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On the Schott Term in the Lorentz-Abraham-Dirac Equation

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Abstract: The equation of motion for a radiating charged particle is known as the Lorentz–Abraham–Dirac (LAD) equation. The radiation reaction force in the LAD equation contains a third time-derivative term, called the Schott term, which leads to a runaway solution and a pre-acceleration solution. Since the Schott energy is the field energy confined to an area close to the particle and reversibly exchanged between particle and fields, the question of how it affects particle motion is of interest. In here we have obtained solutions for the LAD equation with and without the Schott term, and have compared them quantitatively. We have shown that the relative difference between the two solutions is quite small in the classical radiation reaction dominated regime.

Keywords: radiation reaction; Loretnz–Abraham–Dirac equation; schott term

1. Introduction

The laser's focused intensity has been increasing since its invention, and is going to reach 10^{24} W/cm^2 and beyond in the very near future [1]. These intense fields open up new research in high energy-density physics, and researchers are searching for laser-driven quantum beams with unique features [2–4]. One of the important issues in the interactions of these intense fields with matter is the radiation reaction, which is a back reaction of the radiation emission on the charged particle which emits a form of radiation. Numerical investigations showed that high energy photons are emitted from the laser-generated plasmas, which indicates the possibility of generating an intense and collimated laser-driven γ-ray source [5,6]. In order to correctly describe the motions of electrons under these intense field, the radiation reaction effect should be properly treated in the analyses.

The formulation of the equation of motion for a radiating charged particle has caught a lot of attention. Lorentz studied the self-force on an accelerated charged particle [7], and Abraham followed this work by deriving the reaction force in relativistic form [8], which was also derived by von Laue via the transformation of Lorentz's self-force [9]. Thereafter an extension to the Lorenz-covariant form was performed by Dirac [10] and by Pauli [11], leading to the Lorentz–Abraham–Dirac (LAD) equation which is considered to be the fundamental equation of Maxwell-Lorentz theory. Although the LAD equation appears to have solid grounding, its correctness has been questioned due to the mathematical problems—i.e., it allows a runaway solution which exponentially grows in time, and a pre-acceleration solution which violates causality [12–15]. To circumvent these difficulties, modified formulas of self-force were proposed [13,16,17], and particle motions evaluated with these formulas were compared [18]. Recently, new equations of motion have been proposed, where the self-force was reformulated by including the spatial extent of the charged particle [19–21], or a time-delay between the radiation and its reaction [22]. In these analyses, the divergence problem is circumvented by introducing a spatial extent to a particle. However, the validity of the assumption that a charged particle is a rigid relativistic particle with a finite spatial extent has not yet been proven. Then, in the classical electrodynamics whether a charged particle can be treated as a point particle or as a finite sized-particle is still an unsettled problem. In our analysis here, we adopt the model of the electrons as point particles, and focus our study on the LAD equation-based solutions.

The LAD equation of a charged particle with mass m and charge q is written as

$$\frac{du^\mu}{d\tau} = \frac{q}{mc}F^{\mu\nu}u_\nu + \frac{2q^2}{3mc^3}\left(\frac{d^2u^\mu}{d\tau^2} + \frac{a^\nu a_\nu}{c^2}u^\mu\right), \tag{1}$$

where c is the speed of light in vacuum, $u^\mu = (\gamma c, \gamma v^i)$ is the four-velocity with the Lorentz factor γ, a^μ is the four-acceleration which is defined as a derivative of u^μ with respect to the proper time τ, and $F^{\mu\nu}$ is the electromagnetic field tensor. Greek indices run from 0 to 3, and Latin indices run from 1 to 3. The metric of $\mathrm{diag}(+1, -1, -1, -1)$ is adopted. The first term on the RHS represents the Lorentz force, and the second term represents the radiation reaction force. The radiation reaction force is composed of two terms, the Schott term F^μ_{Schott} [23], and a radiation term F^μ_{Rad}:

