fractal and fractional

Article

On the Operator Method for Solving Linear Integro-Differential Equations with Fractional Conformable Derivatives

Batirkhan Kh. Turmetov *,†, Kairat I. Usmanov † and Kulzina Zh. Nazarova †

Department of Mathematics, Khoja Akhmet Yassawi International Kazakh-Turkish University, Turkistan 161200, Kazakhstan; y_kairat@mail.ru (K.I.U.); gjnazarova@mail.ru (K.Z.N.)
* Correspondence: batirkhan.turmetov@ayu.edu.kz
† These authors contributed equally to this work.

Abstract: The methods for constructing solutions to integro-differential equations of the Volterra type are considered. The equations are related to fractional conformable derivatives. Explicit solutions of homogeneous and inhomogeneous equations are constructed, and a Cauchy-type problem is studied. It should be noted that the considered method is based on the construction of normalized systems of functions with respect to a differential operator of fractional order.

Keywords: integro-differential equation; fractional derivatives; fractional conformable derivatives; normalized systems method; explit solutions; Mittag–Leffler funtions; particular solution; Cauchy problem

Citation: Turmetov, B.K.; Usmanov, K.I.; Nazarova, K.Z. On the Operator Method for Solving Linear Integro-Differential Equations with Fractional Conformable Derivatives. *Fractal Fract.* **2021**, *5*, 109. https://doi.org/10.3390/fractalfract5030109

Academic Editor: Haci Mehmet Baskonus

Received: 1 August 2021
Accepted: 31 August 2021
Published: 2 September 2021

1. Introduction

There are many different ways of defining fractional operators, unlike in classical calculus, where there is only one way to define the derivative operation. The most common derivatives are Riemann–Liouville and Caputo derivatives, which were successfully used in the modeling of complex dynamical processes in physics, biology, engineering and many other fields [1–5].

Among the other definitions of fractional calculus, we can mention Hilfer, Riesz, Hadamard, Erd'elyi-Kober, Atangana–Baleanu, Katugampola, fractional conformable derivatives, and many others [2,6–9].

It should be noted that questions related to the theorems of existence and uniqueness of solutions of Cauchy-type and Dirichlet-type problems for linear and nonlinear differential equations of fractional order have been developed in sufficient detail, whereas explicit solutions are only known for certain types of linear differential equations of fractional order.

One of the most widely used methods for constructing solutions to differential equations of fractional order is the method of integral transformations. A detailed description of this method can be found in [2,4,5] and other works. An effective method for constructing explicit solutions and solving the Cauchy problem for differential equations of fractional order is based on the Mikusinski operational calculus. In the papers of Yu. Luchko et al. [10–15], this method was applied to solve linear differential equations of fractional order with constant coefficients and with derivatives of Riemann–Liouville and Caputo type and the general fractional derivative. This method was later used for a general equation with a Hilfer-type operator [16]. In the paper [17] A. Pskhu formulated and solved the initial problem for linear ordinary differential equations of fractional order with Riemann–Liouville derivatives. He reduced the problem to an integral equation and constructed an explicit solution in terms of the Wright function. We also note that in [18,19] the Cauchy problem for differential equations of fractional order has been studied using the Adomian decomposition method.

In this paper, we consider an operator method for constructing solutions to fractional differential equations. This method is based on the construction of normalized systems with respect to operators of fractional differentiation. The method of normalized systems was introduced in [20] and used to construct exact solutions to the Helmholtz equation and the polyharmonic equation. The method of normalized systems was used to solve the Cauchy problem for ordinary differential equations with constant coefficients [21], as well as to construct solutions to differential equations associated with Dunkl operators [22,23]. Later, in [24–27], this method was applied to the construction of an explicit form of solution of fractional differential equations.

Let us first consider the definition of fractional-order integro-differentiation operators that will be used in this paper.

Let $0 < a < b < \infty$. For function $f(x) \in C^1[a,b]$, we define the operator

$$ {}_aT^{\alpha} f(x) = (t-a)^{1-\alpha} f'(x), \alpha > 0. $$

In case $\alpha \in (0,1)$, this operator corresponds to an integral operator of the form

$$ {}_aI^{\alpha} f(x) = \int\limits_a^x f(t) \frac{dt}{(t-a)^{1-\alpha}}. $$

Let $0 < \alpha, \beta$ and ${}_a^nT^{\alpha} = \underbrace{{}_aT^{\alpha} \cdot {}_aT^{\alpha} \cdot \ldots \cdot {}_aT^{\alpha}}_{n}, n = 1, 2, \ldots.$ In [8], the following integro-differential operators were considered:

$$ {}_a^{\beta}J^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int\limits_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} f(t) \frac{dt}{(t-a)^{1-\alpha}}. \quad (1) $$

$$ {}_a^{\beta}D^{\alpha} f(x) = \frac{{}_a^nT^{\alpha}}{\Gamma(\beta)} \int\limits_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{n-\beta-1} f(t) \frac{dt}{(t-a)^{1-\alpha}}, \quad (2) $$

$$ {}_a^{C\beta}D^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{n-\beta-1} {}_a^nT^{\alpha} f(t) \frac{dt}{(t-a)^{1-\alpha}}, n-1 < \beta \le n. \quad (3) $$

In case $\alpha = 1$, operator ${}_a^{\beta}J^{\alpha}$ coincides with the integration operator of the β order in the Riemann–Liouville sense, whereas ${}_a^{\beta}D^{\alpha}$ and ${}_a^{C\beta}D^{\alpha}$ coincide with the differentiation operators of the β order in the Riemann–Liouville and Caputo sense [2].

It should be noted that the methods for solving fractional differential equations with derivatives ${}_a^{\beta}D^{\alpha}$ and ${}_a^{C\beta}D^{\alpha}$ have been studied by many authors, in particular, in works [28–32].

In [28] the theorem on the existence of a unique solution to the Cauchy problem is proved by the method of successive approximations. In [29], using the generalized integral Laplace transform for the case $0 < \beta \le 1$, explicit solutions to the following Cauchy problems were constructed

$$ {}_0^{\beta}D^{\alpha} y(t) = \lambda y(t) + f(t), t > 0, \left({}_0J^{1-\beta} y \right)(0) = b, b \in R, $$

$$ {}_0^{C\beta}D^{\alpha} y(t) = \lambda y(t) + f(t), t > 0, y(0) = b, b \in R. $$

Similar results were obtained in [30–32].

The use of differential equations of fractional order with derivatives ${}_a^{\beta}D^{\alpha}$ and ${}_a^{C\beta}D^{\alpha}$ in the modeling of biological processes (fractional analogue of the Bergman model), electrical circuits, motion of electrons under the action of the electric field (fractional analogue of the Drude model), as well as in the analysis of applied dynamic models (Rabinovich–Fabrikant attractor), is described in [33–37].

Further, in the work of A.A. Kilbas and M. Saigo [38], on the basis of the formula for the composition of operators of integration of fractional order with a three-parameter Mittag–Leffler function $E_{\beta,m,l}(z)$, an algorithm for solving an integral equation of the type

$$\varphi(t) = \frac{\lambda t^{\beta(m-1)}}{\Gamma(\alpha)} \int_0^t \frac{\varphi(\tau)}{(t-\tau)^{1-\beta}} d\tau + f(t), \beta > 0, \lambda \in R$$

was obtained.

In this paper (Section 3), this result is generalized for integral equations with the operator ${}_a^\beta J^\alpha$. In this case, the solution to the integral equation is constructed by a constructive method, i.e., by the method of normalized systems, and it is proved that the solution to the integral equation is represented in terms of Mittag–Leffler-type functions $E_{\beta,m,l}(z)$. The solution to the integral equation is constructed in a closed form when the right-hand side of the equation is a quasi-polynomial. In the particular case of parameters of the considered integral operator, the results obtained in this work agree with the results obtained in [38].

In Section 4 of this work, the method of normalized systems is used to construct solutions to iterated differential equations of fractional order. In contrast with our work [25], in this case, fractional-order differential equations with degeneration are considered. The construction of solutions to such equations has not been studied by other authors. It should be noted that in constructing solutions to these equations, a new class of special functions $E^{p+1}_{\beta,m,l}(z)$, representing a more general form of three-parameter Mittag–Leffler-type functions $E_{\beta,m,l}(z)$, arises.

In the fifth and sixth sections of the work, application of the method of normalized systems to the construction of an explicit solution of one class of fractional-order differential equations with operators ${}_a^\beta D^\alpha$ and ${}_a^{C\beta} D^\alpha$ is considered. Homogeneous and inhomogeneous equations are studied. The considered equations and, therefore, the results obtained, generalize the results obtained in [30–32], as well as the results obtained in the work of A.A. Kilbas and M. Saigo [39].

At the end of the section, an example of solving an equation for electrical circuit simulation is given.

Further, we present some well-known information about the method of normalized systems.

Let L_1 and L_2 be linear operators, acting from the functional space X to X, $L_k X \subset X, k = 1, 2$. Let functions from X be defined in a domain $\Omega \subset R^n$. Let us give the definition of normalized systems [20].

