_k for each j and k.

Therefore, the sum $\sum_j w_{j,k}b_kw_{k,j} = \sum_j w_{j,k}w_kb_kw_kw_{k,j}$ may converge only if it is finite. Thus, the algebra A_G is finite dimensional over G. $\square$

Remark 3. *For a Banach space H over the field F and a set α by $c_0(\alpha, H)$ is denoted a c_0 direct sum of α copies of H such that $c_0(\alpha, H)$ is a Banach space consisting of all vectors $y = (y_j \in H : j \in \alpha)$ with $|y| = \sup_{j\in\alpha} |y_j| < \infty$ and such that for each $t > 0$ a set $\{j \in \alpha : |y_j| > t\}$ is finite. In particular, for the Banach space $X = c_0(\alpha, F)$ over a spherically complete field F, there exists a topological dual space X' of all continuous F-linear functionals $h : X \to F$ (see Ch. 2 and 5 in [11] or Ch. 8 in [29]). Each vector x in X_H has the following decomposition: $x = \sum_{j\in\alpha} e_jx_j$, where $x_j \in H$, $e_j \in X_H$ with $e_j = (\delta_{i,j} : i \in \alpha)$, $\delta_{i,j}$ denotes the Kronecker delta symbol such that $\delta_{i,j} = 0$ for each $i \neq j$ in α, $\delta_{j,j} = 1$ for each $j \in \alpha$.*

Then, for a division algebra H over the spherically complete field F and a Banach H-bimodule $X_H = c_0(\alpha, H)$ we consider a bounded F-linear right H-linear operator C from X_H into X_H , that is $C(xb) = (Cx)b$ for each $x \in X_H$ and $b \in H$. The embedding of F into H as $F1_H$, where 1_H is a unit element in H, induces a F-linear embedding of X into X_H. In this case to each $x \in X$ there corresponds a continuous F-linear right H-linear functional $x' = \theta(x)$ such that $\theta(x)y = \sum_{j\in\alpha} x_jy_j$ for each $x \in X$ and $y \in X_H$. This induces a natural embedding $\theta : X \hookrightarrow L_r(X_H, H)$, where $L_r(X_H, H)$ denotes a space of all bounded F-linear right H-linear operators from X_H into H (see Ch. 3 and 5 in [11], Proposition 23.1 in [16]). Therefore, for the operator C and for each i and j in α, there exists a matrix element $\theta(e_j)Ce_i =: C_{j,i}$. Then, by $L_{r,d}(X_H, X_H)$ is denoted the space of all bounded F-linear right H-linear operators C from X_H into X_H satisfying the condition:

(i) *for each $t > 0$ a finite subset γ in a set α exists such that $|C_{j,k}| < t$ for each j and k with either $j \in \alpha \setminus \gamma$ or $k \in \alpha \setminus \gamma$.*

Theorem 3. *Let A be a spherically complete simple unital B^*-algebra over a spherically complete field F with $R_c(A) = R(A)$. Let also G be a division algebra provided by Theorem 2 in [10] such that $s^{1/2} \in G$ for each $s \in G$, also $G \subset A$ and $G^* = G$. Then a Banach G-bimodule X_G exist such that A and $L_{r,d}(X_G, X_G)$ are isomorphic as the Banach right G-modules and as F-algebras.*

Proof. By the conditions of this theorem, a division algebra G is such that $wAw \subset Gw$ for each irreducible idempotent w in A. Put $H = G \cap G^*$. From $G = G^*$, it follows that $H = G$. If $b \in H$, then $b^{1/2} \in G$ and $(b^{1/2})^* = (b^*)^{1/2} \in G$, since $H^* = H$, consequently, $b^{1/2} \in H$.