$$F^\mu_{RR} = F^\mu_{Schott} + F^\mu_{Rad}, \tag{2}$$

$$F^\mu_{Schott} = \frac{2q^2}{3c^3}\frac{d^2u^\nu}{d\tau^2}, \tag{3}$$

$$F^\mu_{Rad} = \frac{2q^2}{3c^3}\frac{u^\mu}{c^2}a^\nu a_\nu. \tag{4}$$

It is the Schott term which is responsible to the above mathematical problems, and whose physical meaning was not clear until recently. It was found that the Schott term is responsible for the electromagnetic field energy stored around the charged particle [24–26]. This analysis showed that the Schott energy, which is a $\mu = 0$ component of the Schott four-momentum P^μ_{Schott}, is the energy of the electromagnetic field excluding the radiation field, i.e., the interference field of the Coulomb field (velocity field) and the radiation field [24]. Here, the Schott four-momentum is defined as

$$P^\mu_{Schott} = -\frac{2q^2}{3c^3}a^\mu, \tag{5}$$

where $F^\mu_{Schott} = -\frac{dP^\mu_{Schott}}{d\tau}$. Furthermore, it has been shown that the Schott energy is confined to the vicinity of the charged particle, and is reversibly exchanged between particles and fields [25,26].

This intriguing idea is easily understood by considering what is taken into account in the $\mu = 0$ component of the radiation term:

$$F^0_{Rad} = \frac{2q^2\gamma}{3c^4}a^\nu a_\nu = -\frac{\gamma}{c}\left\{\frac{2q^2}{3c}\gamma^4\left[\dot{\boldsymbol{\beta}}^2 + \gamma^2(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right]\right\}. \tag{6}$$

Here, the term inside the curly brackets has the unit of power. The radiation power from the accelerating charged particle is calculated straightforwardly using the Lieénard-Wiechert potential, or the retarded field as

$$\boldsymbol{E}(t) = \frac{q(\boldsymbol{n}-\boldsymbol{\beta})}{r^2\gamma^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3} + \frac{q\boldsymbol{n}\times\left[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{cr(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}, \tag{7}$$

$$\boldsymbol{B}(t) = \boldsymbol{n}\times\boldsymbol{E}. \tag{8}$$

Here, $r(t)$ is the distance between the particle position and the observation point evaluated at t. It should be noted that the value of the RHS is evaluated at the retarded time t' which has the relation $t = t' + r(t')/c$. When calculating the radiation power, only the second term in Equation (7), (i.e., the first time is neglected) is substituted into the following formula [27]:

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}r^2\left|\boldsymbol{E}\right|^2(1-\boldsymbol{n}\cdot\boldsymbol{\beta}). \tag{9}$$

After angular integration, we obtain the radiation power which is exactly the same as the formula inside the curly brackets of Equation (6). This means that in the radiation reaction, term F^{μ}_{Rad} expresses the reaction to the electromagnetic field radiation, and F^{μ}_{Schott} accounts for the reaction to terms other than radiation field, i.e., the neglected terms containing the Coulomb field. Therefore, the Schott term correspond to the field which is bound to the particle, and is not radiated away.

It has been shown that during uniform acceleration, the Schott term plays a major role wherein the radiation energy balances the Schott energy [26]. Furthermore, renormalizing the Schott term into the mass has been proposed [28,29]. That led us to consider the following questions: Can the Schott term can be integrated away when the particle motion is periodic? To what extent does the Schott term affect the particle motion? In this paper, to identify the effect of the Schott term on the particle motion, we quantitatively compare the solutions obtained from the LAD equation with and without the Schott term.

2. The Schott Term for a Stationary Motion

To analyze the solutions, we need to solve the LAD equation both with and without the Schott term for the same problem and procedure. Since numerical time-integration of the LAD equation is not straightforward as it blows up in time, and obtaining a general analytical solution is quite difficult, we look for an exact solution of particle motion under a simple field configuration. We consider the electron motion under a static magnetic field which points along the z-axis and a rotating electric field in the x–y plane with angular frequency ω, which are written as $\boldsymbol{B} = (0, 0, B)$ and $\boldsymbol{E} = (E\cos(\omega t), E\sin(\omega t), 0)$. We look for a stationary solution in which the charge rotates in the x–y plane with the angular frequency ω [30,31]. Then, the four-velocity is written as $\boldsymbol{u} = \gamma\boldsymbol{v} = (u_{\parallel}\cos(\omega t) - u_{\perp}\sin(\omega t), u_{\parallel}\sin(\omega t) + u_{\perp}\cos(\omega t), 0)$. Here, $u_{\parallel}$ and $u_{\perp}$ denote the four-velocity components which are parallel and perpendicular to the electric field, respectively. By substituting these into the $\mu = 1, 2$ components of the LAD equation, and solving for $u_{\perp}$ and $u_{\parallel}$, we obtain