Definition 1. *A sequence of functions $\{f_i(x)\}_{i=0}^\infty, f_i(x) \in X$ is called $f-$normalized with respect to (L_1, L_2) on Ω, having the base $f_0(x)$, if, on this domain, the following equality holds: $L_1 f_0(x) = f(x)$, $L_1 f_i(x) = L_2 f_{i-1}(x)$, $i \geq 1$.*

If $L_2 = E$ is a unit operator, then a system of functions $f-$ normalized with respect to (L_1, I) is called $f-$ normalized with respect to L_1, i.e., $L_1 f_0(x) = f(x), L_1 f_i(x) = f_{i-1}(x), i \geq 1$.

If $f(x) = 0$, then the system of functions $\{f_i(x)\}$ is just called normalized.

The main properties of the systems of functions f-normalized with respect to the operators (L_1, L_2) on Ω have been described in [20]. Let us consider the main property of the f-normalized systems.

Proposition 1. *If a system of functions $\{f_i(x)\}_{i=0}^\infty$ is $f-$normalized with respect to (L_1, L_2) on Ω, then the functional series $y(x) = \sum_{i=0}^\infty f_i(x), x \in \Omega$, is a formal solution of the equation:*

$$(L_1 - L_2)y(x) = f(x), x \in \Omega. \tag{4}$$

The following proposition allows us to construct an $f-$normalized system with respect to a pair of operators (L_1, L_2).

Proposition 2 ([27]). *If for L_1 there exists a right inverse operator L_1^{-1}, i.e., $L_1 \cdot L_1^{-1} = E$, where E is a unit operator and $L_1 f_0(x) = f(x)$, then a system of functions $f_i(x) = \left(L_1^{-1} \cdot L_2\right)^i f_0(x), i \geq 1$ is $f-$normalized with respect to a pair of operators (L_1, L_2) on Ω.*

2. Properties of Integro-Differential Operators

Let us consider some properties of operators ${}_a^{\beta}J^{\alpha}$, ${}_a^{\beta}D^{\alpha}$ and ${}_a^{C\beta}D^{\alpha}$.

Lemma 1. *Let $\alpha, \beta > 0, s > -1$ and $f(x) = (x-a)^{\alpha s}$. Then, the equality holds:*

$$\left({}_a^{\beta}J^{\alpha}(t-a)^{s\alpha}\right)(x) = \frac{1}{\alpha^{\beta}} \frac{\Gamma(s+1)}{\Gamma(s+1+\beta)}(x-a)^{\alpha(s+\beta)}, s > -1. \tag{5}$$

Proof. By the definition of the operator ${}_a^{\beta}J^{\alpha}$, we have

$${}_a^{\beta}J^{\alpha}f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right)^{\beta-1} (t-a)^{s\alpha} \frac{dt}{(t-a)^{1-\alpha}}.$$

After changing variables $\xi = \frac{(t-a)^{\alpha}}{(x-a)^{\alpha}}$ for ${}_a^{\beta}J^{\alpha}f(x)$ we get

$$\begin{aligned} {}_a^{\beta}J^{\alpha}f(x) &= \frac{(x-a)^{\alpha(\beta-1)+\alpha s+\alpha}}{\alpha^{\beta}\Gamma(\beta)} \int_0^1 (1-\xi)^{\beta-1}\xi^s d\xi = \frac{(x-a)^{\alpha s+\alpha\beta}}{\alpha^{\beta}\Gamma(\beta)} \frac{\Gamma(\beta)\Gamma(s+1)}{\Gamma(s+1+\beta)} \\ &= \frac{1}{\alpha^{\beta}} \frac{\Gamma(s+1)}{\Gamma(s+1+\beta)}(x-a)^{\alpha s+\alpha\beta}. \end{aligned}$$

The lemma is proved. □

Lemma 2. *Let $f(t) = (t-a)^{\alpha s}, s > -1$. Then, the following equalities hold:*

$$\left({}_a^{\beta}D^{\alpha}(t-a)^{s\alpha}\right)(x) = \begin{cases} 0, s \in \{\beta-1, \beta-2, ..., \beta-n\} \\ \alpha^{\beta}\frac{\Gamma(s+1)}{\Gamma(s+1-\beta)}(x-a)^{\alpha(s-\beta)}, s > n-1 \end{cases}, \tag{6}$$

$$\left({}_a^{C\beta}D^{\alpha}(t-a)^{s\alpha}\right)(x) = \begin{cases} 0, s \in \{0, 1, ..., n-1\} \\ \alpha^{\beta}\frac{\Gamma(s+1)}{\Gamma(s+1-\beta)}(x-a)^{\alpha(s-\beta)}, s > n-1 \end{cases}, \tag{7}$$

Proof. If $\beta = n$ is an integer, then, by definition, ${}_a^{\beta}D^{\alpha}f(x) = {}_a^{C\beta}D^{\alpha}f(x) = {}_a^{n}T^{\alpha}$ and thus, for all $s > n-1$, we get

$$\begin{aligned} {}_a^{n}T^{\alpha}(x-a)^{\alpha s} &= {}_a^{n-1}T^{\alpha}\left((x-a)^{1-\alpha}\frac{d}{dx}(t-a)^{\alpha s}\right) = \alpha s \cdot {}_a^{n-1}T^{\alpha}\left((x-a)^{\alpha(s-1)}\right) \\ &= \alpha s \cdot \alpha(s-1) \cdot ... \cdot \alpha(s-n+1)(x-a)^{\alpha(s-n)} = \alpha^n s \cdot (s-1) \cdot ... \cdot (s-n+1)(x-a)^{\alpha(s-n)} \\ &= \alpha^n \frac{\Gamma(s+1)}{\Gamma(s+1-n)}(x-a)^{\alpha(s-n)}. \end{aligned}$$

If $s \in \{0, 1, ..., n-1\}$, then ${}_a^{n}T^{\alpha}(x-a)^{\alpha s} = 0$.

Let $n-1<\beta<n, n=1,2,\ldots$. Then, by the definition of the operator ${}_a^{\beta}D^{\alpha}$ and taking into account (5), we get

$$ {}_a^{\beta}D^{\alpha}f(x) = {}_a^{n}T^{\alpha}\left({}_a^{n-\beta}J^{\alpha}(t-a)^{s\alpha}\right)(x) = {}_a^{n}T^{\alpha}\left(\frac{1}{\alpha^{n-\beta}}\frac{\Gamma(s+1)}{\Gamma(s+1+n-\beta)}(x-a)^{\alpha(s+n-\beta)}\right). $$

If in the latter equality the parameter s takes one of the values $s_1=\beta-1, s_2=\beta-2,\ldots,s_n=\beta-n$, then

$$ {}_a^{n}T^{\alpha}\left((x-a)^{\alpha(s_j+n-\beta)}\right)=0 \Rightarrow \left({}_a^{\beta}D^{\alpha}(t-a)^{s_j\alpha}\right)(x)=0, j=1,2,\ldots,n. $$

If $s>\beta-1$, then

$$ {}_a^{\beta}D^{\alpha}f(x) = {}_a^{n}T^{\alpha}\left(\frac{1}{\alpha^{n-\beta}}\frac{\Gamma(s+1)}{\Gamma(s+1+n-\beta)}(x-a)^{\alpha(s+n-\beta)}\right) $$

$$ =\frac{1}{\alpha^{n-\beta}}\frac{\Gamma(s+1)\alpha^{n}(s+n-\beta)\ldots(s+1-\beta)}{\Gamma(s+1+n-\beta)}(x-a)^{\alpha(s-\beta)}=\alpha^{\beta}\frac{\Gamma(s+1)}{\Gamma(s+1-\beta)}(x-a)^{\alpha(s-\beta)}. $$

Similarly, if s takes one of the values $s_1=0, s_2=1,\ldots,s_n=n-1$, we get

$$ {}_a^{n}T^{\alpha}((x-a)^{\alpha s_j})=0, j=1,2,\ldots,n. $$

Then, for these values of s_j, we get $\left({}_a^{C\beta}D^{\alpha}(t-a)^{s_j\alpha}\right)(x)=0$.

If $s>n-1$, then

$$ {}_a^{C\beta}D^{\alpha}f(x) = {}_a^{n-\beta}J^{\alpha}({}_a^{n}T^{\alpha}(t-a)^{s\alpha})(x) = {}_a^{n-\beta}J^{\alpha}(s\alpha(s\alpha-\alpha)\ldots(s\alpha-(n+1)\alpha)(x-a)^{s\alpha-n\alpha}) = $$

$$ =\alpha^{n}s(s-1)\ldots(s+1-n){}_a^{n-\beta}J^{\alpha}\left((x-a)^{\alpha(s-n)}\right)= $$

$$ =\alpha^{n}s(s-1)\ldots(s+1-n)\frac{1}{\alpha^{n-\beta}}\frac{\Gamma(s+1-n)}{\Gamma(s+1-\beta)}(x-a)^{\alpha(s-\beta)}=\alpha^{\beta}\frac{\Gamma(s+1)}{\Gamma(s+1-\beta)}(x-a)^{\alpha(s-\beta)}. $$

The lemma is proved. □

The following assertion was proved in [8].