For each irreducible idempotent w such that $wA_Gw = G$ (see the proof of Theorem 3 in [10]) one gets that $ww^* \neq 0$, since A is the B^*-algebra over F. Then, $(ww^*)(ww^*)^* \neq 0$, hence $ww^*ww^* \neq 0$ and consequently, $w^*ww^* \neq 0$ implying that $ww^*w \neq 0$, since $(w^*ww^*)^* = ww^*w$ and $c^{**} = c$ for each $c \in A$. Therefore, $w^*w \neq 0$ also.

Since w is the irreducible idempotent and $A^* = A$, then w^* is the irreducible idempotent in the B^*-algebra A. Then, we deduce that $w^*ww^* \in (w^*A_Gw^*)w^* \subseteq G^*w^* = (wG)^*$, since $A^* = A$, consequently, an element $s \in G^* \setminus (0)$ exists such that $w^*ww^* = sw^*$, since

$w^*ww^* \neq 0$. The latter implies $w^*ww^*w = sw^*w$. However, the elements w^*ww^*w and w^*w are self-adjoint, hence $sw^*w = w^*ws^*$ and consequently,

$w^*w(s^*)^{-1} = s^{-1}w^*w$.

We put $v = s^{-1}w^*w$, hence

$v^* = w^*w(s^*)^{-1} = s^{-1}w^*w = v$ and

$v^2 = s^{-1}w^*ws^{-1}w^*w = s^{-1}w^*ww^*w(s^*)^{-1}$

$= (s^{-1}(sw^*w))(s^*)^{-1} = w^*w(s^*)^{-1}$

$= s^{-1}w^*w = v$.

Thus, v is the self-adjoint idempotent. On the other hand, $A_Gv = A_Gs^{-1}w^*w \subseteq A_Gw$ and $A_Gv \neq 0$ and the idempotent w is irreducible, hence the idempotent v is also irreducible, since A_Gw is the non-null minimal left ideal in A_G.

Then, from the proof of Theorem 2 in [10] it follows that $(vA_Gv)^* = v^*A_G^*v^* = vA_Gv$ is the self-adjoint division algebra for each such irreducible self-adjoint idempotent v, consequently, $vA_Gv \subseteq Hv$. By the conditions of this theorem we have $A = A_G$.

The algebra A is simple, that is by the definition each its two-sided ideal coincides with either (0) or A.

Next we take a maximal orthogonal system $\{w_j : j \in \Lambda\}$ of self-adjoint idempotents in A and for them elements $w_{j,k}$ as in Theorem 2 in [10], where Λ is a set. Hence, $w_{j,k}w_{j,k}^* \in w_jAw_j$ and $b = b_{j,k} \in H$ exists such that $w_{j,k}w_{j,k}^* = bw_j$. Then, $bw_j = w_jb^*$, since $w_j^* = w_j$ and $(w_{j,k}w_{j,k}^*)^* = w_{j,k}w_{j,k}^*$. Moreover, $b \neq 0$, since $w_{j,k}$ is non null and hence $w_{j,k}w_{j,k}^*$ is non-null. For $v_{j,k} = (b_{j,k})^{-1/2}\, w_{j,k}$, we deduce that $v_{j,k}v_{j,k}^* = w_j$, since

$b^{-1/2}w_{j,k}w_{j,k}^*(b^{-1/2})^* = b^{-1/2}bw_j(b^{-1/2})^*$

$= w_j(b^{1/2})^*(b^{-1/2})^* = w_j(b^{-1/2}b^{1/2})^* = w_j$,

since A is associative and $b^{-1/2} \in H$ for each non null b in H, where $b = b_{j,k}$.

Thus, it is possible to choose an element $w_{j,k}$ such that $w_{j,k}w_{j,k}^* = w_j$ for each k. Taking a marked element $j = j_0$ and setting $w_{k,j} = w_{j,k}^*$ and $w_{l,k} = w_{l,j}w_{j,k}$ for each l and k one gets $w_{l,k}^* = w_{j,k}^*w_{l,j}^* = w_{k,j}w_{j,l} = w_{k,l}$ and $w_{k,k} = w_k$, also $w_{k,l}w_{i,h} = \delta_{l,i}w_{k,h}$ for every h, i, k, l. Thus, elements $w_{l,k}$ can be chosen such that $w_{l,k}^* = w_{k,l}$ for each l and k.

