



Article Intrinsic Metric Formulas on Some Self-Similar Sets via the Code Representation

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Abstract: In recent years, intrinsic metrics have been described on various fractals with different formulas. The Sierpinski gasket is given as one of the fundamental models which defined the intrinsic metrics on them via the code representations of the points. In this paper, we obtain the explicit formulas of the intrinsic metrics on some self-similar sets (but not strictly self-similar), which are composed of different combinations of equilateral and right Sierpinski gaskets, respectively, by using the code representations of their points. We then express geometrical properties of these structures on their code sets and also give some illustrative examples.

Keywords: Sierpinski gasket; self-similar sets; code representation; intrinsic metric

MSC: 28A80; 51F99

1. Introduction

Fractal geometry is one of the most remarkable developments in mathematics in recent years. Since fractals are central to apprehending a wide array of chaotic and nonlinear systems, they have many applications in physics, chemistry, biology, computer science, engineering, economics and so on (for details, see [1-4]). One of the common properties of fractals is self-similarity. A geometric shape is self-similar if there is a point such that every neighborhood of the point contains a copy of the entire shape and, if it is self-similar at every point, then it is called strictly self-similar. For example, the Sierpinski gasket, the Cantor set, the Koch curve and the Sierpinski carpet are strictly self-similar sets (for details, see [5]). There have been different studies on these fractals since the 1970s. Defining the intrinsic metric on these sets is one of them [6-11]. In [12], the intrinsic metric on the code set of the Sierpinski gasket, which is one of the instructive examples of strictly self-similar sets, is formulated by the code representations of its points. Due to this metric formula, important geometrical and topological properties of the Sierpinski gasket are expressed by the code sets, the number of geodesics are determined and the code representations of points are classified according to the number of geodesics (for details, see [13–15]). As seen in these studies, defining the intrinsic metrics by using the code representations of the points on fractals provides some facilities for different works. Hence, our aim is to increase such examples in the literature.

In this paper, we define two new fractals which are self-similar but are not strictly self-similar and then we determine the code representations of points on these fractals. To this end, we first constitute a model by using three classical Sierpinski gaskets with edge length one. We also express code representations of the points on this structure. In Theorem 2, we define the intrinsic metric on this structure and present some geometrical properties on its code set in Propositions 3 and 4. Finally, we formulate the intrinsic metric on a second fractal obtained by two right Sierpinski gaskets

in Proposition 5. As seen in Examples 1 and 3, one of the advantages of these intrinsic metric formulas is to calculate the distance between the points on these sets easily.

2. The Intrinsic Metric on the Code Set of the Sierpinski Gasket

It is well known that the intrinsic metric on a set X is defined as

$$d_{int}(x, y) = \inf\{ \delta \mid \delta \text{ is the length of a rectifiable curve in X joining x and y} \}$$

for $x, y \in X$ (see [16]). By using the code representations of the points, to express the intrinsic metrics on fractals is a valuable problem, but it is not easy to obtain these formulas for every self-similar set. Therefore, the Sierpinski gasket (*S*) is regarded as one of the fundamental models which define the intrinsic metric formulas on fractals (for details, see [12]). It is well known that the attractor of the iterated function system { \mathbb{R}^2 ; f_0, f_1, f_2 } such that

$$f_{0}(x,y) = \left(\frac{x}{2} + \frac{p_{0}}{2}, \frac{y}{2} + \frac{p_{1}}{2}\right),$$

$$f_{1}(x,y) = \left(\frac{x}{2} + \frac{q_{0}}{2}, \frac{y}{2} + \frac{q_{1}}{2}\right),$$

$$f_{2}(x,y) = \left(\frac{x}{2} + \frac{r_{0}}{2}, \frac{y}{2} + \frac{r_{1}}{2}\right)$$
(1)

is the Sierpinski gasket with the vertices $P = (p_0, p_1), Q = (q_0, q_1)$ and $R = (r_0, r_1)$. That is,

$$S = \bigcup_{i=0}^{2} f_i(S).$$

Note that, if these coefficients are taken as $p_0 = p_1 = q_1 = 0$, $q_0 = 1$, $r_0 = \frac{1}{2}$ and $r_1 = \frac{\sqrt{3}}{2}$, then the attractor of the iterated function system is the classical Sierpinski gasket (see Figure 1). In the case of $p_0 = p_1 = q_1 = r_0 = 0$, $q_0 = 1$ and $r_1 = 1$, the so-called right Sierpinski gasket is then obtained as the attractor of the iterated function system.



Figure 1. The classical Sierpinski gasket.

