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The Fractional Derivative of the Dirac Delta Function and Additional Results on the Inverse Laplace Transform of Irrational Functions

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Abstract: Motivated from studies on anomalous relaxation and diffusion, we show that the memory function $M(t)$ of complex materials, that their creep compliance follows a power law, $J(t) \sim t^q$ with $q \in \mathbb{R}^+$, is proportional to the fractional derivative of the Dirac delta function, $\frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q}$ with $q \in \mathbb{R}^+$. This leads to the finding that the inverse Laplace transform of s^q for any $q \in \mathbb{R}^+$ is the fractional derivative of the Dirac delta function, $\frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q}$. This result, in association with the convolution theorem, makes possible the calculation of the inverse Laplace transform of $\frac{s^q}{s^\alpha \mp \lambda}$ where $\alpha < q \in \mathbb{R}^+$, which is the fractional derivative of order q of the Rabotnov function $\varepsilon_{\alpha-1}(\pm\lambda, t) = t^{\alpha-1}E_{\alpha,\alpha}(\pm\lambda t^\alpha)$. The fractional derivative of order $q \in \mathbb{R}^+$ of the Rabotnov function, $\varepsilon_{\alpha-1}(\pm\lambda, t)$ produces singularities that are extracted with a finite number of fractional derivatives of the Dirac delta function depending on the strength of q in association with the recurrence formula of the two-parameter Mittag–Leffler function.

Keywords: generalized functions; laplace transform; anomalous relaxation; diffusion; fractional calculus; mittag–leffler function

MSC: 26A33; 30G99; 44A10; 33E12; 60K50

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1. Introduction

The classical result for the inverse Laplace transform of the function $\mathcal{F}(s) = \frac{1}{s^q}$ is [1]

$$\mathcal{L}^{-1}\left\{\frac{1}{s^q}\right\} = \frac{1}{\Gamma(q)}t^{q-1} \text{ with } q > 0 \tag{1}$$

In Equation (1) the condition $q > 0$ is needed because when $q = 0$, the ratio $\frac{1}{\Gamma(0)} = 0$ and the right-hand side of Equation (1) vanishes, except when $t = 0$, which leads to a singularity. Nevertheless, within the context of generalized functions, when $q = 0$, the right-hand side of Equation (1) becomes the Dirac delta function [2] according to the Gel'fand and Shilov [3] definition of the n^{th} $(n \in \mathbb{N}_0)$ derivative of the Dirac delta function

$$\frac{\mathrm{d}^n\delta(t-0)}{\mathrm{d}t^n} = \frac{1}{\Gamma(-n)}\frac{1}{t^{n+1}} = \Phi_{-n}(t) \text{ with } n \in \{0, 1, 2\ldots\} \tag{2}$$

with a proper interpretation of the quotient $\frac{1}{t^{n+1}}$ as a limit at $t = 0$. Thus, according to the Gel'fand and Shilov [3] definition expressed by Equation (2), Equation (1) can be extended for values of $q \in \{0, -1, -2, -3\ldots\}$, and in this way one can establish the following expression for the inverse Laplace transform of s^n with $n \in \mathbb{N}_0$:

$$\mathcal{L}^{-1}\{s^n\} = \frac{1}{\Gamma(-n)}\frac{1}{t^{n+1}} = \frac{\mathrm{d}^n\delta(t-0)}{\mathrm{d}t^n} \quad n \in \{0, 1, 2\ldots\} \tag{3}$$

For instance when $n = 1$, Equation (3) yields

$$\mathcal{L}^{-1}\{s\} = \frac{1}{\Gamma(-1)}\frac{1}{t^2} = \frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t} \tag{4}$$

which is the correct result, since the Laplace transform of $\frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t}$ is

$$\mathcal{L}\left\{\frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t}\right\} = \int_{0^-}^{\infty}\frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t}e^{-st}\mathrm{d}t = -\frac{\mathrm{d}(e^{-st})}{\mathrm{d}t}\bigg|_{t=0} = -(-s) = s \tag{5}$$

Equation (5) is derived by making use of the property of the Dirac delta function and its higher-order derivatives

$$\int_{0^-}^{\infty}\frac{\mathrm{d}^n\delta(t-0)}{\mathrm{d}t^n}f(t)\,\mathrm{d}t = (-1)^n\frac{\mathrm{d}^n f(0)}{\mathrm{d}t^n} \quad \text{with} \quad n \in \{0,1,2\ldots\} \tag{6}$$

In Equations (5) and (6), the lower limit of integration, 0^- is a shorthand notation for $\lim\limits_{\varepsilon\to 0^+}\int_{-\varepsilon}^{\infty}$, and it emphasizes that the entire singular function $\frac{\mathrm{d}^n\delta(t-0)}{\mathrm{d}t^n}$ $(n \in \mathbb{N}_0)$ is captured by the integral operator. In this paper we first show that Equation (3) can be further extended for the case where the Laplace variable is raised to any positive real power; s^q with $q \in \mathbb{R}^+$. This generalization, in association with the convolution theorem, allows for the derivation of some new results on the inverse Laplace transform of irrational functions that appear in problems with fractional relaxation and fractional diffusion [4–11]. This work complements recent progress on the numerical and approximate Laplace-transform solutions of fractional diffusion equations [12–15].

Most materials are viscoelastic; they both dissipate and store energy in a way that depends on the frequency of loading. Their resistance to an imposed time-dependent shear deformation, $\gamma(t)$, is parametrized by the complex dynamic modulus $\mathcal{G}(\omega) = \frac{\tau(\omega)}{\gamma(\omega)}$ where $\tau(\omega) = \int_{-\infty}^{\infty}\tau(t)e^{-\mathrm{i}\omega t}\mathrm{d}t$ and $\gamma(\omega) = \int_{-\infty}^{\infty}\gamma(t)e^{-\mathrm{i}\omega t}\mathrm{d}t$ are the Fourier transforms of the output stress, $\tau(t)$, and the input strain, $\gamma(t)$, histories. The output stress history, $\tau(t)$, can be computed in the time domain with the convolution integral

$$\tau(t) = \int_{0^-}^{t} M(t-\xi)\gamma(\xi)\mathrm{d}\xi \tag{7}$$

where $M(t-\xi)$ is the memory function of the material [16–18] defined as the resulting stress at time t due to an impulsive strain input at time $\xi(\xi < t)$, and it is the inverse Fourier transform of the complex dynamic modulus

$$M(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{G}(\omega)e^{\mathrm{i}\omega t}\mathrm{d}\omega \tag{8}$$