If the statement of this theorem for the spherical completion $\tilde{H}$ of H is proven, then it will imply the statement of this theorem for H. So the case of the spherically complete division algebra H is sufficient. Then, A and H considered as the Banach spaces over the spherically complete field F are isomorphic with $c_0(\alpha, F)$ and H with $c_0(\beta, F)$ due to Theorems 5.13 and 5.16 in [11], where $\beta \subset \alpha$.

From the proof of Theorem 3 in [10], it follows that the sum $B := \sum_{j,k} w_jAw_k$ is dense in A. Conditions (12), (13), (15) imply that $(xy, z) = (y, x^*z)$, since $t^* = t$ for each $t \in F$. Therefore, from properties (12), (13), (15) it follows that if $j \neq h$ or $k \neq l$, then $(w_jxw_k, w_hzw_l) = 0$ for each x and z in A, since

$(w_jxw_k, w_hzw_l) = (w_jx, w_hzw_lw_k^*) = (w_jx, w_hz(w_lw_k)) = (w_jx, 0) = 0$ for each $k \neq l$,

also

$(w_hzw_l, w_jxw_k) = (zw_l, w_h^*w_jxw_k) = (zw_l, (w_hw_j)xw_k) = (zw_l, 0) = 0$ for each $j \neq h$.

Thus, the set $\{w_{j,k} : j, k\}$ is complete and $(w_{j,k}H, w_{h,l}H) = (0)$ for each $j \neq h$ or $k \neq l$, where the latter property is interpreted as the orthogonality. Together with property (14), this implies that each element $x \in A$ has the form $x = \sum_{j,k\in\Lambda} w_{j,k}x_{j,k}$ with $\lim_{j,k} w_{j,k}x_{j,k} = 0$, since A_H is the right H-module, also A is isomorphic with A_G as the F-algebra and the right G-module, where the series may be infinite, $x_{j,k} \in H$ for each $j, k \in \Lambda$, where Λ denotes the corresponding set.

Take the Banach H-bimodule $X_H = c_0(\Lambda, H)$ and to each element $x \in B$ one can pose the operator T_x such that $e_j'T_xe_k = x_{j,k}\xi_{j,k}$ (see Remark 3), where $\xi_{j,k} \in F$ and $|\xi_{j,k}| = |w_{j,k}|$ for each j and k in Λ, since $|a| \in (\Gamma_F \cup \{0\})$ for each $a \in A$, where $B := \sum_{j,k} w_jAw_k$ (see above). Then, $T_x \in L_{r,d}(X_H, X_H)$ and the mapping $T : B \to L_{r,d}(X_H, X_H)$ is the isometry having the isometrical extension $T : A \to L_{r,d}(X_H, X_H)$. The property $w_{j,k}w_{j,k}^* = w_j \neq 0$ given above provides $|w_{j,k}| \neq 0$ for each j and $k \in \Lambda$, consequently, T is bijective from A onto $L_{r,d}(X_H, X_H)$, since A is simple.

For each S and V in $L_{r,d}(X_H, X_H)$, one has $SV(xb) = S(Vx)b = (SVx)b$ for each $b \in H$ and $x \in X_H$. Moreover, $|(SV)_{j,k}| \leq \sup_m |S_{j,m}||V_{m,k}|$, consequently, SV satisfies condition (i) in Remark 3, that is $SV \in L_{r,d}(X_H, X_H)$. Hence, by verifying other properties, one gets that $L_{r,d}(X_H, X_H)$ also has the F-algebra structure. From the construction of A_H, it follows that A_H is the F-algebra, since H and A are F-algebras. Notice that, moreover, A_H as the F-algebra is isomorphic with the Banach F-algebra $L_{r,d}(X_H, X_H)$. By the conditions of this theorem, A_H is isomorphic with A as the F-algebra and the right H-module. □