$$u_{\parallel} = \frac{eB}{mc\omega\gamma}u_{\parallel} - \frac{2e^2\omega\gamma^3}{3mc^3}u_{\perp}, \tag{10}$$

$$u_{\perp} = \frac{eE}{mc\omega} + \frac{eB}{mc\omega\gamma}u_{\perp} + \frac{2e^2\omega\gamma^3}{3mc^3}u_{\parallel}. \tag{11}$$

Here, we introduce the dimensionless variables $a_0 = eE/(mc\omega), b_0 = eB/(mc\omega)$, and $\varepsilon = 2e^2\omega/(3mc^3)$, and $u_{\perp,\parallel} = u_{\perp,\parallel}/c$. Then, the following equations are obtained:

$$\left(1 - \frac{b_0}{\gamma}\right)u_{\parallel} = -\varepsilon\gamma^3 u_{\perp}, \tag{12}$$

$$\left(1 - \frac{b_0}{\gamma}\right)u_{\perp} = a_0 + \varepsilon\gamma^3 u_{\parallel}. \tag{13}$$

By multiplying $u_{\parallel}$ with Equation (12) and $u_{\perp}$ with Equation (13) and adding them, we obtain $(\gamma^2 - 1)(1 - \frac{b_0}{\gamma}) = a_0 u_{\perp}$. Together with Equation (13), we obtain

$$a_0^2 = (\gamma^2 - 1)\left[\left(1 - \frac{b_0}{\gamma}\right)^2 + \varepsilon^2\gamma^6\right]. \tag{14}$$

Equations (12)–(14) are used to evaluate the electron motion with the radiation reaction effect calculated by the LAD equation. The motion of the electron is shown in Figure 1, where the γ, $u_{\perp}$, and $-u_{\parallel}$ dependencies on a_0 are plotted for the cases of $b = -10$ (red line), 0 (black line), and 10 (blue line), respectively. Here, the frequency of the rotating electric field is $\omega = 2\pi c \times 10^6$, corresponding to $\varepsilon \simeq 1.18 \times 10^{-8}$.

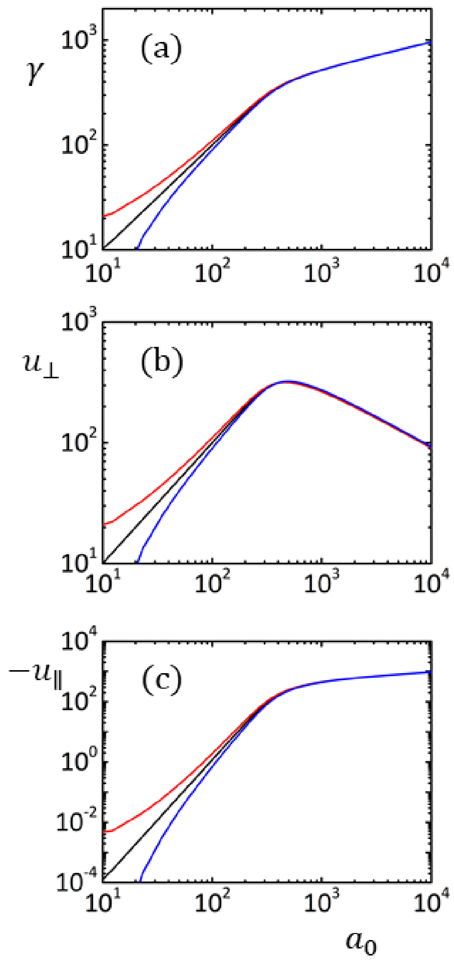

Figure 1. Dependencies of (**a**) the Lorentz factor γ, (**b**) the perpendicular component of the four-velocity $u_\perp$, and (**c**) the parallel component of the four-velocity $-u_\parallel$ on the electric field intensity a_0. In each figure, red, black, and blue lines correspond to the cases of $b_0 = -10$, 0, and 10, respectively. Here, the frequency of the rotating electric field is $\omega = 2\pi c \times 10^6$, corresponding to $\varepsilon \simeq 1.18 \times 10^{-8}$. All variables are dimensionless with normalization given in the text.