2. The Fractional Derivative of the Dirac Delta Function

Early studies on the behavior of viscoelastic materials, that their time-response functions follow power laws, have been presented by Nutting [4], who noticed that the stress response of several fluid-like materials to a step strain decays following a power law, $\tau(t) \sim t^{-q}$ with $0 \leq q \leq 1$. Following Nutting's observation and the early work of Gemant [5,6] on fractional differentials, Scott Blair [19,20] pioneered the introduction of fractional calculus in viscoelasticity. With analogy to the Hookean spring, in which the stress is proportional to the zero-th derivative of the strain and the Newtonian dashpot, in which the stress is proportional to the first derivative of the strain, Scott Blair and his coworkers [19–21] proposed the springpot element—that is a mechanical element in-between a spring and a dashpot with constitutive law

$$\tau(t) = \mu_q \frac{\mathrm{d}^q \gamma(t)}{\mathrm{d}t^q} \tag{9}$$

where q is a positive real number, $0 \leq q \leq 1$, μ_q is a phenomenological material parameter with units $[\mathrm{M}][\mathrm{L}]^{-1}[\mathrm{T}]^{q-2}$ (say $Pa{\cdot}sec^q$), and $\frac{\mathrm{d}^q \gamma(t)}{\mathrm{d}t^q}$ is the fractional derivative of order q of the strain history, $\gamma(t)$.

A definition of the fractional derivative of order q is given through the convolution integral

$$_cI^q\gamma(t) = \frac{1}{\Gamma(q)} \int_c^t (t-\xi)^{q-1}\gamma(\xi)\mathrm{d}\xi \tag{10}$$

where $\Gamma(q)$ is the Gamma function. When the lower limit, $c = 0$, the integral given by Equation (10) is often referred to as the Riemann–Liouville fractional integral $_0I^q\gamma(t)$ [22–25]. The integral in Equation (10) converges only for $q > 0$, or in the case where q is a complex number, the integral converges for $\mathcal{R}(q) > 0$. Nevertheless, by a proper analytic continuation across the line $\mathcal{R}(q) = 0$, and provided that the function $\gamma(t)$ is n times differentiable, it can be shown that the integral given by Equation (10) exists for $n - \mathbb{R}(q) > 0$ [26]. In this case the fractional derivative of order $q \in \mathbb{R}^+$ exists and is defined as

$$\frac{\mathrm{d}^q \gamma(t)}{\mathrm{d}t^q} = {_0D^q}\gamma(t) = \Phi_{-q}(t) * \gamma(t) = \frac{1}{\Gamma(-q)} \int_{0^-}^t \frac{\gamma(\xi)}{(t-\xi)^{q+1}}\mathrm{d}\xi,\ q \in \mathbb{R}^+ \tag{11}$$

where $\mathbb{R}^+$ is the set of positive real numbers, and $\Phi_{-q}(t) = \frac{1}{\Gamma(-q)}\frac{1}{t^{q+1}}$ is a generalization of the Gel'fand and Shilov kernel given by Equation (2) for $q \in \mathbb{R}^+$. Gorenflo and Mainardi [27] and subsequently Mainardi [28] concluded that within the context of generalized functions, Equation (11) is indeed a formal definition of the fractional derivative of order $q \in \mathbb{R}^+$ of a sufficiently differentiable function by making use of the property of the Dirac delta function and its higher-order derivatives given by Equation (6) in association with the 1964 Gel'fand and Shilov [3] definition of the Dirac delta function and its higher-order derivatives given by Equation (2). Accordingly, the nth-order derivative, $n \in \{0, 1, 2, \ldots\}$, of a sufficiently differentiable function $\gamma(t)$ is the convolution of $\gamma(t)$ with $\Phi_{-n}(t)$ defined by Equation (2)

$$\frac{\mathrm{d}^n \gamma(t)}{\mathrm{d}t^n} = \Phi_{-n}(t) * \gamma(t) = \int_{0^-}^t \frac{\mathrm{d}^n \delta(t-\xi)}{\mathrm{d}t^n}\gamma(\xi)\,\mathrm{d}\xi = \frac{1}{\Gamma(-n)} \int_{0^-}^t \frac{\gamma(\xi)}{(t-\xi)^{n+1}}\,\mathrm{d}\xi, \quad t > 0 \tag{12}$$

By replacing $n \in \{0, 1, 2, \ldots\}$ with $q \in \mathbb{R}^+$ in Equation (12), Gorenflo and Mainardi [27] and Mainardi [28] explain that Equation (11) is a formal definition of the fractional derivative of order $q \in \mathbb{R}^+$ and should be understood as the convolution of the function $\gamma(t)$ with the kernel $\Phi_{-q}(t) = \frac{1}{\Gamma(-q)}\frac{1}{t^{q+1}}$ as indicated in Equation (11) [22–24,27,28]. The formal character of Equation (11) is evident in that the kernel $\Phi_{-q}(t)$ is not locally absolutely integrable in the classical sense; therefore, the integral appearing in Equation (11) is in general divergent. Nevertheless, when dealing with classical functions (not just generalized functions) the integral can be regularized as shown in [27,28].

The Fourier transform of the fractional derivative of a function defined by Equation (11) is

$$\mathcal{F}\left\{\frac{\mathrm{d}^q \gamma(t)}{\mathrm{d}t^q}\right\} = \int_{-\infty}^{\infty} \frac{\mathrm{d}^q \gamma(t)}{\mathrm{d}t^q} e^{-\mathrm{i}\omega t}\mathrm{d}t = \int_0^{\infty} \frac{\mathrm{d}^q \gamma(t)}{\mathrm{d}t^q} e^{-\mathrm{i}\omega t}\mathrm{d}t = (\mathrm{i}\omega)^q \gamma(\omega) \tag{13}$$

where $\mathcal{F}$ indicates the Fourier transform operator [1,24,28]. The one-sided integral appearing in Equation (13) that results from the causality of the strain history, $\gamma(t)$ is also the Laplace transform of the fractional derivative of the strain history, $\gamma(t)$

$$\mathcal{L}\left\{\frac{\mathrm{d}^q\gamma(t)}{\mathrm{d}t^q}\right\} = \int_0^\infty \frac{\mathrm{d}^q\gamma(t)}{\mathrm{d}t^q}e^{-st}\mathrm{d}t = s^q\gamma(s) \tag{14}$$

where $s = \mathrm{i}\omega$ is the Laplace variable, and $\mathcal{L}$ indicates the Laplace transform operator [28,29].