The intrinsic metric on the code set of the scalene Sierpinski gasket with the vertices $P = (p_0, p_1), Q = (q_0, q_1)$ and $R = (r_0, r_1)$ is formulated in [15] as follows:

Let $a_i = b_i$ for i = 1, 2, ..., k - 1 and $a_k \neq b_k$, where $a_i, b_i \in \{0, 1, 2\}$ for i = 1, 2, 3, ... If the code representations points *A* and *B* on this set are $a_1a_2...a_{k-1}a_ka_{k+1}...$ and $b_1b_2...b_{k-1}b_kb_{k+1}...$, respectively, then the intrinsic metric formula is defined as follows:

Theorem 1 ([15], Theorem 1). *Let* $a_k \neq c_k \neq b_k$ *and* $c_k \in \{0, 1, 2\}$ *and*

$$\kappa = \begin{cases} |PQ|, & (a_k = 0, b_k = 1) \text{ or } (a_k = 1, b_k = 0), \\ |PR|, & (a_k = 0, b_k = 2) \text{ or } (a_k = 2, b_k = 0), \\ |QR|, & (a_k = 1, b_k = 2) \text{ or } (a_k = 2, b_k = 1). \end{cases}$$

Then, the formula

$$d(A,B) = \min\left\{\sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \frac{\kappa}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i}\right\},\tag{2}$$

such that

$$\alpha_{i} = \begin{cases} 0, & a_{i} = b_{k}, \\ |PQ|, & (a_{i} = 0, b_{k} = 1) \text{ or } (a_{i} = 1, b_{k} = 0), \\ |PR|, & (a_{i} = 0, b_{k} = 2) \text{ or } (a_{i} = 2, b_{k} = 0), \\ |QR|, & (a_{i} = 1, b_{k} = 2) \text{ or } (a_{i} = 2, b_{k} = 1), \end{cases} \beta_{i} = \begin{cases} 0, & b_{i} = a_{k}, \\ |PQ|, & (b_{i} = 0, a_{k} = 1) \text{ or } (b_{i} = 1, a_{k} = 0), \\ |PR|, & (b_{i} = 0, a_{k} = 2) \text{ or } (b_{i} = 2, a_{k} = 0), \\ |QR|, & (b_{i} = 1, a_{k} = 2) \text{ or } (b_{i} = 2, a_{k} = 1), \end{cases}$$

$$\gamma_{i} = \begin{cases} 0, & a_{i} = c_{k}, \\ |PQ|, & (a_{i} = 0, c_{k} = 1) \text{ or } (a_{i} = 1, c_{k} = 0), \\ |PR|, & (a_{i} = 0, c_{k} = 2) \text{ or } (a_{i} = 2, c_{k} = 0), \\ |QR|, & (a_{i} = 1, c_{k} = 2) \text{ or } (a_{i} = 2, c_{k} = 1), \end{cases} \delta_{i} = \begin{cases} 0, & b_{i} = c_{k}, \\ |PQ|, & (b_{i} = 0, c_{k} = 1) \text{ or } (b_{i} = 1, c_{k} = 0), \\ |PR|, & (b_{i} = 0, c_{k} = 2) \text{ or } (b_{i} = 2, c_{k} = 0), \\ |QR|, & (b_{i} = 1, c_{k} = 2) \text{ or } (b_{i} = 2, c_{k} = 1), \end{cases}$$

gives the length of the shortest path between the points A and B on the scalene Sierpinski gasket.

Remark 1. The intrinsic metric formula on the right Sierpinski gasket is defined as follows:

$$d(A,B) = \min\left\{\sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \frac{\kappa}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i}\right\},\tag{3}$$

such that

$$\alpha_{i} = \begin{cases} 0, & a_{i} = b_{k}, \\ \sqrt{2}, & (a_{i} = 1, b_{k} = 2) \text{ or } (a_{i} = 2, b_{k} = 1), \\ 1, & otherwise, \end{cases} \beta_{i} = \begin{cases} 0, & b_{i} = a_{k}, \\ \sqrt{2}, & (b_{i} = 1, a_{k} = 2) \text{ or } (b_{i} = 2, a_{k} = 1), \\ 1, & otherwise, \end{cases}$$

$$\gamma_{i} = \begin{cases} 0, & a_{i} = c_{k}, \\ \sqrt{2}, & (a_{i} = 1, c_{k} = 2) \text{ or } (a_{i} = 2, c_{k} = 1), \\ 1, & otherwise, \end{cases} \delta_{i} = \begin{cases} 0, & b_{i} = c_{k}, \\ \sqrt{2}, & (b_{i} = 1, c_{k} = 2) \text{ or } (b_{i} = 2, c_{k} = 1), \\ 1, & otherwise, \end{cases}$$

and

$$\kappa = \begin{cases} \sqrt{2}, & (a_k = 1, b_k = 2) \text{ or } (a_k = 2, b_k = 1), \\ 1, & otherwise. \end{cases}$$

The intrinsic metric formula on the classical Sierpinski gasket is obtained by

$$d(A,B) = \min\left\{\sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i}\right\},$$
(4)

where

$$\alpha_{i} = \begin{cases} 0, & a_{i} = b_{k}, \\ 1, & a_{i} \neq b_{k}, \end{cases} \qquad \beta_{i} = \begin{cases} 0, & b_{i} = a_{k}, \\ 1, & b_{i} \neq a_{k}, \end{cases}$$
$$\gamma_{i} = \begin{cases} 0, & a_{i} \neq a_{k} \text{ and } a_{i} \neq b_{k}, \\ 1, & otherwise, \end{cases} \qquad \delta_{i} = \begin{cases} 0, & b_{i} \neq b_{k} \text{ and } b_{i} \neq a_{k}, \\ 1, & otherwise \end{cases}$$

(for details, see [12,15]).