Theorem 4. *Let A be a spherically complete simple unital B^*-algebra over the spherically complete field F with $R_c(A) = R(A)$ and $Z(A) = F$. Let also G be a division algebra provided by Theorem 2 in [10] such that $s^{1/2} \in G$ for each $s \in G$. Then a division subalgebra H of G and a Banach H-bimodule X_H exist such that A_H and $L_{r,d}(X_H, X_H)$ are isomorphic as the Banach right H-modules and as F-algebras.*

Proof. In this case, $H = G \cap G^*$ and instead of A we consider $A_H = A\hat{\otimes}_F H$.

The B^*-algebra A is simple and central, $Z(A) = F$, hence the right H-module A_H is simple due to Satz 5.9 in [7] and Theorem 2 above. We denote A_H shortly by A and the rest of the proof is similar to that of Theorem 3. □

From Theorems 1, 3 and 4, the corollary follows.

Corollary 1. *Suppose that A is a spherically complete unital B^*-algebra over the spherically complete field F with $R_c(A) = R(A)$ and G is the division algebra given by Theorem 2 in [10] so that $s^{1/2} \in G$ for each $s \in G$ such that either*

(19) *$G \subset A$ and $G^* = G$ or*

(20) *$Z(A) = F$.*

Then, a division subalgebra H in G with $H^* = H$ and H-bimodules $X_{k,H}$ exist such that A_H as the right H-module and the F-algebra is the direct sum of $L_{r,d}(X_{k,H}, X_{k,H})$.

Example 6. *Let A be a B^*-algebra over a spherically complete normed field F (see Definition 4 and Introduction). Evidently, the algebra A also has the structure of the Banach A-bimodule. Hence, there exists a Banach space H over F such that A can be embedded into the normed algebra $L(H, H)$ of all bounded F-linear operators $D : H \to H$. In view of Theorems 5.13 and 5.16 in [11], there exists a set α such that H is isomorphic with the Banach space $c_0(\alpha, F)$ (see Remark 3). Therefore, each element D of A is characterized by the corresponding to it matrix $[D]$, which is unique relative to a fixed basis in H. This matrix is infinite, if $card(\alpha) \geq \aleph_0$.*

5. Conclusions

The results obtained in this article can be used for further studies of infinite matrices structure. Moreover, it provides new tools for investigations of their algebras over normed fields, linear operator algebras on Banach spaces, spectral theory of linear operators, the representation theory of groups, algebraic geometry, PDEs, mathematical physics. Then, studies of relations with symplectic structures may be of some interest [13]. It is important also for their applications in the sciences, including quantum mechanics, quantum field theory, informatics, etc. (see [1,4,5,7,8,11,14,15,19,20,23–25] and references therein).