For the elastic Hookean spring with elastic modulus, G, its memory function as defined by Equation (8) is $M(t) = G\delta(t-0)$, which is the zero-order derivative of the Dirac delta function; whereas, for the Newtonian dashpot with viscosity, η, its memory function is $M(t) = \eta\,\frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t}$, which is the first-order derivative of the Dirac delta function [16]. Since the springpot element defined by Equation (9) with $0 \le q \le 1$ is a constitutive model that is in-between the Hookean spring and the Newtonian dashpot, physical continuity suggests that the memory function of the springpot model given by Equation (9) shall be of the form of $M(t) = \mu_q\,\frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q}$, which is the fractional derivative of order q of the Dirac delta function [22,25].

The fractional derivative of the Dirac delta function emerges directly from the property of the Dirac delta function [2]

$$\int_{-\infty}^{\infty} \delta(t-\xi)f(t)\mathrm{d}t = f(\xi) \tag{15}$$

By following the Riemann–Liouville definition of the fractional derivative of a function given by the convolution appearing in Equation (11), the fractional derivative of order $q \in \mathbb{R}^+$ of the Dirac delta function is

$$\frac{\mathrm{d}^q\delta(t-\xi)}{\mathrm{d}t^q} = \frac{1}{\Gamma(-q)}\int_{0^-}^{t}\frac{\delta(\tau-\xi)}{(t-\tau)^{1+q}}\mathrm{d}\tau,\ q\in\mathbb{R}^+ \tag{16}$$

and by applying the property of the Dirac delta function given by Equation (15); Equation (16) gives

$$\frac{\mathrm{d}^q\delta(t-\xi)}{\mathrm{d}t^q} = \frac{1}{\Gamma(-q)}\frac{1}{(t-\xi)^{1+q}},\ q\in\mathbb{R}^+ \tag{17}$$

The result of Equation (17) has been presented in [22,25] and has been recently used to study problems in anomalous diffusion [11] and anomalous relaxation [30]. Equation (17) offers the remarkable result that the fractional derivative of the Dirac delta function of any order $q \in \{\mathbb{R}^+ - \mathbb{N}\}$ is finite everywhere other than at $t = \xi$; whereas, the Dirac delta function and its integer-order derivatives are infinite-valued, singular functions that are understood as a monopole, dipole and so on; and we can only interpret them through their mathematical properties as the one given by Equations (6) and (15). Figure 1 plots the fractional derivative of the Dirac delta function at $\xi = 0$

$$\frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q} = \frac{1}{\Gamma(-q)}\frac{1}{t^{1+q}} = \Phi_{-q}(t) \quad \text{with } q\in\mathbb{R}^+,\ t>0 \tag{18}$$

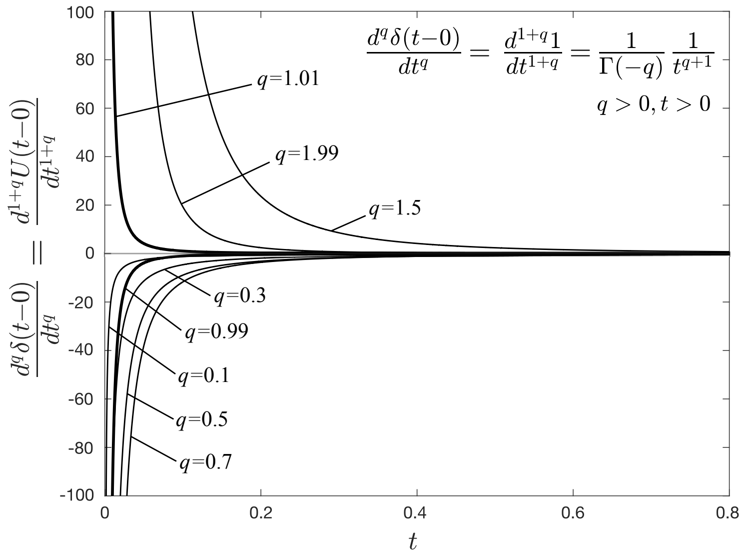

Figure 1. Plots of the fractional derivative of the Dirac delta function of order $q \in \{\mathbb{R}^+ - \mathbb{N}\}$, which are the $1+q$ order derivative of the constant 1 for positive times. The functions are finite everywhere other than the time origin, $t = 0$. Figure 1 shows that the fractional derivatives of the singular Dirac delta function, and these of the constant unit at positive times are expressed with the same family of functions.

The result of Equation (18) for $q \in \mathbb{R}^+$ is identical to the Gel'fand and Shilov [3] definition of the n^{th} $(n \in \mathbb{N}_0)$ derivative of the Dirac delta function given by Equation (2), where $\mathbb{N}_0$ is the set of positive integers including zero and shows that the fractional derivative of the Dirac delta function $\frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q}$ with $q \in \mathbb{R}^+$ is merely the kernel $\Phi_{-q}(t)$ in the formal definition of the fractional derivative given by Equation (11). Accordingly, in analogy with Equation (12) the fractional derivative of order $q \in \mathbb{R}^+$ of a sufficiently differentiable function is

$$\frac{\mathrm{d}^q\gamma(t)}{\mathrm{d}t^q} = \Phi_{-q}(t) * \gamma(t) = \int_{0^-}^{t} \frac{\mathrm{d}^q\delta(t-\xi)}{\mathrm{d}t^q}\gamma(\xi)\,\mathrm{d}\xi = \frac{1}{\Gamma(-q)}\int_{0^-}^{t}\frac{\gamma(\xi)}{(t-\xi)^{q+1}}\,\mathrm{d}\xi, \quad q \in \mathbb{R}^+ \tag{19}$$

3. The Inverse Laplace Transform of s^q with $q \in \mathbb{R}^+$

The memory function, $M(t)$ appearing in Equation (7) of the Scott–Blair (springpot when $0 \le q \le 1$) element expressed by Equation (9) results directly from the definition of the fractional derivative expressed with the Reimann–Liouville integral given by Equation (11). Substitution of Equation (11) into Equation (9) gives

$$\tau(t) = \frac{\mu_q}{\Gamma(-q)}\int_{0^-}^{t}\frac{\gamma(\xi)}{(t-\xi)^{q+1}}\mathrm{d}\xi,\ q \in \mathbb{R}^+ \tag{20}$$

By comparing Equation (20) with Equation (7), the memory function, $M(t)$, of the Scott–Blair element is merely the kernel of the Riemann–Liouville convolution multiplied with the material parameter μ_q

$$M(t) = \frac{\mu_q}{\Gamma(-q)}\frac{1}{t^{q+1}} = \mu_q\frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q},\ q \in \mathbb{R}^+ \tag{21}$$

where the right-hand side of Equation (21) is from Equation (18). Equation (21) shows that the memory function of the springpot element is the fractional derivative of order $q \in \mathbb{R}^+$ of the Dirac delta function as was anticipated by using the argument of physical continuity given that the springpot element interpolates the Hookean spring and the Newtonian dashpot.