Lemma 3. *Let $n-1<\beta\le n, n=1,2,\ldots$ and $f(x)\in C[a,b]$. Then, the equality*

$$ {}_a^{C\beta}D^{\alpha}\left({}_a^{\beta}J^{\alpha}f(x)\right)=f(x) \tag{8} $$

is valid.

3. Construction of a Solution to an Integral Equation

Let $\alpha,\beta>0, m=1,2,\ldots$. Let us consider in the domain $x>a$ the following integral equation

$$ \varphi(x)=\frac{\lambda(x-a)^{\alpha\beta(m-1)}}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\varphi(t)\frac{dt}{(t-a)^{1-\alpha}}+f(x). \tag{9} $$

It should be noted that for the case of the Riemann–Liouville operator, i.e., for $\alpha=1$, integral Equation (9) was studied in [38]. In this work, in the case when $\alpha=1$, based on the properties of a special Mittag–Leffler type function

$$ E_{\beta,m,l}(z)=\sum_{i=0}^{\infty}c_i z^i, c_0=1, c_i=\prod_{k=0}^{i-1}\frac{\Gamma[\beta(km+l)+1]}{\Gamma[\beta(km+l+1)+1]}, i\ge 1, \tag{10} $$

an algorithm for constructing a solution to Equation (9) was proposed for the cases when $f(x)$is a polynomial or a quasi-polynomial. The properties of the function $E_{\beta,m,l}(z)$ were also studied in [39–42].

In our case, to construct a solution to Equation (9), we use the method of normalized systems. For this purpose, we introduce the notations $L_1 = E$ and $L_2 = \lambda(x-a)^{\alpha(m-1)} \cdot {}_a^\beta J^\alpha$,where L_1 is the unit operator. Then, Equation (9) can be rewritten in the form (4).

Let $f(x) \in C[a,b]$. Let us denote $\varphi_0(x) = f(x)$ and

$$\varphi_k(x) = \left(\lambda(x-a)^{\alpha\beta(m-1)} \cdot {}_a^\beta J^\alpha\right)^k \varphi_0(x), k = 1,2,.... \tag{11}$$

It is known (see, for example, [8]) that the operator ${}_a^\beta J^\alpha$ is bounded from the space $C[a,b]$ to the space $C[a,b]$, and therefore, for each $k = 1,2,...$, an inclusion $\varphi_k(x) \in C[a,b]$ occurs.

It is obvious that

$$L_1\varphi_0(x) = E\varphi_0(x) = f(x), L_1\varphi_k(x) = \varphi_k(x) = L_2^k\varphi_0(x) = L_2\left(L_2^{k-1}\varphi_0(x)\right) = L_2\varphi_{k-1}(x).$$

Hence, the system of functions $\varphi_k(x)$ from (11) is f-normalized with respect to the pair of operators $L_1 = E, L_2 = \lambda(x-a)^{\alpha(m-1)} \cdot {}_a^\beta J^\alpha$.

The following assertion is valid.

Theorem 1. *Let $\alpha, \beta > 0, m > 0, f(x) \in C[a,b]$ and $\varphi_k(x)$ be defined by equality (11). Then, the function*

$$\varphi(x) = \sum_{k=0}^{\infty} \varphi_k(x) \tag{12}$$

is a solution to Equation (9) from class $C[a,b]$.

Proof. Let $f(x) \in C[a,b]$. Then, formally applying operators L_1 and L_2 to the series (11), we have

$$(L_1 - L_2)\varphi(x) = L_1\sum_{k=0}^{\infty}\varphi_k(x) - L_2\sum_{k=0}^{\infty}\varphi_k(x) = f(x) + \sum_{k=1}^{\infty}\varphi_k(x) - \sum_{k=0}^{\infty} L_2^{k+1} f(x)$$

$$= f(x) + \sum_{k=1}^{\infty}\varphi_k(x) - \sum_{i=1}^{\infty} L_2^i f(x) = f(x) + \sum_{k=1}^{\infty}\varphi_k(x) - \sum_{i=1}^{\infty}\varphi_i(x) = f(x).$$

Hence, function $\varphi(x)$ from (12) formally satisfies Equation (9). It remains to study the convergence of series (11). For this, let us estimate functions $\varphi_k(x)$.

For $k = 1$ we get

$$|\varphi_1(x)| = \left| \frac{\lambda(x-a)^{\alpha\beta(m-1)}}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\beta-1} f(t) \frac{dt}{(t-a)^{1-\alpha}} \right|$$

$$\leq \frac{|\lambda|(x-a)^{\alpha\beta(m-1)}}{\alpha^{\beta-1}\Gamma(\beta)} \int_a^x ((x-a)^\alpha - (t-a)^\alpha)^{\beta-1} |f(t)| \frac{dt}{(t-a)^{1-\alpha}}$$

$$\leq ||f||_{C[a,b]} \lambda(x-a)^{\alpha\beta(m-1)}\, {}_a^\beta J^\alpha(1)(x)$$

$$= ||f||_{C[a,b]} |\lambda|(x-a)^{\alpha\beta(m-1)} \frac{1}{\alpha^\beta} \frac{\Gamma(1)}{\Gamma(1+\beta)}(x-a)^{\alpha\beta} = ||f||_{C[a,b]} \frac{|\lambda|}{\alpha^\beta} \frac{\Gamma(1)}{\Gamma(1+\beta)}(x-a)^{\alpha\beta m}.$$

For $k = 2$, we get

$$|\varphi_2(x)| \leq \frac{|\lambda|(x-a)^{\alpha\beta(m-1)}}{\Gamma(\beta)} \int\limits_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} |\varphi_1(t)| \frac{dt}{(t-a)^{1-\alpha}}$$

$$\leq ||f||_{C[a,b]} \frac{|\lambda|^2}{\alpha^\beta} \frac{\Gamma(1)}{\Gamma(1+\beta)} (x-a)^{\alpha\beta(m-1)} \left({}_a^\beta J^\alpha (t-a)^{\alpha\beta m} \right)(x) =$$

$$= ||f||_{C[a,b]} \frac{|\lambda|^2}{\alpha^\beta} \frac{\Gamma(1)}{\Gamma(1+\beta)} (x-a)^{\alpha\beta(m-1)} \frac{1}{\alpha^\beta} \frac{\Gamma(\beta m+1)}{\Gamma(\beta(m+1)+1)} (x-a)^{\alpha\beta(m+1)}$$

$$= ||f||_{C[a,b]} \frac{|\lambda|^2}{\alpha^{2\beta}} \frac{\Gamma(1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta m+1)}{\Gamma(\beta m+\beta+1)} (x-a)^{2\alpha\beta m}.$$

In the general case, using the method of mathematical induction, one can prove that the inequality

$$|\varphi_k(x)| \leq ||f||_{C[a,b]} \left(\prod_{i=0}^{k-1} \frac{\Gamma[\beta i m+1]}{\Gamma[\beta(im+1/\beta+1)+1]} \right) \frac{|\lambda|^k}{\alpha^{k\beta}} (x-a)^{k\alpha\beta m}, k \geq 1$$

is satisfied. Then,

$$|\varphi(x)| \leq \sum_{k=0}^\infty |\varphi_k(x)| \leq ||f||_C \left(1 + \sum_{k=0}^\infty \left(\prod_{i=0}^{k-1} \frac{\Gamma[\beta i m+1]}{\Gamma[\beta(im+1/\beta+1)+1]} \right) \frac{|\lambda|^k}{\alpha^{k\beta}} (x-a)^{k\alpha\beta m} \right)$$

$$= ||f||_{C[a,b]} E_{\beta,m,1/\beta} \left(|\lambda| \frac{(x-a)^{\alpha\beta m}}{\alpha^\beta} \right).$$

This implies an absolute and uniform convergence of series (12) and the inclusion $\varphi(x) \in C[a,b]$. The theorem is proved. □

Now, let us construct explicit solutions of Equation (9) for particular cases of function $f(x)$.