Since the Lorentz factor is constant in time,

$$a^0 = \frac{du^0}{d\tau} = \gamma \frac{d(\gamma c)}{dt} = 0; \tag{15}$$

i.e., the Schott energy vanishes for this stationary motion. Then, the $\mu = 0$ component of the LAD equation becomes

$$0 = -eF^{0i}u_i + \frac{2e^2\gamma a^\nu a_\nu}{3c^3}. \tag{16}$$

This equation shows that the work done by the electric field per unit time equals the power radiated away, with the Schott term in the $\mu = 0$ component of the LAD equation not playing any role. This is confirmed in Figure 2, where the work rate done by the electric field W_E is plotted in black and the power emitted as radiation P_R in red is plotted for $b_0 = 0$, which are almost identical to the precision of the calculation.

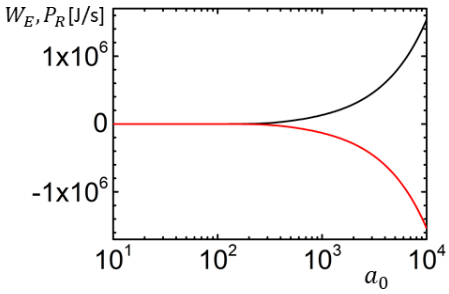

Figure 2. The work rate of the electron done by the electric field W_K (black) and the radiation power P_R (red) in units of J/s are plotted as functions of a_0.

3. Stationary Solution of LAD Equation Without the Schott Term

Although the Schott energy disappears from the LAD equation in the case of periodic motion, the spatial component of the Schott term, the Schott momentum, does not vanish from the $\mu = i$ component of the LAD equation, and may play a role in determining the electron motion. To see the effect of the Schott momentum, we consider the LAD equation without the Schott term;

$$\frac{du^\mu}{d\tau} = \frac{q}{mc} F^{\mu\nu} u_\nu + \frac{2q^2}{3mc^3} \frac{a^\nu a_\nu}{c^2} u^\mu. \tag{17}$$

With this equation, we perform the same analysis as that in the previous section for the electron under static magnetic and rotating electric fields. This leads to the following solution of the electron motion:

$$\left(1 - \frac{b}{\gamma}\right) u_\parallel = -\varepsilon\gamma(\gamma^2 - 1) u_\perp, \tag{18}$$

$$\left(1 - \frac{b}{\gamma}\right) u_\perp = a_0 + \varepsilon\gamma(\gamma^2 - 1) u_\parallel, \tag{19}$$

$$a^2 = \left(1 - \frac{b}{\gamma}\right)^2 (\gamma^2 - 1) + \varepsilon^2 (\gamma^2 - 1)^3. \tag{20}$$

The obtained solutions of γ, $u_\perp$, and $u_\parallel$ are almost identical to those calculated by the LAD equation. When plotted together in Figure 1, they almost completely overlap. To see the difference in γ calculated by the LAD equation with and without the Schott term, we introduce the relative difference of the Lorentz factor:

$$\delta\gamma = \frac{|\gamma - \gamma_{LAD}|}{\gamma_{LAD}}. \tag{21}$$

Here, γ denotes the Lorentz factor calculated by the equation without the Schott term. The dependence of $\delta\gamma$ on a_0 is plotted in Figure 3, where $b_0 = 0$ and $\varepsilon \simeq 1.18 \times 10^{-8}$.

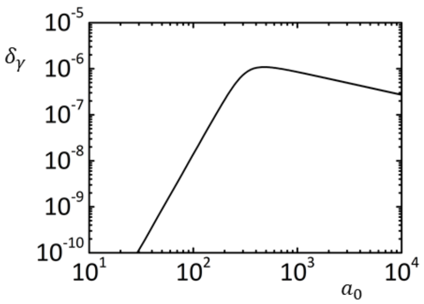

Figure 3. The relative difference between the Lorentz factor calculated from the LAD with and without the Schott term, δ_γ, for $b_0 = 0$ and $\varepsilon \simeq 1.18 \times 10^{-8}$.