In this study we adopt the name "Scott–Blair element" rather than the more restrictive "springpot" element given that the fractional order of differentiation $q \in \mathbb{R}^+$ is allowed to take values larger than one. The complex dynamic modulus, $\mathcal{G}(\omega)$, of the Scott–Blair fluid described by Equation (9) with now $q \in \mathbb{R}^+$ derives directly from Equation (13)

$$\mathcal{G}(\omega) = \frac{\tau(\omega)}{\gamma(\omega)} = \mu_q(\mathrm{i}\,\omega)^q \tag{22}$$

and its inverse Fourier transform is the memory function, $M(t)$, as indicated by Equation (8). With the introduction of the fractional derivative of the Dirac delta function expressed by Equations (17) or (21), the definition of the memory function given by Equation (8) offers a new (to the best of our knowledge) and useful result regarding the Fourier transform of the function $\mathcal{F}(\omega) = (\mathrm{i}\,\omega)^q$ with $q \in \mathbb{R}^+$

$$\mathcal{F}^{-1}(\mathrm{i}\,\omega)^q = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathrm{i}\,\omega)^q e^{\mathrm{i}\,\omega t}\mathrm{d}\omega = \frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q} = \frac{1}{\Gamma(-q)}\frac{1}{t^{q+1}},\ q \in \mathbb{R}^+,\ t > 0 \tag{23}$$

In terms of the Laplace variable $s = \mathrm{i}\,\omega$ (see equivalence of Equations (13) and (14)), Equation (23) gives that

$$\mathcal{L}^{-1}\{s^q\} = \frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q} = \frac{1}{\Gamma(-q)}\frac{1}{t^{q+1}},\ q \in \mathbb{R}^+,\ t > 0 \tag{24}$$

where $\mathcal{L}^{-1}$ indicates the inverse Laplace transform operator [1,28,29].

When $t > 0$ the right-hand side of Equations (23) or (24) is non-zero only when $q \in \{\mathbb{R}^+ - \mathbb{N}\}$; otherwise it vanishes because of the poles of the Gamma function when q is zero or any positive integer. The validity of Equation (23) can be confirmed by investigating its limiting cases. For instance, when $q = 0$, then $(\mathrm{i}\,\omega)^q = 1$; and Equation (23) yields that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\,\omega t}\mathrm{d}\omega = \delta(t-0)$; which is the correct result. When $q = 1$, Equation (23) yields that $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{i}\,\omega e^{\mathrm{i}\,\omega t}\mathrm{d}\omega = \frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t}$. Clearly, the function $\mathcal{F}(\omega) = \mathrm{i}\,\omega$ is not Fourier integrable in the classical sense, yet the result of Equation (23) can be confirmed by evaluating the Fourier transform of $\frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t}$ in association with the properties of the higher-order derivatives of the Dirac delta function given by Equation (6). By virtue of Equation (6), the Fourier transform of $\frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t}$ is

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t}e^{-\mathrm{i}\,\omega t}\mathrm{d}t = -(-\mathrm{i}\,\omega)e^{-\mathrm{i}\,\omega 0} = \mathrm{i}\,\omega \tag{25}$$

therefore, the functions $\mathrm{i}\,\omega$ and $\frac{\mathrm{d}\delta(t-0)}{\mathrm{d}t}$ are Fourier pairs, as indicated by Equation (23).

More generally, for any $q = n \in \mathbb{N}$, Equation (23) yields that $\frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathrm{i}\,\omega)^n e^{\mathrm{i}\,\omega t}\mathrm{d}\omega$ $= \frac{\mathrm{d}^n\delta(t-0)}{\mathrm{d}t^n}$ and by virtue of Equation (6), the Fourier transform of $\frac{\mathrm{d}^n\delta(t-0)}{\mathrm{d}t^n}$ is

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}^n\delta(t-0)}{\mathrm{d}t^n}e^{-\mathrm{i}\,\omega t}\mathrm{d}t = (-1)^n(-\mathrm{i}\,\omega)^n = (\mathrm{i}\,\omega)^n \tag{26}$$

showing that the functions $(\mathrm{i}\,\omega)^n$ and $\frac{\mathrm{d}^n\delta(t-0)}{\mathrm{d}t^n}$ are Fourier pairs, which is a special result for $q \in \mathbb{N}_0$ of the more general result offered by Equation (23). Consequently, fractional calculus and the memory function of the Scott–Blair element with $q \in \mathbb{R}^+$ offer an alternative avenue to reach the Gel'fand and Shilov [3] definition of the Dirac delta function and its integer-order derivatives given by Equation (2). By establishing the inverse Laplace transform of s^q with $q \in \mathbb{R}^+$ given by Equation (24) we proceed by examining the inverse Laplace transform of $\frac{s^q}{(s\mp\lambda)^\alpha}$ with $\alpha < q \in \mathbb{R}^+$.