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References

1. Beckenstein, E.; Narici, L.; Suffel, C. *Topological Algebras*; North-Holland Publishing Company: Amsterdam, The Netherlands, 1977.
2. Dunford, N.; Schwartz, J.C. *Linear Operators*; J. Wiley and Sons, Inc.: New York, NY, USA, 1966.
3. Kadison, R.V.; Ringrose, J.R. *Fundamentals of the Theory of Operator Algebras*; Acad. Press: New York, NY, USA, 1983.
4. Naimark, M.A. *Normed Rings*; Nauka: Moscow, Russia, 1968.
5. Bourbaki, N. *Algèbre. Ch. 1–3*; Springer: Berlin/Heidelberg, Germany, 2007.
6. Bourbaki, N. *Première Partie. Les Structures Fondamentales de l'Analyse. XI. Livre II. Algèbre. Ch. IV. Polynomes et Fractions Rationnelles. Ch. V. Corps Commutatifs*; Hermann: Paris, France, 1950.
7. Kersten, I. *Brauergruppen von Körpern*; Friedr. Vieweg and Sons: Braunschweig, Germany, 1990.
8. Ludkowski, S.V. Non-archimedean antiderivations and calculus of operators with local spectra. *Far East J. of Mathem. Sci.* **2016**, 99, 455–489.
9. Ludkowski, S.V. Normed dual algebras. *Mathematics* **2019**, *7*, 1–14. [CrossRef]
10. Ludkowski, S.V. Structure of normed simple annihilator algebras. *Mathematics* **2019**, *7*, 347. [CrossRef]
11. van Rooij, A.C.M. *Non-Archimedean Functional Analysis*; Marcel Dekker Inc.: New York, NY, USA, 1978.
12. Weil, A. *Basic Number Theory*; Springer: Berlin/Heidelberg, Germany, 1973.
13. Lam, T.Y.; Swan, R.G. Symplectic modules and von Neumann regular matrices over commutative rings. In *Advances in Ring Theory*; Huyn, D.V., López-Permouth, S.R., Eds.; Birkhäser: Basel, Swizerland, 2010; pp. 213–227.
14. Robert, A. Representations p-adiques irréductibles de sous-groupes ouverts de $SL_2(\mathbf{Z_p})$. C.R. *Acad. Sci. Paris Sér. I Math.* **1984**, *298*, 237–240.
15. van Rooij, A.C.M.; Schikhof, W.H. Groups representations in non-Archimedean Banach spaces. *Bull. Soc. Math. France. Memoire* **1974**, *39–40*, 329–340. [CrossRef]
16. Schikhof, W.H. *Ultrametric Calculus*; Cambr. Univ. Press: Cambridge, UK, 1984.
17. Ludkovsky, S.V. *Quasi-Invariant and Pseudo-Differentiable Measures in Banach Spaces*; Nova Science Publishers, Inc.: New York, NY, USA, 2008.
18. Ludkovsky, S.V. *Stochastic Processes in Non-Archimedean Banach Spaces, Manifolds and Topological Groups*; Nova Science Publishers, Inc.: New York, NY, USA, 2010.
19. Anashin, V. Automata finitness criterion in terms of van der Put series of automata functions. *p-Adic Numbers Ultrametric Anal. Appl.* **2012**, *4*, 151–160. [CrossRef]
20. Aref'eva, I.Y. Holographic relation between p-adic effective action and string field theory. P*roc. Steklov Inst. Math.* **2014**, *285*, 26–29. [CrossRef]
21. Dragovich, B. On measurements, numbers and p-adic mathematical physics. *p-Adic Numbers Ultrametric Anal. Appl.* **2012**, *4*, 102–108. [CrossRef]
22. Isham, C.J. Topological and global aspects of quantum theory. In *Relativity, Groups and Topology*; Hauches, L., II; Stora, R., De Witt, B.S., Eds.; Elsevier Sci. Publ.: Amsterdam, The Netherlands, 1984; pp. 1059–1290.
23. Mikhailov, V.M. The effectiveness of monitoring as a necessary condition for the adoption of correct decisions on the security of technosphere. *Russ. Technol. J.* **2020**, *8*, 23–32. [CrossRef]
24. Viswanathan, K.S. Colliding gravitational plane waves and black hole creation. *p-Adic Numbers Ultrametric Anal. Appl.* **2012**, *4*, 143–150. [CrossRef]
25. Vladimirov, V.S.; Volovich, I.V.; Zelenov, E.I. *p-adic Analysis and Mathematical Physics*; Nauka: Moscow, Russia, 1994.
26. Gantmaher, F.R. *Matrix Theory*; Nauka: Moscow, Russia, 1988.
27. Pierce, R.S. *Associative Algebras*; Springer: New York, NY, USA, 1982.
28. van der Put, M.; van Tiel, J. Espaces nucléaires non-archimédiens. *Indag. Mathem.* **1967**, *29*, 556–561.
29. Narici, L.; Beckenstein, E. *Topological Vector Spaces*; Marcel-Dekker Inc.: New York, NY, USA, 1985.