3. The Code Representations of Points on the Sierpinski Propellers

Let us first consider identical classical Sierpinski gaskets whose colors are red, yellow, blue, black and purple. As seen in Figure 2, we combine *m* copies of the identical classical Sierpinski gaskets (for m = 2, 3, 4, 5, respectively) at one touching point *T*. Note that these shapes are similar to a propeller. Thus, we call these new structures as the Sierpinski propeller (briefly *SP*). Since the fractal dimension of the Sierpinski gasket is $\frac{\ln 3}{\ln 2}$, the fractal dimensions of the Sierpinski propellers are also $\frac{\ln 3}{\ln 2}$. Moreover, the areas of these sets are 0.



Figure 2. The classical Sierpinski propellers.

The Sierpinski gasket is a strictly self-similar. Although the Sierpinski propellers are self-similar, they are not strictly self-similar. It can easily be seen that *T* is a special point since every neighborhood of this point contains the scaled copies of the Sierpinski propellers (see Figure 3 and Proposition 4).



Figure 3. The scaled copies of the classical Sierpinski propeller for m = 3.

In this section, we only express the code representations of points on the classical Sierpinski propeller for m = 3 and we define the intrinsic metric on the code set of this fractal in Section 4. The intrinsic metric formulas and the code sets of the classical Sierpinski propellers for m = 2, 4, 5 can be similarly obtained. Note that the intrinsic metric formula can be expressed more simply for m = 3 when a different code representation is used (see Remark 4). However, the following method can be more convenient for the proofs in cases of m = 4 and m = 5 (the numbers of the code representations of the point *T* are 4 and 5, respectively).

Let us denote the red Sierpinski gasket, the yellow Sierpinski gasket and the blue Sierpinski gasket of the Sierpinski propeller by $SP_{\tilde{1}}$, $SP_{\tilde{1}}$ and $SP_{\tilde{2}}$, respectively. It is obvious that

$$SP_{\widetilde{0}} \cap SP_{\widetilde{1}} \cap SP_{\widetilde{2}} = T$$

and

$$SP_{\widetilde{0}} \cup SP_{\widetilde{1}} \cup SP_{\widetilde{2}} = SP.$$

Let us state the left-bottom part, the right-bottom part and the upper part of SP_{a_0} by SP_{a_00} , SP_{a_01} and SP_{a_02} , respectively, where $a_0 \in \{0, 1, 2\}$. The smaller triangular pieces of SP are denoted by $SP_{a_0a_1a_2...a_k}$, where $a_i \in \{0, 1, 2\}$ and i = 1, 2, ..., k. We thus obtain

$$SP_{a_0} \supset SP_{a_0a_1} \supset SP_{a_0a_1a_2} \supset SP_{a_0a_1a_2a_3} \supset \ldots \supset SP_{a_0a_1a_2\dots a_n} \supset \ldots$$

and for a sequence of nested sets SP_{a_0} , $SP_{a_0a_1}$, $SP_{a_0a_1a_2}$, $SP_{a_0a_1a_2a_3}$, ..., the Cantor intersection theorem states that the infinite intersection of these sets contains exactly one point:

$$\bigcap_{k=0}^{\infty} SP_{a_0a_1a_2\dots a_k} = \{A\}.$$

We denote the point $A \in SP$ by $a_0a_1a_2...a_n...$ where $a_0 \in \{0, 1, 2\}$ and $a_n \in \{0, 1, 2\}$ for n = 1, 2, ...

The following proposition shows that the points on the Sierpinski propeller have the code representations different from the code representations of points on the Sierpinski gasket:

Proposition 1. *The Sierpinski propeller has points whose numbers of the code representations are* 1, 2 or 3. *Moreover, the point T is the unique point whose number of the code representations is* 3.