4. The Inverse Laplace Transform of $\frac{s^q}{(s\mp\lambda)^\alpha}$ with $\alpha < q \in \mathbb{R}^+$

The inverse Laplace transform of $\mathcal{F}(s) = \frac{s^q}{(s\mp\lambda)^\alpha}$ with $\alpha < q \in \mathbb{R}^+$ is evaluated with the convolution theorem [29]

$$f(t) = \mathcal{L}^{-1}\{\mathcal{F}(s)\} = \mathcal{L}^{-1}\{\mathcal{H}(s)\mathcal{G}(s)\} = \int_0^t h(t-\xi)g(\xi)\mathrm{d}\xi \tag{27}$$

where $h(t) = \mathcal{L}^{-1}\{\mathcal{H}(s)\} = \mathcal{L}^{-1}\{s^q\}$ given by Equation (24) and $g(t) = \mathcal{L}^{-1}\{\mathcal{G}(s)\}$ $= \mathcal{L}^{-1}\left\{\frac{1}{(s\mp\lambda)^\alpha}\right\} = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{\pm\lambda t}$ shown in entry (2) of Table 1 [1] which summarizes selective known inverse Laplace transforms of functions with arbitrary power. Accordingly, Equation (27) gives

$$\mathcal{L}^{-1}\left\{\frac{s^q}{(s\mp\lambda)^\alpha}\right\} = \frac{1}{\Gamma(-q)}\int_0^t \frac{1}{(t-\xi)^{q+1}}\frac{1}{\Gamma(\alpha)}\xi^{\alpha-1}e^{\pm\lambda\xi}\mathrm{d}\xi \tag{28}$$

Table 1. Known inverse Laplace transforms of irrational functions with an arbitrary power.

	$\mathcal{F}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\mathrm{d}t$	$f(t) = \mathcal{L}^{-1}\{\mathcal{F}(s)\}$
(1)	$\frac{1}{s^\alpha}$ $\alpha \in \mathbb{R}^+$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$
(2)	$\frac{1}{(s\mp\lambda)^\alpha}$ $\alpha \in \mathbb{R}^+$	$\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{\pm\lambda t}$
(3)	$\frac{1}{s^\alpha(s\mp\lambda)}$ $\alpha \in \mathbb{R}^+$	$t^\alpha E_{1,1+\alpha}(\pm\lambda t) = I^\alpha e^{\pm\lambda t}$
(4)	$\frac{s^\alpha}{s\mp\lambda}$ $0 < \alpha < 1$	$t^{-\alpha}E_{1,1-\alpha}(\pm\lambda t) = \frac{\mathrm{d}^\alpha e^{\pm\lambda t}}{\mathrm{d}t^\alpha}$
(5)	$\frac{s^{\alpha-\beta}}{s^\alpha\mp\lambda}$ $\alpha, \beta \in \mathbb{R}^+$	$t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^\alpha)$
(6) Special case of (5) for $\beta = 1$	$\frac{s^{\alpha-1}}{s^\alpha\mp\lambda}$ $\alpha \in \mathbb{R}^+$	$E_\alpha(\pm\lambda t^\alpha)$
(7) Special case of (5) for $\alpha = \beta$	$\frac{1}{s^\alpha\mp\lambda}$ $\alpha \in \mathbb{R}^+$	$t^{\alpha-1}E_{\alpha,\alpha}(\pm\lambda t^\alpha) = \mathcal{E}_{\alpha-1}(\pm\lambda, t)$
(8) Special case of (5) $\alpha - \beta = -1$	$\frac{1}{s(s^\alpha\mp\lambda)}$ $\alpha \in \mathbb{R}^+$	$t^\alpha E_{\alpha,\alpha+1}(\pm\lambda t^\alpha)$
(9) Special case of (5) with $0 < \alpha - \beta = q < \alpha$	$\frac{s^q}{s^\alpha\mp\lambda}$ $0 < q < \alpha \in \mathbb{R}^+$	$t^{\alpha-q-1}E_{\alpha,\alpha-q}(\pm\lambda t^\alpha)$

With reference to Equation (11), Equation (28) is expressed as

$$\mathcal{L}^{-1}\left\{\frac{s^q}{(s\mp\lambda)^\alpha}\right\} = \frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}^q}{\mathrm{d}t^q}\left[t^{\alpha-1}e^{\pm\lambda t}\right], \ \alpha, q \in \mathbb{R}^+ \tag{29}$$

For the special case where $\lambda = 0$ and after using that $\frac{\mathrm{d}^q t^{\alpha-1}}{\mathrm{d}t^q} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t^{\alpha-1-q}$ [24], Equation (29) reduces to

$$\mathcal{L}^{-1}\{s^{q-\alpha}\} = \frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}^q}{\mathrm{d}t^q} t^{\alpha-1} = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{\Gamma(-q+\alpha)} \frac{1}{t^{q-\alpha+1}} = \frac{\mathrm{d}^{q-\alpha}\delta(t-0)}{\mathrm{d}t^{q-\alpha}}, \ \alpha < q \in \mathbb{R}^+ \quad (30)$$

and the result of Equation (24) is recovered. Equation (30) also reveals the intimate relation between the fractional derivative of the Dirac delta function and the fractional derivative of the power law.

$$\frac{\mathrm{d}^{q-\alpha}\delta(t-0)}{\mathrm{d}t^{q-\alpha}} = \frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}^q}{\mathrm{d}t^q} t^{\alpha-1}, \ \alpha < q \in \mathbb{R}^+ \quad (31)$$

For the special case where $q = \alpha$, Equation (31) yields $\delta(t-0) = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{\Gamma(0)} t^{-1} = \frac{1}{\Gamma(0)} \frac{1}{t}$ and the Gel'fand and Shilov [3] definition of the Dirac delta function given by Equation (2) is recovered. The new results, derived in this paper, on the inverse Laplace transform of irrational functions with arbitrary powers are summarized in Table 2.

Table 2. New results on the inverse Laplace transform of irrational functions with arbitrary powers.