Theorem 2. *Let $\alpha, \beta > 0, m > 0$ and $f(x) = f_0(x-a)^{\alpha\mu}, \mu > -1$, where f_0 is a real number. Then, the solution to Equation (9) is the function*

$$\varphi(x) = f_0 \cdot (x-a)^{\alpha\mu} E_{\beta,m,\mu/\beta} \left(\lambda \alpha^{-\beta} (x-a)^{\alpha\beta m} \right). \tag{13}$$

Proof. Under the conditions of this theorem, system (11) can be written as

$$\varphi_k(x) = \left(\lambda (x-a)^{\alpha\beta(m-1)} \cdot {}_a^\beta J^\alpha \right)^k f_0 (x-a)^\mu, k = 1, 2, ...$$

Find the explicit form of $\varphi_k(x)$. For $k = 1$, we get

$$\varphi_1(x) = \lambda (x-a)^{\alpha\beta(m-1)} {}_a^\beta J^\alpha (f_0 (t-a)^{\alpha\mu})(x)$$

$$= f_0 \frac{\lambda (x-a)^{\alpha\beta(m-1)}}{\alpha^\beta} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\beta)} (x-a)^{\alpha(\mu+\beta)} = f_0 \frac{\lambda}{\alpha^\beta} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\beta)} (x-a)^{\alpha\mu+\alpha\beta m}.$$

For $k = 2$, we get

$$\varphi_2(x) = \lambda (x-a)^{\alpha\beta(m-1)} {}_a^\beta J^\alpha (\varphi_1)(x)$$

$$= \lambda (x-a)^{\alpha\beta(m-1)} f_0 \frac{\lambda}{\alpha^\beta} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\beta)} \, {}_a^\beta J^\alpha \left((t-a)^{\alpha(\mu+\beta m)} \right)(x)$$

$$= (x-a)^{\alpha\beta(m-1)} f_0 \frac{\lambda^2}{\alpha^{2\beta}} \frac{\Gamma(\mu+1)}{\Gamma(\beta+\mu+1)} \frac{\Gamma(\beta m+\mu+1)}{\Gamma(\beta m+\beta+\mu+1)} (t-a)^{\alpha\mu+\alpha\beta m+\alpha\beta}$$

$$= f_0 \frac{\lambda^2}{\alpha^{2\beta}} \frac{\Gamma(\mu+1)}{\Gamma(\beta+\mu+1)} \frac{\Gamma(\beta m+\mu+1)}{\Gamma(\beta m+\beta+\mu+1)} (x-a)^{2\alpha\beta m+\alpha\mu}.$$

In the general case, for an arbitrary $k \geq 1$, we get

$$\varphi_k(x) = f_0 \frac{\lambda^k}{\alpha^{k\beta}} \left(\prod_{i=0}^{k-1} \frac{\Gamma[\beta(im+\mu/\beta)+1]}{\Gamma[\beta(im+\mu/\beta+1)+1]} \right) (x-a)^{k\alpha\beta m+\alpha\mu}.$$

Hence, for the solution of Equation (9), we obtain representation (13). The theorem is proved. □

Corollary 1. *Let $\alpha, \beta > 0, m > 0$ and $f(x) = \sum_{k=0}^{l} f_k (x-a)^{\alpha\mu_k}, \mu_k > -1$, where f_k is a real number. Then the solution to Equation (9) is written as*

$$\varphi(x) = \sum_{k=0}^{l} f_k (x-a)^{\alpha\mu_k} \cdot E_{\beta,m,\mu_k/\beta}\left(\lambda\alpha^{-\beta}(x-a)^{\alpha\beta m}\right). \tag{14}$$

Remark 1. *For case $\alpha = 1, a = 0$ representation (14) was obtained in [38].*

Corollary 2. *Let $\alpha, \beta > 0, m = 1$ and $f(x) = \sum_{k=0}^{l} f_k (x-a)^{\alpha k}$, where f_k is a real number. Then, the solution to Equation (9) is written as*

$$\varphi(x) = \sum_{k=0}^{l} k! f_k (x-a)^{\alpha k} \cdot E_{\beta,k+1}\left(\lambda\alpha^{-\beta}(x-a)^{\alpha\beta}\right),$$

where $E_{\beta,\gamma}(z)$ is a Mittag-Leffler type function [2].

4. Construction of Solutions for Homogeneous Fractional Differential Equations

Let $\alpha > 0, n-1 < \beta \leq n, \gamma > 0$. Let us introduce the notations ${}_{RL}B^{\gamma}_{\alpha,\beta} = (x-a)^{-\alpha\gamma}{}^{\beta}_{a}D^{\alpha}$, ${}_{C}B^{\alpha,\beta}_{\gamma} = (x-a)^{-\alpha\gamma}{}^{C\beta}_{a}D^{\alpha}$. Consider in the domain $x > a$ a differential equation of the type

$$\left(B^{\alpha,\beta}_{\gamma} - \lambda\right)^m y(x) = 0, x > a, \tag{15}$$

where $m = 1, 2, \ldots, B^{\alpha,\beta}_{\gamma}$ is one of the operators ${}_{RL}B^{\gamma}_{\alpha,\beta}$ or ${}_{C}B^{\alpha,\beta}_{\gamma}$.

Let $m = 1$. If we introduce the notations $L_1 = B^{\alpha,\beta}_{\gamma}$, $L_2 = \lambda$, Equation (15) can be rewritten in the form (4), and to construct a solution to this equation we have to construct a 0-normalized system with respect to operators $\left(B^{\alpha,\beta}_{\gamma}, \lambda\right)$. In this case, we will use the method proposed in [25].

Definition 2 ([25])**.** *Operator D_μ is called generalized-homogeneous of the μ order with respect to the variable t, if*

$$D_\mu t^a = C_{\mu,a} t^{a-\mu}, t > 0, \tag{16}$$

where $0 < a \leq \mu$ is a real number, $C_{\mu,a}$ is a constant.

Let $s \in R$ and D_μ be a generalized-homogeneous operator of order μ. Let us suppose that operator D_μ can be applied to the monomial $t^{\mu k+s}$. Based on equality (16), we introduce the following coefficients

$$C(\mu,s,0)=1, C(\mu,s,i)=\prod_{k=1}^{i}\left(t^{-\mu k-s+\mu}D_\mu t^{\mu k+s}\right), i\geq 1. \tag{17}$$

Let us assume that $C(\mu,s,i)\neq 0, i\geq 1$.

From (17) it follows that for coefficients $C(\mu,s,i)$ the equalities

$$\frac{1}{C(\mu,s,i)}=\frac{\left(t^{-\mu i-s}D_\mu t^{\mu i+s+\mu}\right)}{C(\mu,s,i+1)}, i\geq 1$$

hold.

Let $p=0,1,\ldots$. Consider the function

$$y_{s,p}(t)=\sum_{i=p}^{\infty}\lambda^{i-p}\begin{pmatrix} i \\ p \end{pmatrix}\frac{t^{\mu i+s}}{C(\mu,s,i)}, \tag{18}$$

where $\begin{pmatrix} i \\ p \end{pmatrix}=\frac{i!}{p!(i-p)!}$.

Theorem 3. *Let the series (18) converge and the operator D_μ can be applied term-by-term to it. If there exist such values of parameter s for which the equality $\left(t^{-\mu i-s+\mu}D_\mu t^{\mu i+s}\right)\big|_{i=0}=0$, then functions $y_{s,p}(t)$ for all such values of parameters and for all $p=0,1,\ldots,m-1$ satisfy the equation*

$$(D_\mu-\lambda)^m y(t)=0, t>0, m=1,2,\ldots.$$

From equalities (6) and (7) it follows that operators ${}_{RL}B^{\gamma}_{\alpha,\beta}$ and ${}_{C}B^{\alpha,\beta}_{\gamma}$ are generalized-homogeneous of order $\alpha(\beta+\gamma)$ with respect to $(x-a)$. Let us construct function (18) for these operators.

First consider the case for operator ${}_{C}B^{\alpha,\beta}_{\gamma}$. Let $\alpha>0, n-1<\beta\leq n$, $s_j=j, j=0,1,\ldots,n-1$ and $f_{k,j}(x)=(x-a)^{\alpha(\beta+\gamma)k+\alpha s_j}$, $k=0,1,\ldots$.

As in case (17), consider the coefficients

$$C\big(\alpha(\beta+\gamma),s_j,0\big)=1,$$

$$C\big(\alpha(\beta+\gamma),s_j,i\big)=\prod_{k=1}^{i}\left((x-a)^{-\alpha(\beta+\gamma)k-\alpha s_j+a(\beta+\gamma)}{}^{\beta}_{a}D^{\alpha}f_{k,j}(x)\right), i\geq 1.$$

By virtue of equality (7) for $k\geq 1$, we get

$$(x-a)^{-\alpha\gamma}\,{}^{C\beta}_{a}D^{\alpha}f_{k,j}(x)=\alpha^{\beta}\frac{\Gamma\big((\beta+\gamma)k+s_j+1\big)}{\Gamma\big((\beta+\gamma)k+s_j+1-\beta\big)}(x-a)^{\alpha(\beta+\gamma)k+\alpha s_j-\alpha(\beta+\gamma)}, k\geq 1.$$

Hence,

$$C\big(\alpha(\beta+\gamma),s_j,i\big)=\alpha^{i\beta}\prod_{k=1}^{i}\left(\frac{\Gamma\big((\beta+\gamma)k+s_j+1\big)}{\Gamma\big((\beta+\gamma)k+s_j+1-\beta\big)}\right). \tag{19}$$

By analogy with (18), we construct the functions

$$y_j(x)=\sum_{i=0}^{\infty}\lambda^{i}\frac{(x-a)^{\alpha(\beta+\gamma)i+s_j\alpha}}{C\big(\alpha(\beta+\gamma),s_j,i\big)}, \tag{20}$$

$$y_{j,p}(x) = \sum_{i=p}^{\infty} \lambda^{i-p} \begin{pmatrix} i \\ p \end{pmatrix} \frac{(x-a)^{\alpha(\beta+\gamma)i+s_j\alpha}}{C\big(\alpha(\beta+\gamma), s_j, i\big)}, j = 1, 2, ..., n. \tag{21}$$