Proof. *T* is the unique point which has three code representations such that 0111..., 1000... and 2222... since all of the set sequences

$$SP_{\tilde{0}}, SP_{\tilde{0}1}, SP_{\tilde{0}11}, SP_{\tilde{0}111}, \dots$$

 $SP_{\tilde{1}}, SP_{\tilde{1}0}, SP_{\tilde{1}00}, SP_{\tilde{1}000}, \dots$

and

$$SP_{\widetilde{2}}, SP_{\widetilde{2}2}, SP_{\widetilde{2}22}, SP_{\widetilde{2}222}, \ldots$$

contain the point *T*. Note that, for $A \neq T$, if *A* is the intersection point of any two sub-triangles in the same level of $S_{a_0a_1a_2...a_k}$, then *A* has two different representations such that $a_0a_1a_2...a_k\beta\alpha\alpha\alpha\alpha$... and $a_0a_1a_2...a_k\alpha\beta\beta\beta\beta\ldots$ where $\alpha,\beta\in\{0,1,2\}$. Otherwise, the code representation of *A* is unique. \Box

In Figure 4, it can be seen that the set of code representations of points on the red Sierpinski gasket is

$$SP_{\widetilde{0}} = \{\widetilde{0}a_1a_2a_3\dots \mid a_i \in \{0, 1, 2\}\},\$$

the set of code representations of points on the yellow Sierpinski gasket is

$$SP_{\tilde{1}} = \{\tilde{1}a_1a_2a_3\dots \mid a_i \in \{0, 1, 2\}\}$$

and the set of code representations of points on the blue Sierpinski gasket is

$$SP_{\widetilde{2}} = \{\widetilde{2}a_1a_2a_3\dots \mid a_i \in \{0, 1, 2\}\}.$$

Generally, the sub-triangles of S_{σ} for $\sigma = a_0 a_1 a_2 \dots a_{k-1}$ are expressed as

$$SP_{\sigma} = \{ \sigma a_k a_{k+1} a_{k+2} \dots \mid a_i \in \{0, 1, 2\}, \ i = k, k+1, k+2, \dots \}.$$

In addition, the code sets of the Sierpinski propellers are

 $\{\tilde{0}1\dots 1a_{k+1}\dots \mid a_i \in \{0,1,2\}\} \cup \{\tilde{1}0\dots 0a_{k+1}\dots \mid a_i \in \{0,1,2\}\} \cup \{\tilde{2}2\dots 2a_{k+1}\dots \mid a_i \in \{0,1,2\}\},\$

where i = k + 1, k + 2, k + 3... for k = 1, 2, 3, ...



Figure 4. Some code sets of the Sierpinski propeller.

4. The Construction of the Intrinsic Metric on the Code Set of SP

In the following theorem, we formulate the intrinsic metric on *SP* by using the code representations of its points.

Theorem 2. Suppose that the code representations of the points A and B on the Sierpinski propeller are $a_0a_1a_2...a_{k-1}a_ka_{k+1}...and b_0b_1b_2...b_{k-1}b_kb_{k+1}...$ respectively such that $a_i = b_i$ for i = 1, 2, ..., k-1 and $a_k \neq b_k$ where $a_0, b_0 \in \{\tilde{0}, \tilde{1}, \tilde{2}\}$ and $a_i, b_i \in \{0, 1, 2\}$ for i = 1, 2, 3, ...

(*i*) If $a_0 \neq b_0$, then the shortest distance between A and B is determined by

$$d(A,B) = \sum_{i=1}^{\infty} \frac{\widetilde{\alpha}_i + \widetilde{\beta}_i}{2^i},$$
(5)

such that

$$\tilde{\alpha}_{i} = \begin{cases} 0, & a_{i} = 1 \\ 1, & a_{i} \neq 1 \end{cases}, \quad \tilde{\beta}_{i} = \begin{cases} 0, & b_{i} = 0 \\ 1, & b_{i} \neq 0 \end{cases} \quad if (a_{0} = \tilde{0}, b_{0} = \tilde{1}),$$
(6)

$$\widetilde{\alpha}_{i} = \begin{cases} 0, & a_{i} = 1 \\ 1, & a_{i} \neq 1 \end{cases}, \quad \widetilde{\beta}_{i} = \begin{cases} 0, & b_{i} = 2 \\ 1, & b_{i} \neq 2 \end{cases} \quad if(a_{0} = \widetilde{0}, b_{0} = \widetilde{2}),$$
(7)

$$\widetilde{\alpha}_{i} = \begin{cases} 0, & a_{i} = 0 \\ 1, & a_{i} \neq 0 \end{cases}, \quad \widetilde{\beta}_{i} = \begin{cases} 0, & b_{i} = 2 \\ 1, & b_{i} \neq 2 \end{cases}, \quad if(a_{0} = \widetilde{1}, b_{0} = \widetilde{2}).$$
(8)

(ii) If $a_0 = b_0$, then Equation (4) gives the desired distance.

Proof. (i) Suppose that *A* and *B* are two different points of the Sierpinski propeller with the code representations $a_0a_1a_2...a_n...$ and $b_0b_1b_2...b_n...$, respectively, where $a_0 \neq b_0$. It is clear that the shortest path between *A* and *B* must pass through the point *T* since $SP_{a_0} \cap SP_{b_0} = \{T\}$.