	$\mathcal{F}(s)=\mathcal{L}\{f(t)\}=\int_0^\infty f(t)e^{-st}\mathrm{d}t$	$f(t)=\mathcal{L}^{-1}\{\mathcal{F}(s)\}$
(1)	$s^q \quad q\in\mathbb{R}^+$	$\frac{1}{\Gamma(-q)}\frac{1}{t^{q+1}}=\frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q}$
(2)	$\frac{s^q}{(s\mp\lambda)^\alpha} \quad \alpha,q\in\mathbb{R}^+$	$\frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}^q}{\mathrm{d}t^q}\left[t^{\alpha-1}e^{\pm\lambda t}\right]$
(3) Extension of entry (9) of Table 1 for $\alpha<q<2\alpha\in\mathbb{R}^+$	$\frac{s^q}{s^\alpha\mp\lambda} \quad \alpha<q<2\alpha\in\mathbb{R}^+$	$\frac{1}{\Gamma(-q+\alpha)}\frac{1}{t^{q-\alpha+1}}\pm\lambda t^{2\alpha-q-1}E_{\alpha,2\alpha-q}(\pm\lambda t^\alpha)=\frac{\mathrm{d}^{q-\alpha}}{\mathrm{d}t^{q-\alpha}}\delta(t-0)\pm\lambda t^{2\alpha-q-1}E_{\alpha,2\alpha-q}(\pm\lambda t^\alpha)$
(4) Special case of (3) for $\alpha=1$	$\frac{s^q}{s\mp\lambda} \quad 1<q<2$	$\frac{1}{\Gamma(-q+1)}\frac{1}{t^q}\pm\lambda t^{1-q}E_{1,2-q}(\pm\lambda t)=\frac{\mathrm{d}^{q-1}\delta(t-0)}{\mathrm{d}t^{q-1}}\pm\lambda t^{1-q}E_{1,2-q}(\pm\lambda t)$
(5) General case of (3) for any $q\in\mathbb{R}^+$ with $n\alpha<q<(n+1)\alpha$, $n\in\mathbb{N}$	$\frac{s^q}{s^\alpha\mp\lambda} \quad \alpha<q\in\mathbb{R}^+$	$\sum_{j=1}^{n}(\pm\lambda)^{j-1}\frac{1}{\Gamma(-q+j\alpha)}\frac{1}{t^{q-j\alpha+1}}+(\pm\lambda)^n t^{(n+1)\alpha-q-1}E_{\alpha,(n+1)\alpha-q}(\pm\lambda t^\alpha)$ $=\sum_{j=1}^{n}(\pm\lambda)^{j-1}\frac{\mathrm{d}^{q-j\alpha}}{\mathrm{d}t^{q-j\alpha}}\delta(t-0)+(\pm\lambda)^n t^{(n+1)\alpha-q-1}E_{\alpha,(n+1)\alpha-q}(\pm\lambda t^\alpha),\quad n\alpha<q<(n+1)\alpha$ $=\frac{\mathrm{d}^q}{\mathrm{d}t^q}\mathcal{E}_{\alpha-1}(\pm\lambda,t)$
(6) Special case of (5) for $\alpha=1$ with $n<q<n+1$, $n\in\mathbb{N}$	$\frac{s^q}{s\mp\lambda} \quad 1<q\in\mathbb{R}^+$	$\sum_{j=1}^{n}(\pm\lambda)^{j-1}\frac{1}{\Gamma(-q+j)}\frac{1}{t^{q-j+1}}+(\pm\lambda)^n t^{n-q}E_{1,n+1-q}(\pm\lambda t)$ $=\sum_{j=1}^{n}(\pm\lambda)^{j-1}\frac{\mathrm{d}^{q-j}}{\mathrm{d}t^{q-j}}\delta(t-0)+(\pm\lambda)^n t^{n-q}E_{1,n+1-q}(\pm\lambda t),\quad n<q<n+1$ $=\frac{\mathrm{d}^q}{\mathrm{d}t^q}e^{\pm\lambda t}$

5. The Inverse Laplace Transform of $\frac{s^q}{s^\alpha \mp \lambda}$ with $\alpha, q \in \mathbb{R}^+$

We start with the known result for the inverse Laplace transform of the function $\mathcal{Q}(s) = \frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda}$ with $\alpha, \beta \in \mathbb{R}^+$ [25,27]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda}\right\} = t^{\beta-1} E_{\alpha,\beta}(\pm\lambda t^\alpha), \; \lambda, \alpha, \beta \in \mathbb{R}^+ \tag{32}$$

where $E_{\alpha,\beta}(z)$ is the two-parameter Mittag–Leffler function [31–33]

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha + \beta)}, \;\; \alpha, \beta > 0 \tag{33}$$

When $\beta = 1$, Equation (32) reduces to the result of the Laplace transform of the one-parameter Mittag–Leffler function, originally derived by Mittag–Leffler [33]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha \mp \lambda}\right\} = E_{\alpha,1}(\pm\lambda t^\alpha) = E_\alpha(\pm\lambda t^\alpha), \; \lambda, \alpha \in \mathbb{R}^+ \tag{34}$$

When $\alpha = \beta$, the right-hand side of Equation (32) is known as the Rabotnov function, $\varepsilon_{\alpha-1}(\pm\lambda, t) = t^{\alpha-1} E_{\alpha,\alpha}(\pm\lambda t^\alpha)$ [11,28,30,34,35]; and Equation (32) yields

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha \mp \lambda}\right\} = t^{\alpha-1} E_{\alpha,\alpha}(\pm\lambda t^\alpha) = \varepsilon_{\alpha-1}(\pm\lambda, t), \; \lambda, \alpha \in \mathbb{R}^+ \tag{35}$$

Figure 2 plots the function $E_\alpha(-\lambda t^\alpha)$ (left) and the function $\varepsilon_{\alpha-1}(-\lambda, t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$ (right) for various values of the parameter $\alpha \in \mathbb{R}^+$. For $\alpha = 1$ both functions contract to $e^{-\lambda t}$. When $\alpha - \beta = -1$ Equation (32) gives

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^\alpha \mp \lambda)}\right\} = t^\alpha E_{\alpha,\alpha+1}(\pm\lambda t^\alpha), \; \lambda, \alpha \in \mathbb{R}^+ \tag{36}$$

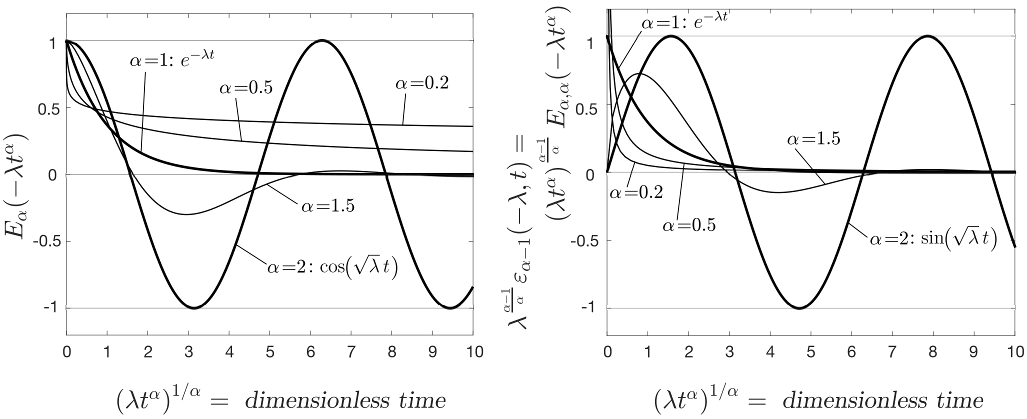

Figure 2. The one-parameter Mittag–Leffler function $E_\alpha(-\lambda t^\alpha)$ (left) and the Rabotnov function $\varepsilon_{\alpha-1}(-\lambda, t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$ (right) for various values of the parameter $\alpha \in \mathbb{R}^+$.