Further, as

$$(\beta+\gamma)k + s_j + 1 = \beta\left[\left(1+\frac{\gamma}{\beta}\right)k + \frac{s_j}{\beta}\right] + 1 = \beta\left[\left(1+\frac{\gamma}{\beta}\right)(k-1) + \frac{s_j+\gamma}{\beta} + 1\right] + 1$$

and

$$(\beta+\gamma)k + s_j + 1 - \beta = \beta\left[\left(1+\frac{\gamma}{\beta}\right)k + \frac{s_j-\beta}{\beta}\right] + 1 = \beta\left[\left(1+\frac{\gamma}{\beta}\right)(k-1) + \frac{s_j+\gamma}{\beta}\right] + 1$$

then, introducing notations $m = 1 + \frac{\gamma}{\beta}, \ell = \frac{s_j+\gamma}{\beta}$ for coefficients $\frac{1}{C\left(\alpha(\beta+\gamma), s_j, i\right)}$ we get:

$$\begin{aligned}\frac{1}{C\big(\alpha(\beta+\gamma), s_j, i\big)} &= \frac{1}{\alpha^{i\beta}} \prod_{k=1}^{i} \left(\frac{\Gamma\big[(\beta+\gamma)k + s_j + 1 - \beta\big]}{\Gamma\big[(\beta+\gamma)k + s_j + 1\big]} \right) \\ &= \frac{1}{\alpha^{i\beta}} \prod_{k=1}^{i} \left(\frac{\Gamma[\beta(m(k-1)+\ell)+1]}{\Gamma[\beta(m(k-1)+\ell+1)+1]} \right).\end{aligned}$$

If we now change the index k to $k+1$, we finally obtain the equality

$$\frac{1}{C\big(\alpha(\beta+\gamma), s_j, i\big)} = \frac{1}{\alpha^{i\beta}} \prod_{k=0}^{i-1} \left(\frac{\Gamma[\beta(mk+\ell)+1]}{\Gamma[\beta(mk+\ell+1)+1]} \right), m = 1 + \frac{\gamma}{\beta}, \ell = \frac{s_j+\gamma}{\beta}.$$

Hence, function $y_j(x)$ in (20) satisfies the representation

$$y_j(x) = \sum_{i=0}^{\infty} \lambda^i \frac{(x-a)^{\alpha(\beta+\gamma)i+s_j\alpha}}{C\big(\alpha(\beta+\gamma), s_j, i\big)} = (x-a)^{s_j\alpha} E_{\beta,1+\frac{\gamma}{\beta},\frac{s_j+\gamma}{\beta}}\left(\lambda\alpha^{-\beta}(x-a)^{\alpha(\beta+\gamma)}\right).$$

Thus, the following assertion is valid.

Theorem 4. *Let $\alpha > 0, n-1 < \beta \le n, \gamma \ge 0, m = 1, s_j = j, j = 0, 1, ..., n-1$. Then, in case of operator $B_\gamma^{\alpha,\beta} =_C B_\gamma^{\alpha,\beta}$ solutions of Equation (15) are the functions*

$$y_j(x) = (x-a)^{\alpha j} E_{\beta,1+\frac{\gamma}{\beta},\frac{j+\gamma}{\beta}}\left(\lambda\alpha^{-\beta}(x-a)^{\alpha(\beta+\gamma)}\right), j = 0, 1, ..., n-1.$$

We can similarly transform the functions $y_{j,p}(x)$ from (21). We get

$$y_{j,p}(x) = \sum_{i=p}^{\infty} \lambda^{i-p} \begin{pmatrix} i \\ p \end{pmatrix} \frac{1}{\alpha^{i\beta}} \prod_{k=0}^{i-1} \left(\frac{\Gamma[\beta(mk+\ell)+1]}{\Gamma[\beta(mk+\ell+1)+1]} \right) (x-a)^{\alpha(\beta+\gamma)i+s_j\alpha}$$

$$\begin{aligned}&= (x-a)^{j\alpha} \sum_{n=0}^{\infty} \lambda^n \begin{pmatrix} n+p \\ p \end{pmatrix} \frac{1}{\alpha^{(n+p)\beta}} \prod_{k=0}^{n+p-1} \left(\frac{\Gamma[\beta(mk+\ell)+1]}{\Gamma[\beta(mk+\ell+1)+1]} \right) (x-a)^{\alpha(\beta+\gamma)(n+p)} \\ &= \frac{(x-a)^{\alpha(\beta+\gamma)p+j\alpha}}{\alpha^{p\beta}} \sum_{n=0}^{\infty} \frac{(p+1)_n}{n!} \prod_{k=0}^{n+p-1} \left(\frac{\Gamma[\beta(mk+\ell)+1]}{\Gamma[\beta(mk+\ell+1)+1]} \right) \left(\lambda\alpha^{-\beta}(x-a)^{\alpha(\beta+\gamma)}\right)^n \\ &= \frac{(x-a)^{\alpha(\beta+\gamma)p+j\alpha}}{\alpha^{p\beta}} E^{p+1}_{\beta,1+\gamma/\beta,(j+\gamma)/\beta}\left(\lambda\alpha^{-\beta}(x-a)^{\alpha(\beta+\gamma)}\right),\end{aligned}$$

where function $E^{p+1}_{\beta,m,\ell}(z)$ is defined by the equality

$$E^{p+1}_{\beta,m,\ell}(z) = \sum_{n=0}^{\infty} \frac{(p+1)_n}{n!} c_{n+p} z^n, c_0 = 1, c_n = \prod_{k=0}^{n-1} \frac{\Gamma[\beta(km+l)+1]}{\Gamma[\beta(km+l+1)+1]}, n \geq 1. \tag{22}$$

Theorem 5. *Let $\alpha > 0, n-1 < \beta \leq n, \gamma \geq 0, m = 1,2,..., s_j = j, j = 0,1,...,n-1$. Then, in case of operator $B^{\alpha,\beta}_{\gamma} =_C B^{\alpha,\beta}_{\gamma}$ solutions to Equation (15) are the functions*

$$y_{j,p}(x) = \frac{(x-a)^{\alpha(\beta+\gamma)p+j\alpha}}{\alpha^{p\beta}} E^{p+1}_{\beta,1+\gamma/\beta,(j+\gamma)/\beta}\left(\lambda\alpha^{-\beta}(x-a)^{\alpha(\beta+\gamma)}\right), j = 0,1,...,n-1, p = 0,$$

$1,...,m-1.$

Remark 2. *Note that for $p = 0$, we get $E^1_{\beta,m,\ell}(z) = E_{\beta,m,\ell}(z)$. In addition, the equality holds:*

$$\frac{1}{p!}\frac{\partial^p}{\partial\lambda^p} E_{\beta,m,\ell}(\lambda z) = z^p E^{p+1}_{\beta,m,\ell}(z).$$

Solutions to differential equations with the operator ${}_{RL}B^{\gamma}_{\alpha,\beta} = (x-a)^{-\alpha\gamma}\,{}^{\beta}_{a}D^{\alpha}$ are constructed in a similar way.

The following assertion is valid.

Theorem 6. *Let $\alpha > 0, n-1 < \beta \leq n, \gamma \geq 0, m = 1,2,..., s_j = \beta - j, j = 1,2,...,n$. Then in case of operator ${}_{RL}B^{\gamma}_{\alpha,\beta} = (x-a)^{-\alpha\gamma}\,{}^{\beta}_{a}D^{\alpha}$ solutions to Equation (15) are the functions*

$$y_{j,p}(x) = \frac{(x-a)^{\alpha(\beta+\gamma)p+(\beta-j)\alpha}}{\alpha^{p\beta}} E^{p+1}_{\beta,1+\gamma/\beta,(\gamma+\beta-j)/\beta}\left(\lambda\alpha^{-\beta}(x-a)^{\alpha(\beta+\gamma)}\right),$$

$j = 1,2,...,n, p = 0,1,...,m-1.$

Remark 3. *For case $\alpha = 1, a = 0$ this theorem was proved in [39], and for the case $\gamma = 0, a = 0$ it was proved in [25].*

5. Construction of Solutions to Inhomogeneous Differential Equations of Fractional Order

In this section, we consider a method for constructing a solution to inhomogeneous differential equations of fractional order with operators ${}^{\beta}_{a}D^{\alpha}$ and ${}^{C\beta}_{a}D^{\alpha}$.

Let $\alpha > 0, n-1 < \beta \leq n, \gamma, \nu \geq 0$. Consider the equation

$${}^{C\beta}_{a}D^{\alpha} y(x) = \lambda(x-a)^{\alpha\gamma}\,{}^{\nu}_{a}J^{\alpha} y(x) + f(x), a < x. \tag{23}$$

Let us introduce the notations $L_1 = {}^{C\beta}_{a}D^{\alpha}, L_2 = \lambda(x-a)^{\alpha\gamma} \cdot {}^{\nu}_{a}J^{\alpha}$. Then Equation (23) can be rewritten in the form (4).