Firstly, let us consider the case of $a_0 = \tilde{0}$ and $b_0 = \tilde{1}$ (the other cases are done similarly). If $a_1 = 0$ or $a_1 = 2$, then the shortest path between A and T must pass through the points $SP_{\tilde{0}0} \cap SP_{\tilde{0}1}$ or $SP_{\tilde{0}1} \cap SP_{\tilde{0}2}$, respectively. These points have the code representations $\tilde{0}1000 \dots$ (equivalently $\tilde{0}0111 \dots$) or $\tilde{0}1222 \dots$ (equivalently $\tilde{0}2111 \dots$), respectively. Furthermore, T and $SP_{\tilde{0}0} \cap SP_{\tilde{0}1}$ (similarly T and $SP_{\tilde{0}1} \cap SP_{\tilde{0}2}$) are the points which are the vertices in the sub-triangle $SP_{\tilde{0}1}$ of the Sierpinski gasket. Hence, the length of the shortest path between these points is $\frac{1}{2}$. We compute the length of the shortest path between A and T as $\frac{1}{2}$ if the code representations of A are $\tilde{0}1000 \dots$ (equivalently $\tilde{0}0111 \dots$) or $\tilde{0}1222 \dots$ (equivalently $\tilde{0}2111 \dots$). If $a_1 = 1$ and the code representations of A are not $\tilde{0}1000 \dots$ and $\tilde{0}1222 \dots$, then the length of the shortest path between A and T is less than $\frac{1}{2}$. Consequently, in (5), we get $\tilde{\alpha}_1 = 1$ if $a_1 = 0$ or $a_1 = 2$ and we get $\tilde{\alpha}_1 = 0$ if $a_1 = 1$.

Now, consider $a_1 = 0$ and let us take $a_2 = 0$ or $a_2 = 2$ (see Figure 5). Then, the shortest path between *A* and $SP_{\tilde{0}0} \cap SP_{\tilde{0}1}$ must pass through $SP_{\tilde{0}00} \cap SP_{\tilde{0}01}$ or $SP_{\tilde{0}01} \cap SP_{\tilde{0}02}$, respectively. Note that these intersections have the code representations 001000... (equivalently 000111...) or 001222...(equivalently 002111...), respectively. In addition, 00111... and $SP_{\tilde{0}00} \cap SP_{\tilde{0}01}$ (similarly 00111...) and $SP_{\tilde{0}01} \cap SP_{\tilde{0}02}$) are the points that are the vertices in the sub-triangle $SP_{\tilde{0}01}$ of the Sierpinski gasket. Thus, the length of the shortest path between these points is $\frac{1}{2^2}$. We compute the length of the shortest path between *A* and *T* as $\frac{1}{2} + \frac{1}{2^2}$ if the code representations of *A* are 000111... or 002111... If $a_2 = 1$ and the code representations of *A* are not 001000... and 001222..., then the length of the shortest path between *A* and *T* is less than $\frac{1}{2} + \frac{1}{2^2}$. Consequently, in (5), we get $\tilde{a}_2 = 1$ if $a_2 = 0$ or $a_2 = 2$ and we get $\tilde{a}_2 = 0$ if $a_2 = 1$. Generally, for i = 1, 2, 3, ..., we obtain $\tilde{a}_i = 1$ if $a_i = 0$ or $a_i = 2$ and we get $\tilde{a}_i = 0$ if $a_i = 1$. A similar procedure is also valid for the computation of the length of the shortest path between the points *T* and *B*. \Box



Figure 5. One of the shortest paths between *A* and *B* on the Sierpinski propeller for $a_0 = \tilde{0}$ and $b_0 = \tilde{1}$.

Remark 2. The point T has three code representations such as 0111..., 1000... and 2222... Due to Formula (5), it is easily seen that the distance (which is computed by using different codes of this point) is 0. Thus, using a similar method given in [12], the following proposition can be proven.

Proposition 2. *The metric d defined in Theorem 2 does not depend on the choice of the code representations of the points.*

Example 1. Assume that A and B are the points of SP whose code representations are 1010202... and 2010101..., respectively. For the computation of d(A, B), we must use Formula (8) owing to the fact that the first terms of the code representations of A and B are 1 and 2, respectively. Thus, we get

$$\tilde{\alpha}_i = 0$$
 for $i = 2, 4, 6, \dots \tilde{\alpha}_i = 1$, for $i = 1, 3, 5, \dots$

and

$$\beta_i = 1$$
 for $i = 1, 2, 3, \dots$

It follows that

$$d(A,B) = \sum_{i=1}^{\infty} \frac{\widetilde{\alpha}_i + \widetilde{\beta}_i}{2^i} = \frac{0+1}{2} + \frac{1+1}{2^2} + \frac{0+1}{2^3} + \frac{1+1}{2^4} + \cdots$$
$$= \frac{1}{2} + \frac{2}{2^2} + \frac{1}{2^3} + \frac{2}{2^4} + \cdots$$
$$= \frac{4}{3}.$$