The inverse Laplace transform of $\mathcal{F}(s) = \frac{s^q}{s^\alpha \mp \lambda}$ with $\alpha, q \in \mathbb{R}^+$ is evaluated with the convolution theorem expressed by Equation (27) where $h(t) = \mathcal{L}^{-1}\{\mathcal{H}(s)\} = \mathcal{L}^{-1}\{s^q\}$ given by Equation (24) and $g(t) = \varepsilon_{\alpha-1}(\pm\lambda, t)$ is given by Equation (35). Accordingly, Equation (27) gives

$$\mathcal{L}^{-1}\left\{\frac{s^q}{s^\alpha \mp \lambda}\right\} = \frac{1}{\Gamma(-q)} \int_0^t \frac{1}{(t-\xi)^{q+1}} \xi^{\alpha-1} E_{\alpha,\alpha}(\pm\lambda\xi^\alpha) \mathrm{d}\xi \tag{37}$$

With reference to Equation (11), Equation (37) indicates that $\mathcal{L}^{-1}\left\{\frac{s^q}{s^\alpha \mp \lambda}\right\}$ is the fractional derivative of order q of the Rabotnov function $\varepsilon_{\alpha-1}(\pm\lambda, t)$

$$\mathcal{L}^{-1}\left\{\frac{s^q}{s^\alpha \mp \lambda}\right\} = \frac{\mathrm{d}^q}{\mathrm{d}t^q}\left[t^{\alpha-1}E_{\alpha,\alpha}(\pm\lambda t^\alpha)\right] = t^{\alpha-q-1}E_{\alpha,\alpha-q}(\pm\lambda t^\alpha) \tag{38}$$

For the case where $q < \alpha \in \mathbb{R}^+$, the exponent q can be expressed as $q = \alpha - \beta$ with $0 < \beta \leq \alpha \in \mathbb{R}^+$, and Equation (38) returns the known result given by Equation (32). For the case where $q > \alpha \in \mathbb{R}^+$, the numerator of the fraction $\frac{s^q}{s^\alpha \mp \lambda}$ is more powerful than the denominator, and the inverse Laplace transform expressed by Equation (38) is expected to yield a singularity that is manifested with the second parameter of the Mittag–Leffler function $E_{\alpha,\alpha-q}(\pm\lambda t^\alpha)$, being negative $(\alpha - q < 0)$. This embedded singularity in the right-hand side of Equation (38) when $q > \alpha$ is extracted by using the recurrence relation [31–33]

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z) \tag{39}$$

By employing the recurrence relation (39) to the right-hand side of Equation (38), then Equation (38) for $q > \alpha \in \mathbb{R}^+$ assumes the expression

$$\mathcal{L}^{-1}\left\{\frac{s^q}{s^\alpha \mp \lambda}\right\} = \frac{1}{\Gamma(-q+\alpha)}\frac{1}{t^{q-\alpha+1}} \pm \lambda t^{2\alpha-q-1}E_{\alpha,2\alpha-q}(\pm\lambda t^\alpha) \tag{40}$$

Recognizing that according to Equation (18), the first term in the right-hand side of Equation (39) is $\frac{\mathrm{d}^{q-\alpha}\delta(t-0)}{\mathrm{d}t^{q-\alpha}}$, the inverse Laplace transform of $\frac{s^q}{s^\alpha \mp \lambda}$ with $q > \alpha \in \mathbb{R}^+$ can be expressed in the alternative form

$$\mathcal{L}^{-1}\left\{\frac{s^q}{s^\alpha \mp \lambda}\right\} = \frac{\mathrm{d}^{q-\alpha}}{\mathrm{d}t^{q-\alpha}}\delta(t-0) \pm \lambda t^{2\alpha-q-1}E_{\alpha,2\alpha-q}(\pm\lambda t^\alpha),\ \alpha < q < 2\alpha \tag{41}$$

in which the singularity $\frac{\mathrm{d}^{q-\alpha}\delta(t-0)}{\mathrm{d}t^{q-\alpha}}$ has been extracted from the right-hand side of Equation (38), and now the second index of the Mittag–Leffler function appearing in Equation (40) or (41) has been increased to $2\alpha - q$. In the event that $2\alpha - q$ remains negative $(q > 2\alpha)$, the Mittag–Leffler function appearing on the right-hand side of Equation (40) or (41) is replaced again by virtue of the recurrence relation (39) and results in

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^q}{s^\alpha \mp \lambda}\right\} &= \frac{\mathrm{d}^{q-\alpha}}{\mathrm{d}t^{q-\alpha}}\delta(t-0) \\ &\pm \lambda\frac{\mathrm{d}^{q-2\alpha}}{\mathrm{d}t^{q-2\alpha}}\delta(t-0) + (\pm\lambda)^2 t^{3\alpha-q-1}E_{\alpha,3\alpha-q}(\pm\lambda t^\alpha),\ 2\alpha < q < 3\alpha\end{aligned} \tag{42}$$