First, we construct a solution to the homogeneous equation. To do this, we will construct 0-normalized systems with respect to the pair of operators $\left({}^{C\beta}_{a}D^{\alpha}, \lambda(x-a)^{\alpha\gamma} \cdot {}^{\nu}_{a}J^{\alpha}\right)$. From Proposition 2, it follows that for this purpose, it is necessary to find all solutions of the equation ${}^{C\beta}_{a}D^{\alpha} y(x) = 0$ and the right inverse for the operator ${}^{C\beta}_{a}D^{\alpha}$. By the proposition of Lemma 3, the right inverse to the operator ${}^{C\beta}_{a}D^{\alpha}$ is the operator ${}^{\beta}_{a}D^{\alpha}$, and by virtue of equality (8), linearly independent solutions of the equation ${}^{C\beta}_{a}D^{\alpha} y(t) = 0$ are functions $(x-a)^{\alpha j}, j = 0,1,...,n-1$.

Let $f_{0,j}(x) = (x-a)^{\alpha j}, j = 0, 1, ..., n-1$. Consider a system of functions

$$f_{i,j}(x) = \left({}_a^\beta J^\alpha \cdot \lambda (x-a)^{\alpha\gamma} \cdot {}_a^\nu J^\alpha \right)^i f_{0,j}(x), i = 1, 2, ... \tag{24}$$

Let us find an explicit form of the system of functions $f_{i,j}(x)$.
The following assertion is valid.

Lemma 4. *For functions $f_{i,j}(x)$ equalities hold*

$$f_{i,j}(x) = \frac{1}{\alpha^{i(\beta+\nu)}} C_{\beta,\nu}(\beta+\nu+\gamma, j, i)(x-a)^{\alpha(\beta+\nu+\gamma)i+\alpha j}, i = 1, 2, ..., \tag{25}$$

where

$$C_{\beta,\nu}(\beta+\nu+\gamma, j, i) = \prod_{k=0}^{i-1} \frac{\Gamma[(\beta+\nu+\gamma)k+j+1]}{\Gamma[(\beta+\nu+\gamma)k+j+1+\nu]} \times \frac{\Gamma[(\beta+\nu+\gamma)(k+1)+j+1-\beta]}{\Gamma[(\beta+\nu+\gamma)(k+1)+j+1]}. \tag{26}$$

Proof. By virtue of equality (8), we get

$${}_a^\nu J^\alpha (x-a)^{\alpha j} = \frac{1}{\alpha^\nu} \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} (x-a)^{\alpha j + \alpha\nu}.$$

Hence, for function $f_{1,j}(x)$, we obtain

$$\begin{aligned}
f_{1,j}(x) &= \left({}_a^\beta J^\alpha \cdot \lambda (x-a)^{\alpha\gamma} \cdot {}_a^\nu J^\alpha \right) f_{0,j}(x) = \frac{\lambda}{\alpha^\nu} \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} \, {}_a^\beta J^\alpha (x-a)^{\alpha(j+\nu+\gamma)} \\
&= \frac{\lambda}{\alpha^{\nu+\beta}} \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} \frac{\Gamma(\nu+\gamma+j+1)}{\Gamma(\beta+\gamma+j+1+\nu)} \, (x-a)^{\alpha(j+\nu+\gamma+\beta)} \\
&= \frac{\lambda}{\alpha^{\nu+\beta}} \frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} \frac{\Gamma((\beta+\gamma+\nu)+j+1-\beta)}{\Gamma((\beta+\gamma+\nu)+j+1)} \, (x-a)^{\alpha(\beta+\nu+\gamma+j)} \\
&= \frac{\lambda}{\alpha^{\nu+\beta}} C_{\beta,\nu}(\beta+\gamma+\nu, j, 1)(x-a)^{\alpha(\beta+\gamma+\nu+j)}.
\end{aligned}$$

If $i = 2$, then

$$\begin{aligned}
f_{1,j}(x) &= \left({}_a^\beta J^\alpha \cdot \lambda (x-a)^{\alpha\gamma} {}_a^\nu J^\alpha \right)^2 f_{0,j}(x) \\
&= \frac{\lambda}{\alpha^{\nu+\beta}} C_{\beta,\nu}(\beta+\nu+\gamma, j, 1) \left({}_a^\beta J^\alpha \cdot \lambda (x-a)^{\alpha\gamma} {}_a^\nu J^\alpha \right) (x-a)^{\alpha(\beta+\nu+\gamma+j)} \\
&= \frac{\lambda^2}{\alpha^{2\nu+\beta}} C_{\beta,\nu}(\beta+\nu+\gamma, j, 1) \frac{\Gamma(\beta+\nu+\gamma+j+1)}{\Gamma(\beta+\nu+\gamma+j+1+\nu)} \, {}_a^\beta J^\alpha (x-a)^{\alpha(\beta+2\nu+2\gamma+j)} \\
&= \frac{\lambda^2}{\alpha^{2(\nu+\beta)}} C_{\beta,\nu}(\beta+\nu+\gamma, j, 1) \frac{\Gamma(\beta+\nu+\gamma+j+1)}{\Gamma(\beta+2\nu+\gamma+j+1)} \frac{\Gamma(\beta+2\nu+2\gamma+j+1)}{\Gamma(2(\beta+\nu+\gamma)+j+1+\nu)} \\
&\cdot (x-a)^{\alpha(2\beta+2\nu+2\gamma+j)} = \frac{\lambda^2}{\alpha^{2(\nu+\beta)}} C_{\beta,\nu}(\beta+\nu+\gamma, j, 1) \frac{\Gamma((\beta+\nu+\gamma)+j+1))}{\Gamma((\beta+\nu+\gamma)+j+1+\nu)} \\
&\cdot \frac{\Gamma(2(\beta+\nu+\gamma)+j+1-\beta)}{\Gamma(2(\beta+\nu+\gamma)+j+1)} \, (x-a)^{2\alpha(\beta+\nu+\gamma)+\alpha j} \\
&= \frac{\lambda^2}{\alpha^{2(\nu+\beta)}} C_{\beta,\nu}(\beta+\nu+\gamma, j, 2)(x-a)^{\, 2\alpha(\beta+\nu+\gamma)+\alpha j}.
\end{aligned}$$

Further, let equality (25) hold for a natural number r. Then, for $r+1$, we get

$$f_{r+1,j}(x) = \left({}_a^\beta J^\alpha \cdot \lambda(x-a)^{\alpha\gamma} \cdot {}_a^\nu J^\alpha\right)^{r+1} f_{0,j}(x) = \left({}_a^\beta J^\alpha \cdot \lambda(x-a)^{\alpha\gamma} \cdot {}_a^\nu J^\alpha\right)$$

$$\cdot\left({}_a^\beta J^\alpha \cdot \lambda(x-a)^{\alpha\gamma} \cdot {}_a^\nu J^\alpha\right)^r f_{0,j}(x) = \frac{\lambda^r}{\alpha^{r(\nu+\beta)}} C_{\beta,\nu}(\beta+\nu+\gamma,j,r)\left({}_a^\beta J^\alpha \cdot \lambda(x-a)^{\alpha\gamma} \cdot {}_a^\nu J^\alpha\right)$$

$$\cdot(x-a)^{\alpha(\beta+\nu+\gamma)r+\alpha j} = \frac{\lambda^{r+1}}{\alpha^{r(\nu+\beta)+\nu}} C_{\beta,\nu}(\beta+\nu+\gamma,j,r)\frac{\Gamma[(\beta+\nu+\gamma)r+j+1]}{\Gamma[(\beta+\nu+\gamma)r+j+1+\nu]}$$

$$\cdot{}_a^\beta J^\alpha (x-a)^{\alpha(\beta+\nu+\gamma)r+\alpha j+\alpha\nu+\alpha\gamma} = \frac{\lambda^{r+1}}{\alpha^{(r+1)(\nu+\beta)}} C_{\beta,\nu}(\beta+\nu+\gamma,j,r)$$

$$\cdot\frac{\Gamma((\beta+\nu+\gamma)r+j+1)}{\Gamma((\beta+\nu+\gamma)r+j+1+\nu)}\frac{\Gamma((\beta+\nu+\gamma)r+\nu+\gamma+j+1)}{\Gamma((\beta+\nu+\gamma)r+\nu+\gamma+\beta+j+1+\nu)}$$

$$\cdot(x-a)^{\alpha(\beta+\nu+\gamma)r+\alpha j+\alpha\nu+\alpha\gamma+\alpha\beta}$$

$$= \frac{\lambda^{r+1}}{\alpha^{(r+1)(\nu+\beta)}} C_{\beta,\nu}(\beta+\nu+\gamma,j,r)\frac{\Gamma((\beta+\nu+\gamma)r+j+1)}{\Gamma((\beta+\nu+\gamma)r+j+1+\nu)}$$

$$\cdot\frac{\Gamma((\beta+\nu+\gamma)(r+1)+j+1-\beta)}{\Gamma((\beta+\nu+\gamma)(r+1)+j+1+\nu)}(x-a)^{\alpha(\beta+\nu+\gamma)r+\alpha j+\alpha\nu+\alpha\gamma+\alpha\beta}$$

$$= \frac{\lambda^{r+1}}{\alpha^{(r+1)(\nu+\beta)}} C_{\beta,\nu}(\beta+\nu+\gamma,j,r+1)(x-a)^{\alpha(\beta+\nu+\gamma)(r+1)+\alpha j}.$$

Thus, equality (25) also holds for the case $r+1$. Obviously, for the given values of parameters α,β,γ,ν, for any $i \geq 1$, the inequality $C_{\beta,\nu}(\beta+\nu+\gamma,j,i) \neq 0$ is satisfied. The lemma is proved. □

Let $C_{\beta,\nu}(\beta+\nu+\gamma,j,0) = 1$ and consider functions

$$u_j(z) = \sum_{i=0}^{\infty} C_{\beta,\nu}(\beta+\nu+\gamma,j,i)z^i, j = 0,1,...,n-1, \tag{27}$$

where $z-$ is a complex number.