Proposition 3. If *d* is the intrinsic metric defined in (5) on SP, then

$$diam(SP) = \max\{d(A, B) \mid A, B \in SP\} = 2.$$
(9)

Moreover, d(A, B) = 2 if and only if the code representations of A are an element of the set $\{\tilde{0}a_1a_2a_3... | a_i \in \{0,2\}\}$ and the code representations of B are an element of the sets $\{\tilde{1}b_1b_2b_3... | b_i \in \{1,2\}\}$ or $\{\tilde{2}b_1b_2b_3... | b_i \in \{0,1\}\}$, or if the code representations of A are an element of the set $\{\tilde{1}a_1a_2a_3... | a_i \in \{0,1\}\}$, or if the code representations of A are an element of the set $\{\tilde{1}a_1a_2a_3... | a_i \in \{0,1\}\}$, or if the code representations of A are an element of the set $\{\tilde{1}a_1a_2a_3... | a_i \in \{0,1\}\}$.

 $\{1,2\}$ and the code representations of B are an element of the sets $\{\tilde{0}b_1b_2b_3... | b_i \in \{0,2\}\}$ or $\{\tilde{2}b_1b_2b_3... | b_i \in \{0,1\}\}$.

Proof. For the computation of the maximum value of the distances between any two points *A* and *B* of *SP*, \tilde{a}_i and $\tilde{\beta}_i$ must be 1 for i = 1, 2, 3, ... (see Formula (5)). Thus, if $a_0 = \tilde{0}$ and $b_0 = \tilde{1}$, then $a_i \neq 1$ and $b_i \neq 0$ or if $a_0 = \tilde{0}$ and $b_0 = \tilde{2}$, then $a_i \neq 1$ and $b_i \neq 2$ for i = 1, 2, 3, ... (see Formulas (6) and (7)). In addition, if $a_0 = \tilde{1}$ and $b_0 = \tilde{2}$, then $a_i \neq 0$ and $b_i \neq 2$ (see Formula (8)). In these cases, we obtain

$$d(A,B) = \sum_{i=1}^{\infty} \frac{\widetilde{\alpha}_i + \beta_i}{2^i} = \frac{1+1}{2} + \frac{1+1}{2^2} + \frac{1+1}{2^3} + \frac{1+1}{2^4} + \cdots$$
$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$
$$= 2.$$

Proposition 4. For n = 1, 2, 3, ..., the closed discs of radii $\frac{1}{2^n}$ with center T are the scaled copy of Sierpinski propeller. Especially, if n = 0, then D(T, 1) = SP. In addition, the code sets of circles of radii $\frac{1}{2^n}$ with center T are

 $\{\tilde{0}1\dots 1a_{n+1}\dots \mid a_i \in \{0,2\}\} \cup \{\tilde{1}0\dots 0a_{n+1}\dots \mid a_i \in \{1,2\}\} \cup \{\tilde{2}2\dots 2a_{n+1}\dots \mid a_i \in \{0,1\}\}.$

Proof. We first compute the code representations of the points *A* satisfying $d(A, T) = \frac{1}{2^n}$. Without loss of generality, we take $\tilde{2}222...$ as the code representation of *T*. Let the code representation of *A* be $a_0a_1a_2...$ where $a_0 \in \{\tilde{0}, \tilde{1}, \tilde{2}\}$ and $a_i \in \{0, 1, 2\}$ for i = 1, 2, 3, ...

Case 1: Suppose that $a_0 = 0$. Since the terms of code representation of *T* (for i=1,2,3,...) are 2, we get $\tilde{\beta}_i = 0$ for i = 1, 2, 3, ... In addition, a_i must be 1 for $i \le n - 1$ (see Formula (7)). Otherwise, we compute $d(A, T) \ge \frac{1}{2^{n-1}}$. Now, consider that a_i are 1 for all i = 1, 2, 3, ..., n. This means that $\tilde{\alpha}_i = 0$ for i = 1, 2, 3, ..., n. If $a_i \ne 1$ for i = n + 1, n + 2, n + 3, ..., then we obtain $\tilde{\alpha}_i = 1$. Therefore, we compute

$$d(A,T) = \sum_{i=n+1}^{\infty} \frac{1+0}{2^i} = \frac{1}{2^n}$$

and this shows that $\{\tilde{0}1...1a_{n+1}a_{n+2}... \mid a_i \in \{0,2\}\}$ is one of the parts of the set of the code representations of *A* satisfying $d(A, T) = \frac{1}{2^n}$ for $a_0 = \tilde{0}$.

From the construction above, it can be easily seen that, if $d(A, T) \leq \frac{1}{2^n}$ for $a_0 = \tilde{0}$, then $\{\tilde{0}1...1a_{n+1}a_{n+2}... \mid a_i \in \{0, 1, 2\}\}$ is one of the parts of the set of the code representations of *A*.