More generally, for any $q \in \mathbb{R}^+$ with $n\alpha < q < (n+1)\alpha$ with $n \in \mathbb{N} = \{1, 2, ...\}$ and $\alpha \in \mathbb{R}^+$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^q}{s^\alpha \mp \lambda}\right\} &= \frac{\mathrm{d}^q}{\mathrm{d}t^q}\varepsilon_{\alpha-1}(\pm\lambda, t) = \\ &\sum_{j=1}^{n}(\pm\lambda)^{j-1}\frac{\mathrm{d}^{q-j\alpha}}{\mathrm{d}t^{q-j\alpha}}\delta(t-0) + (\pm\lambda)^n t^{(n+1)\alpha-q-1}E_{\alpha,(n+1)\alpha-q}(\pm\lambda t^\alpha)\end{aligned} \tag{43}$$

and all singularities from the Mittag–Leffler function have been extracted. For the special case where $\alpha = 1$ Equation (43) gives for $n < q < n+1$ with $n \in \mathbb{N} = \{1, 2, ...\}$

$$\mathcal{L}^{-1}\left\{\frac{s^q}{s \mp \lambda}\right\} = \frac{\mathrm{d}^q}{\mathrm{d}t^q}e^{\pm\lambda t} = \sum_{j=1}^{n}(\pm\lambda)^{j-1}\frac{\mathrm{d}^{q-j}}{\mathrm{d}t^{q-j}}\delta(t-0) + (\pm\lambda)^n t^{n-q}E_{1,n+1-q}(\pm\lambda t) \tag{44}$$

which is the extension of entry (4) of Table 1 for any $q \in \mathbb{R}^+$. As an example, for $1 < q < 2$ Equation (44) is expressed in its dimensionless form

$$\frac{1}{\lambda^q}\mathcal{L}^{-1}\left\{\frac{s^q}{s+\lambda}\right\} = \frac{1}{\Gamma(-q+1)}\frac{1}{(\lambda t)^q} - \frac{1}{(\lambda t)^{q-1}}E_{1,2-q}(-\lambda t),\ 1 < q < 2 \tag{45}$$

whereas for $2 < q < 3$, Equation (44) yields

$$\begin{aligned}\frac{1}{\lambda^q}\mathcal{L}^{-1}\left\{\frac{s^q}{s+\lambda}\right\} = &\frac{1}{\Gamma(-q+1)}\frac{1}{(\lambda t)^q} - \frac{1}{\Gamma(-q+2)}\frac{1}{(\lambda t)^{q-1}} \\ &+ \frac{1}{(\lambda t)^{q-2}}E_{1,3-q}(-\lambda t),\ 2 < q < 3\end{aligned} \tag{46}$$

Figure 3 plots the results of Equation (45) for $q = 1.3$, 1.7, 1.9 and 1.99 together with the results of Equation (46) for $q = 2.01$, 2.1, 2.3 and 2.7. When q tends to 2 from below, the curves for $\frac{1}{\lambda^q}\mathcal{L}^{-1}\left\{\frac{s^q}{s+\lambda}\right\}$ approach $e^{-\lambda t}$ from below; whereas when q tends to 2 from above, the curves of the inverse Laplace transform approach $e^{-\lambda t}$ from above.

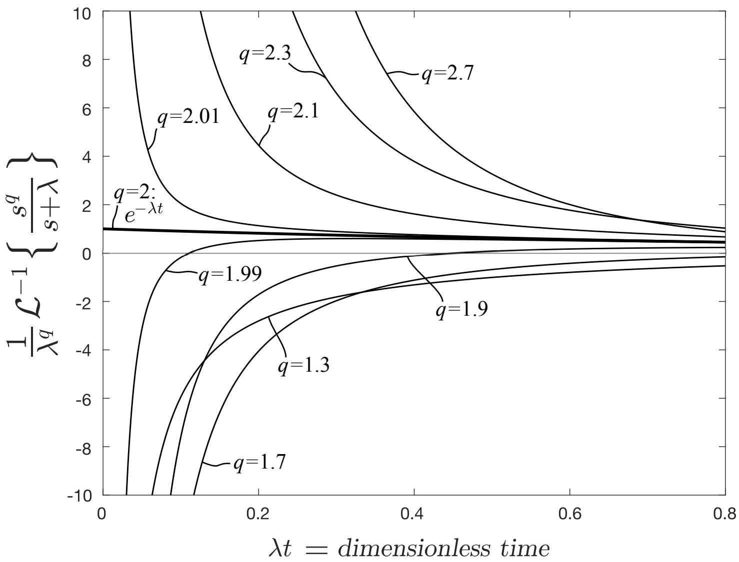

Figure 3. Plots . of $\mathcal{L}^{-1}\left\{\frac{s^q}{s+1}\right\} = \frac{\mathrm{d}^q}{\mathrm{d}t^q}e^{-t}$ by using Equation (45) for $1 < q < 2$ and Equation (46) for $2 < q < 3$. When q tends to 2 from below, the curves for $\frac{1}{\lambda^q}\mathcal{L}^{-1}\left\{\frac{s^q}{s+\lambda}\right\}$ approach $e^{-\lambda t}$ from below; whereas when q tends to 2 from above, the curves of the inverse Laplace transform approach $e^{-\lambda t}$ from above.

6. Summary

In this paper we first show that the memory function, $M(t)$, of the fractional Scott–Blair fluid, $\tau(t) = \mu_q\frac{\mathrm{d}^q\gamma(t)}{\mathrm{d}t^q}$ with $q \in \mathbb{R}^+$ (springpot when $0 \le q \le 1$) is the fractional derivative of the Dirac delta function $\frac{\mathrm{d}^q\delta(t-0)}{\mathrm{d}t^q} = \frac{1}{\Gamma(-q)}\frac{1}{t^{q+1}}$ with $q \in \mathbb{R}^+$. Given that the memory function $M(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{G}(\omega)e^{\mathrm{i}\,\omega t}\mathrm{d}t$ is the inverse Fourier transform of the complex dynamic modulus, $\mathcal{G}(\omega)$, in association with that $M(t)$ is causal ($M(t) = 0$ for $t < 0$), we showed that the inverse Laplace transform of s^q for any $q \in \mathbb{R}^+$ is the fractional derivative of order q of the Dirac delta function. This new finding in association with the convolution theorem makes possible the calculation of the inverse Laplace transform of $\frac{s^q}{s^\alpha \mp \lambda}$ when $\alpha < q \in \mathbb{R}^+$, which is the fractional derivative of order q of the Rabotnov function $\varepsilon_{\alpha-1}(\pm\lambda, t) = t^{\alpha-1}E_{\alpha,\alpha}(\pm\lambda t^\alpha)$. The fractional derivative of order $q \in \mathbb{R}^+$ of the Rabotnov function $\varepsilon_{\alpha-1}(\pm\lambda, t)$ produces singularities that are extracted with a finite number of fractional derivatives of the Dirac delta function depending on the strength of the order of differentiation q in association with the recurrence formula of the two-parameter Mittag–Leffler function. The number of singularities, $n \in \mathbb{N}$, that need to be extracted when $\alpha < q \in \mathbb{R}^+$ is the lowest integer so that $n\alpha < q < (n+1)\alpha$.

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