If, in equality (26) $\gamma = 0$, then

$$C_{\beta,\nu}(\beta+\nu,j,i) = \prod_{k=0}^{i-1}\frac{\Gamma[(\beta+\nu)k+j+1]}{\Gamma[(\beta+\nu)k+j+1+\nu]}\cdot\frac{\Gamma[(\beta+\nu)(k+1)+j+1-\beta]}{\Gamma[(\beta+\nu)(k+1)+j+1]}$$

$$= \frac{\Gamma(j+1)}{\Gamma[i(\beta+\nu)+j+1]},$$

and

$$u_j(z) = \sum_{i=0}^{\infty}\frac{\Gamma(j+1)}{\Gamma[i(\beta+\nu)+j+1]}z^i = \Gamma(j+1)E_{\beta+\nu,j+1}(z).$$

Moreover, for $\nu = 0$ the following equality holds:

$$C_{\beta,0}(\beta+\gamma,j,i) = \prod_{k=0}^{i-1}\frac{\Gamma[(\beta+\gamma)k+j+1]}{\Gamma[(\beta+\gamma)k+j+1]}\cdot\frac{\Gamma[(\beta+\gamma)(k+1)+j+1-\beta]}{\Gamma[(\beta+\gamma)(k+1)+j+1]}$$

$$= \prod_{k=0}^{i-1}\frac{\Gamma[(\beta+\gamma)(k+1)+j+1-\beta]}{\Gamma[(\beta+\gamma)(k+1)+j+1]} = \prod_{k=0}^{i-1}\frac{\Gamma[(\beta+\gamma)k+\beta+\gamma+j+1-\beta]}{\Gamma[(\beta+\gamma)k+\beta+\gamma+j+1]}$$

$$= \prod_{k=0}^{i-1}\frac{\Gamma[\beta((1+\gamma/\beta)k+(\gamma+j)/\beta)+1]}{\Gamma[\beta((1+\gamma/\beta)k+(\gamma+j)/\beta+1)+1]} = c_i,$$

i.e., these coefficients coincide with the expansion coefficients of the function$E_{\beta,m,l}(z)$with the indices $m = 1 + \frac{\gamma}{\beta}, \ell = \frac{\gamma+j}{\beta}$. It was shown in [30] that the coefficients of the function $E_{\beta,m,l}(z)$ satisfy the asymptotic estimate

$$\frac{c_i}{c_{i+1}} = (\beta m i)^{\beta} \to \infty (i \to \infty),$$

from whence it follows that the function$E_{\beta,m,l}(z)$ is an integral function.

Let us introduce the notation $\delta = \beta + \gamma + \nu$ and rewrite coefficients $C_{\beta,\nu}(\beta + \nu + \gamma, j, i)$ as

$$C_{\beta,\nu}(\delta, j, i) = \prod_{k=0}^{i-1} \frac{\Gamma[\delta k + j + 1]}{\Gamma[\delta k + j + 1 + \nu]} \cdot \frac{\Gamma[\delta k + \delta + j + 1 - \beta]}{\Gamma[\delta k + \delta + j + 1]}$$

$$= \prod_{k=0}^{i-1} \frac{\Gamma\left[\nu\left(k\frac{\delta}{\nu} + \frac{j}{\nu}\right) + 1\right]}{\Gamma\left[\nu\left(k\frac{\delta}{\nu} + \frac{j}{\nu} + 1\right) + 1\right]} \cdot \frac{\Gamma\left[\beta\left(k\frac{\delta}{\beta} + \frac{\delta+j-\beta}{\beta}\right) + 1\right]}{\Gamma\left[\beta\left(k\frac{\delta}{\beta} + \frac{\delta+j-\beta}{\beta} + 1\right) + 1\right]}, i \geq 1, \nu > 0.$$

Further, from the asymptotic estimate

$$\frac{C_{\beta,\nu}(\delta, j, i)}{C_{\beta,\nu}(\delta, j, i+1)} = (\delta i)^{\nu+\beta} \to \infty (i \to \infty),$$

it follows that $u_j(z), j = 1, 2, ..., m$, from equality (27) it also follows that they are integral functions.

Lemma 4 and Proposition 2 imply the following lemma.

Lemma 5. *Let $\alpha > 0, n-1 < \beta \leq n, \gamma, \nu \geq 0$. Then for all values $j = 0, 1, ..., n-1$ the system of functions (25) is 0-normalized with respect to the pair of operators $\left({}^{C\beta}_{a}D^{\alpha}, \lambda(x-a)^{\alpha\gamma} \cdot {}^{\nu}_{a}J^{\alpha}\right)$ in the domain $x > a$.*

Using the main property of normalized systems, we obtain the following assertion.

Theorem 7. *Let $\alpha > 0, n-1 < \beta \leq n, \gamma, \nu \geq 0$ Then, for all values $j = 0, 1, ..., n-1$ the functions*

$$y_j(x) = \sum_{i=0}^{\infty} f_{i,j}(x) \equiv (x-a)^{\alpha j} \sum_{i=0}^{\infty} \frac{\lambda^i}{\alpha^{i(\beta+\nu)}} C_{\beta,\nu}(\beta + \nu + \gamma, j, i)(x-a)^{\alpha(\beta+\nu+\gamma)i} \tag{28}$$

are linearly independent solutions of the homogeneous Equation (23) where $y_j(t) \in C[a,b]$, ${}^{C\beta}_{a}D^{\alpha} y_j(x) \in C[a,b]$.

Proof. Consider the function

$$u_j(x) = \sum_{i=0}^{\infty} \lambda^i C_{\beta,\nu}(\beta + \nu + \gamma, j, i)(x-a)^i, j = 0, 1, ..., n-1.$$

Since function (27) is an integral function, it is obvious that

$$y_j(x) = (x-a)^{\alpha j} u_j\left(\lambda \alpha^{-(\beta+\nu)} (x-a)^{\alpha(\beta+\nu+\gamma)}\right) \in C[a,b]$$

and

$${}^{C\beta}_{a}D^{\alpha} y_j(x) = \lambda y_j(x) \in C[a,b],$$

for $j = 0, 1, ..., n-1$. Therefore, functions $y_j(x)$ from (28) are solutions to the homogeneous Equation (23). The proof of the linear independence of solutions (28) will be shown below in Theorem 11. The theorem is proved. □

Corollary 3. *Let the conditions of Theorem 7 be satisfied and $\nu = 0$. Then, the solutions to the homogeneous Equation (23) are represented as*

$$y_j(x) = (x-a)^{\alpha j} E_{\beta, 1+\frac{\gamma}{\beta}, \frac{\gamma+j}{\beta}} \left(\lambda \frac{(x-a)^{\alpha(\beta+\gamma)}}{\alpha^{(\beta+\gamma)}} \right), j = 0, 1, ..., n-1.$$

Corollary 4. *Let the conditions of Theorem 7 be satisfied and $\gamma = 0$. Then, the solutions of the homogeneous Equation (23) are represented as*

$$y_j(x) = \Gamma(j+1)(x-a)^{\alpha j} E_{\beta+\nu, j+1} \left(\lambda \frac{(x-a)^{\alpha\beta}}{\alpha^{\beta}} \right), j = 0, 1, ..., n-1.$$

Further, we will consider a method for constructing a solution to the inhomogeneous equation. Let $f(x) \in C[a,b]$. Then, by the proposition of Lemma 3, the function $f_0(x) = {}_a^{\beta} J^{\alpha} f(x)$ satisfies the equality

$$L_1 f_0(x) = {}_a^{C\beta} D^{\alpha} \left[{}_a^{\beta} J^{\alpha} f \right](x) = f(x).$$

Consider the system

$$f_i(x) = \left({}_a^{\beta} J^{\alpha} \cdot \lambda (x-a)^{\alpha\gamma} \cdot {}_a^{\nu} J^{\alpha} \right)^i f_0(x) \equiv \lambda^i \left({}_a^{\beta} J^{\alpha} (x-a)^{\alpha\gamma} \cdot {}_a^{\nu} J^{\alpha} \right)^i f_0(x), i = 1, 2, ... \quad (29)$$

Lemma 6. *Let $f(x) \in C[a,b]$. Then the system of function (29) is $f(x)$-normalized with respect to the pair of operators $\left({}_a^{C\beta} D^{\alpha}, \lambda (x-a)^{\alpha\gamma} \cdot {}_a^{\nu} J^{\alpha} \right)$ in the domain $x > a$.*