Case 2: Let $a_0 = \tilde{1}$. By using Formula (8) and following a method similar to Case 1, we obtain that $\{\tilde{1}0...0a_{n+1}a_{n+2}... | a_i \in \{1,2\}\}$ is one of the parts of the set of the code representations of *A* satisfying $d(A,T) = \frac{1}{2^n}$. Moreover, we compute if $d(A,T) \leq \frac{1}{2^n}$ for $a_0 = \tilde{1}$, then $\{\tilde{1}0...0a_{n+1}a_{n+2}... | a_i \in \{0,1,2\}\}$ is one of the parts of the set of the code representations of *A*.

Case 3: Assume that $a_0 = \tilde{2}$. Then, the points *A* and *T* are in the same Sierpinski gasket. Applying Formula (4), we obtain that $\{\tilde{2}2...2a_{n+1}a_{n+2}... | a_i \in \{0,1\}\}$ and $\{\tilde{2}2...2a_{n+1}a_{n+2}... | a_i \in \{0,1,2\}\}$ are one of the parts of the set of the code representations of *A* satisfying $d(A,T) = \frac{1}{2^n}$ and $d(A,T) \leq \frac{1}{2^n}$, respectively.

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Thus, the unions of sets obtained in Cases 1, 2 and 3 give us the code representations of *A* satisfying $d(A,T) = \frac{1}{2^n}$ and $d(A,T) \le \frac{1}{2^n}$. That is,

$$S(A,T) = \{ \widetilde{0}1 \dots 1a_{n+1}a_{n+2} \dots \mid a_i \in \{0,2\} \} \cup \{ \widetilde{1}0 \dots 0a_{n+1}a_{n+2} \dots \mid a_i \in \{1,2\} \} \\ \cup \{ \widetilde{2}2 \dots 2a_{n+1}a_{n+2} \dots \mid a_i \in \{0,1\} \}$$

and

$$D(A,T) = \{01\dots 1a_{n+1}a_{n+2}\dots \mid a_i \in \{0,1,2\}\} \cup \{10\dots 0a_{n+1}a_{n+2}\dots \mid a_i \in \{0,1,2\}\} \cup \{\widetilde{2}2\dots 2a_{n+1}a_{n+2}\dots \mid a_i \in \{0,1,2\}\}.$$

Example 2. Consider the point A of SP which has the code representation $\tilde{0}000...$ Obviously, the code set of the closed disc of radius $\frac{1}{2}$ with center A is

$$D(\widetilde{0}000\ldots,\frac{1}{2}) = \{\widetilde{0}0a_2a_3a_4\ldots \mid a_i \in \{0,1,2\}\}.$$

Remark 3. Proposition 4 and Example 2 show that SP is a self-similar set but not strictly self-similar.

Remark 4. In the Sierpinski propeller for m = 3, if the red Sierpinski gasket is coded by $SP_{\tilde{1}}$ and the yellow Sierpinski gasket is coded by $SP_{\tilde{0}}$, then the code representations of the point T are determined by $\tilde{w}www...$, where $w \in \{0, 1, 2\}$. Thus, the intrinsic metric formula can be expressed by

$$d(A,B) = \sum_{i=1}^{\infty} \frac{\widetilde{\alpha}_i + \widetilde{\beta}_i}{2^i}$$
(10)

such that

$$\widetilde{\alpha}_{i} = \begin{cases} 0, & a_{i} = a_{0}, \\ 1, & a_{i} \neq a_{0}, \end{cases}, \quad \widetilde{\beta}_{i} = \begin{cases} 0, & b_{i} = b_{0}, \\ 1, & b_{i} \neq b_{0}, \end{cases}$$
(11)

for $a_0 \neq b_0$. Moreover, the code sets of circles of radii $\frac{1}{2^n}$ with center T can be determined by

$$\{\widetilde{a_0}a_0a_0\dots a_0a_{n+1}\dots \mid a_i \neq a_0, \ i \ge n+1\}$$

5. The Intrinsic Metric Formula on Two Adjacent Right Sierpinski Gaskets

By using two right Sierpinski gaskets with vertices *P*, *Q* and *R* where |PQ| = |PR| = 1 and $|QR| = \sqrt{2}$, we now define a second fractal which is self-similar but not strictly self-similar. Consider two identical right Sierpinski gakets whose colors of sub-triangles are red, blue and yellow, and then combine these two triangles at the point *Q* as in Figure 6. We call this structure adjacent right Sierpinski gaskets, briefly *AS*.