The Schott momentum does play a role in determining the particle motion, but its effect on the motion seems quite small in our case, where $\delta\gamma$ has a maximum of about 10^{-6}. The magnitudes of the spatial component of F^i_{Schott} and F^i_{Rad} are compared:

$$\left|\frac{F^i_{Schott}}{F^i_{Rad}}\right| = \left|\frac{\frac{d^2u^i}{d\tau^2}}{\frac{a^\nu a_\nu u^i}{c^2}}\right| = \frac{c^2\omega^2\gamma^2 u^i}{c^2\omega^2\gamma^2(\gamma^2-1)u^i} = \frac{1}{\gamma^2-1}. \tag{22}$$

The radiation reaction term in the LAD equation without the Schott term is obtained from the LAD equation by substitution of $\gamma^2 \to \gamma^2 - 1$. Therefore, the effect of the Schott term on the electron motion is quite small for wide range of a_0.

The radiation power from the electron is evaluated by the Larmor formula $P = \frac{2e^2a^2}{3c^3}$, or simply the work rate done by the electric field on the electron, as

$$P = -e\boldsymbol{v}\cdot\boldsymbol{E} = -\frac{eu_\parallel E}{\gamma}. \tag{23}$$

This is plotted in Figure 4a. The radiation power sharply increases with a_0 for $a_0 \le 400$, and gradually for $a_0 \ge 400$. The relative differences in the radiation power evaluated by the LAD with and without the Schott term are plotted in Figure 4b, which confirms the radiation power is evaluated by the LAD equation with and without the Schott term with a high degree of accuracy.

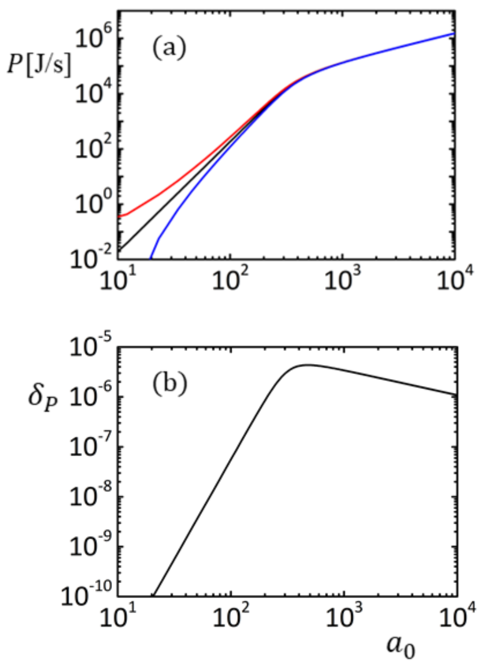

Figure 4. (**a**) The radiation power from the charged particle under a static magnetic and a rotating electric field as a function of electric field intensity a_0 for different magnetic field intensities of $b_0 = 10$ (red), $b_0 = 0$ (black), and $b_0 = -10$ (blue). (**b**) The relative difference of radiation power evaluated using the LAD equation with and without the Schott term.

4. Conclusions

The effect of the Schott term in the LAD equation on the charged particle motion was investigated. We obtained exact solutions of the LAD equation with and without the Schott term for a relatively simple system where the particle performs periodic motion under static magnetic and rotating electric fields. Since the Schott energy vanishes for the stationary solution, the $\mu = 0$ components of the two equations become equal, which simply leads to the power balance between the work rate of the field and the power of the radiation. The Schott momentum, however, does not vanish, so the spatial components of two equations are different, and the solutions also differ to each other. We compared the two solutions and found that they agree with high precision. The relative difference between the Lorentz factor and radiation power is less than 10^{-6} for an electric field intensity of $10 \leq a_0 \leq 10^4$. This analysis shows that for periodic motion, the effect of the Schott term on particle motion is relatively small, which leads to that the renormalized LAD equation, which does not suffer from the mathematical problems, so could be used for analyzing the interactions of intense fields and charged particles in this particular case. Analysis in a more general geometry is left for a future study.

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