Let us denote the triangle that is left of the point Q by $AS_{\tilde{0}}$ and the triangle that is the right of the point Q by $AS_{\tilde{1}}$. In addition, let us denote the red Sierpinski gaskets by $AS_{\tilde{0}0}$ and $AS_{\tilde{1}0}$, respectively, the blue Sierpinski gaskets by $AS_{\tilde{0}1}$ and $AS_{\tilde{1}1}$, respectively, and the yellow Sierpinski gaskets by $AS_{\tilde{0}2}$ and $AS_{\tilde{1}2}$, respectively. It is clear that

 $AS_{\widetilde{0}1} \cap AS_{\widetilde{1}1} = Q$

and

$$AS_{\widetilde{0}} \cup AS_{\widetilde{1}} = AS.$$

Thus, two different code representations of Q are 0111... and 1111... It is also easily seen that a point of AS has either one code representation or two code representations.



Figure 6. Some code sets and code representations of *AS*.

In the following proposition, we express the intrinsic metric on the code set of *AS*:

Proposition 5. Suppose that the code representations of the points A and B on AS are $a_0a_1a_2...a_{k-1}a_ka_{k+1}...$ and $b_0b_1b_2...b_{k-1}b_kb_{k+1}...$ respectively such that $a_i = b_i$ for i = 1, 2, ..., k - 1 and $a_k \neq b_k$, where $a_0, b_0 \in \{\tilde{0}, \tilde{1}\}$ and $a_i, b_i \in \{0, 1, 2\}$ for i = 1, 2, 3, ...

(i) If $a_0 \neq b_0$, then the shortest distance between A and B is determined by

$$d(A,B) = \sum_{i=1}^{\infty} \frac{\widetilde{\alpha}_i + \widetilde{\beta}_i}{2^i},$$
(12)

such that

$$\widetilde{\alpha}_{i} = \begin{cases} 0, & a_{i} = 1, \\ 1, & a_{i} = 0, \\ \sqrt{2}, & a_{i} = 2, \end{cases} \qquad \widetilde{\beta}_{i} = \begin{cases} 0, & b_{i} = 1, \\ 1, & b_{i} = 0, \\ \sqrt{2}, & b_{i} = 2. \end{cases}$$
(13)

(ii) If $a_0 = b_0$, then Equation (3) gives the desired distance.

Proof. The proof is omitted since it is similar to the proof of Theorem 2. \Box

Example 3. Let the code representation of A be 0020202... and let the code representation of B be 1101010...We must use Formula (12) to compute d(A, B) since the first terms of the code representations of A and B are different. Then, we obtain

$$\widetilde{\alpha}_i = 1$$
, for $i = 1, 3, 5, ...$ $\widetilde{\alpha}_i = \sqrt{2}$ for $i = 2, 4, 6, ...$

and

$$\widetilde{\beta}_i = 0 \text{ for } i = 1, 3, 5, \dots \quad \widetilde{\beta}_i = 1 \text{ for } i = 2, 4, 6, \dots$$

Thus, we get

$$d(A,B) = \sum_{i=1}^{\infty} \frac{\widetilde{\alpha}_i + \widetilde{\beta}_i}{2^i} = \frac{1+0}{2} + \frac{\sqrt{2}+1}{2^2} + \frac{1+0}{2^3} + \frac{\sqrt{2}+1}{2^4} + \cdots$$
$$= \left(\frac{1}{2} + \frac{2}{2^3} + \frac{1}{2^5} + \cdots\right) + (\sqrt{2}+1)\left(\frac{1}{2^2} + \frac{2}{2^4} + \frac{1}{2^6} + \cdots\right)$$
$$= 1 + \frac{\sqrt{2}}{3}.$$

Proposition 6. If *d* is the intrinsic metric defined in (12) on AS, then

$$diam(AS) = \max\{d(A, B) \mid A, B \in AS\} = 2\sqrt{2}.$$
(14)

Moreover, the code representations of A and B satisfying $d(A, B) = 2\sqrt{2}$ are $\tilde{0}222...$ and $\tilde{1}222...$, respectively.

Proof. As seen in Formula (12), to obtain the maximum value of the distances between any two points *A* and *B* of *AS*, $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ must be $\sqrt{2}$ for i = 1, 2, 3, ... Then, we get $\tilde{a}_0 = 0$, $\tilde{b}_0 = 1$, $a_i = 2$ and $b_i = 2$ for i = 1, 2, 3, ... We also have

$$d(A,B) = \sum_{i=1}^{\infty} \frac{\widetilde{\alpha}_i + \widetilde{\beta}_i}{2^i} = \frac{\sqrt{2} + \sqrt{2}}{2} + \frac{\sqrt{2} + \sqrt{2}}{2^2} + \frac{\sqrt{2} + \sqrt{2}}{2^3} + \frac{\sqrt{2} + \sqrt{2}}{2^4} + \cdots$$
$$= \sqrt{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2^2} + \frac{\sqrt{2}}{2^3} + \cdots$$
$$= 2\sqrt{2}.$$

6. Conclusions

In this paper, we express the intrinsic metrics on two new self-similar (but not strictly self-similar) sets created with different combinations of the Sierpinski gaskets. With a similar way, various self-similar sets can be defined by using the different fractals and the intrinsic metrics on these sets can be formulated.

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