Proof. Let $f(x) \in C[a,b]$, then

$$|f_0(x)| = \left| {}_a^{\beta} J^{\alpha} f(x) \right| \le ||f||_{C[a,b]} \, {}_a^{\beta} J^{\alpha}(1)(x) \le ||f||_{C[a,b]} \frac{1}{\alpha^{\beta}} \frac{\Gamma(1)}{\Gamma(\beta+1)} (x-a)^{\alpha\beta}.$$

Hence

$$|f_0(x)| \le \frac{||f||_{C[a,b]}}{\alpha^{\beta}} \frac{\Gamma(1)}{\Gamma(\beta+1)} (x-a)^{\alpha\beta}, \ \|f_0(x)\|_{C[a,b]} \frac{||f||_{C[a,b]}}{\alpha^{\beta}} \frac{\Gamma(1)}{\Gamma(\beta+1)} (b-a)^{\alpha\beta}.$$

Further, we use the notation $M = \frac{||f||_{C[a,b]}}{\alpha^{\beta}} \frac{\Gamma(1)}{\Gamma(\beta+1)}$. Then $|f_0(x)| \le M(x-a)^{\alpha\beta}$ From the latter estimate, it follows that

$$\left| \left({}_a^{\beta} J^{\alpha} \cdot \lambda (x-a)^{\alpha\gamma} \cdot {}_a^{\nu} J^{\alpha} \right) f_0(t) \right| \le M|\lambda| \left({}_a^{\beta} J^{\alpha} \cdot (x-a)^{\alpha\gamma} \cdot {}_a^{\nu} J^{\alpha} \right) (x-a)^{\alpha\beta}.$$

Hence, for any $i \ge 1$, the estimate is valid:

$$|f_i(x)| = \left| \left({}_a^{\beta} J^{\alpha} \cdot \lambda (x-a)^{\alpha\gamma} \cdot {}_a^{\nu} J^{\alpha} \right)^i f_0(x) \right| \le M|\lambda|^i \left({}_a^{\beta} J^{\alpha} \cdot (x-a)^{\alpha\gamma} \cdot {}_a^{\nu} J^{\alpha} \right)^i (x-a)^{\alpha\beta}.$$

Let us calculate the value of the function $g_i(x) = \left({}_a^{\beta} J^{\alpha} \cdot (x-a)^{\alpha\gamma} \cdot {}_a^{\nu} J^{\alpha} \right)^i (x-a)^{\alpha\beta}$. Due to equality (25), we get

$$g_i(x) = \frac{1}{\alpha^{i(\beta+\nu)}} C_{\beta,\nu}(\beta+\nu+\gamma, \beta, i)(x-a)^{\alpha(\beta+\nu+\gamma)i+\alpha\beta}, i = 1, 2, ...$$

Hence, for any $i \geq 1$ the relation $f_i(x) \in C[a,b]$ and estimates

$$|f_i(x)| \leq M \frac{|\lambda|^i}{\alpha^{i(\beta+\nu)}} C_{\beta,\nu}(\beta+\nu+\gamma,\beta,i)(x-a)^{\alpha(\beta+\nu+\gamma)i+\alpha\beta} \tag{30}$$

are valid. Moreover,

$$\left({}_a^{\beta}J^{\alpha} \cdot \lambda(x-a)^{\alpha\gamma} \cdot {}_a^{\nu}J^{\alpha}\right)^i f_0(x) = \left({}_a^{\beta}J^{\alpha} \cdot \lambda(x-a)^{\alpha\gamma} \cdot {}_a^{\nu}J^{\alpha}\right) f_{i-1}(x) = {}_a^{\beta}J^{\alpha} g(x)$$

where

$$g(x) = \lambda(x-a)^{\alpha\gamma} \cdot {}_a^{\nu}J^{\alpha} f_{i-1}(x).$$

As $f_{i-1}(x) \in C[a,b]$, then $g(x) = \lambda(x-a)^{\alpha\gamma} \cdot {}_a^{\nu}J^{\alpha} f_{i-1}(x)$ also belongs to class $C[a,b]$ and the equality is satisfied:

$$L_1 f_i(x) = {}_a^{C\beta}D^{\alpha}\left({}_a^{\beta}J^{\alpha} \cdot \lambda(x-a)^{\alpha\gamma} \cdot {}_a^{\nu}J^{\alpha}\right)^i f_0(x) = {}_a^{C\beta}D^{\alpha}\left[{}_a^{\beta}J^{\alpha} g\right](x) = g(x)$$

$$= \lambda(x-a)^{\alpha\gamma} \cdot {}_a^{\nu}J^{\alpha} f_{i-1}(x) = L_2 f_{i-1}(x), i \geq 1.$$

It is obvious that $L_1 f_0(x) = {}_a^{C\beta}D^{\alpha}(f_0)(x) = f(x)$. Thus, in the class of functions $X = C[a,b]$, the equalities

$$L_1 f_0(x) = f(x), L_1 f_i(x) = L_2 f_{i-1}(x), i \geq 1$$

hold, i.e., system (29) is $f-$ normalized with respect to the pair of operators

$$\left({}_a^{C\beta}D^{\alpha}, \lambda(x-a)^{\alpha\gamma} \cdot {}_a^{\nu}J^{\alpha}\right).$$

The lemma is proved. □

Theorem 8. *Let $f(x) \in C[a,b]$, $f_0(x) = {}_a^{\beta}J^{\alpha} f(x)$ and function $f_i(x), i \geq 1$ are defined by equality (29). Then, the function*

$$y_f(x) = \sum_{i=0}^{\infty} f_i(x) \tag{31}$$

is a particular solution of Equation (23) from the class $C[a,b]$.

Proof. Let us estimate the series (31). By virtue of estimate (30), we have

$$|y_f| \leq \sum_{i=0}^{\infty} |f_i(t)| \leq M||f||_{C[0,d]}(x-a)^{\alpha\beta}\left[1 + \sum_{i=1}^{\infty} |\lambda|^i C_{\beta,\nu}(\beta+\gamma+\nu,\beta,i)(x-a)^{i(\beta+\gamma+\nu)}\right]$$

As the latter series converges uniformly in the domain $a \leq t \leq b$, the sum of this series, and hence the function $y_f(t)$, belong to class $C[a,b]$. The theorem is proved.

Let us investigate the representation of function (31) for some special cases of function $f(x)$. □

Lemma 7. *Let $f(x) = (x-a)^{\alpha\mu}, \mu \geq 0$. Then, the particular solution of Equation (23) is written as*

$$y_{\mu}(t) = \frac{1}{\alpha^{\beta}} \frac{\Gamma(\mu+1)(x-a)^{\alpha(\mu+\beta)}}{\Gamma(\mu+1+\beta)} \sum_{i=0}^{\infty} \frac{\lambda^i}{\alpha^{i(\beta+\nu)}} C_{\beta,\nu}(\beta+\gamma+\nu,\mu+\beta,i)(x-a)^{\alpha i(\beta+\gamma+\nu)}.$$

Proof. In this case, for $f_i(x)$ from (29), we get

$$f_i(x) = \left({}^{\beta}_{a}J^{\alpha} \cdot \lambda(x-a)^{\alpha\gamma} \cdot {}^{\nu}_{a} J^{\alpha}\right)^{i} {}^{\beta}_{a}J^{\alpha}(x-a)^{\alpha\mu} = \frac{1}{\alpha^{\beta}}\frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\beta)}\left({}^{\beta}_{a}J^{\alpha} \cdot \lambda(x-a)^{\alpha\gamma} \cdot {}^{\nu}_{a} J^{\alpha}\right)^{i}$$

$$\cdot(x-a)^{\alpha(\mu+\beta)} = \frac{1}{\alpha^{\beta}}\frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\beta)}\frac{1}{\alpha^{i(\beta+\nu)}}C_{\beta,\nu}(\beta+\gamma+\nu,\mu+\beta,i)(x-a)^{\alpha i(\beta+\gamma+\nu)+\alpha(\mu+\beta)}.$$

The lemma is proved. □

This lemma implies the following assertion.

Theorem 9. *Let* $f(x) = \sum\limits_{j=1}^{p}\lambda_j(x-a)^{\mu_j}, \mu_j \geq 0\ \mu_j > -1$. *Then the particular solution of Equation* (23) *is written as*

$$y_f(t) = \sum_{j=1}^{p}\frac{\lambda_j\Gamma(\mu_j+1)(x-a)^{\alpha(\beta+\mu_j)}}{\alpha^{\beta}\Gamma(\mu_j+1+\beta)} \times \sum_{i=0}^{\infty}\frac{\lambda^i}{\alpha^{i(\beta+\nu)}}C_{\beta,\nu}(\beta+\gamma+\nu,\mu_j+\beta,i)(x-a)^{i\alpha(\beta+\gamma+\nu)}. \quad (32)$$

Remark 4. *In case* $\alpha = 1$ *representation* (32) *of a particular solution of Equation* (23) *coincides with the result of* [27].

Further, let us investigate the following Cauchy-type problem

$${}^{C\beta}_{a}D^{\alpha}y(x) = \lambda(x-a)^{\alpha\gamma} \cdot {}^{\nu}_{a} J^{\alpha}y(x) + \sum_{j=1}^{p}\lambda_j(x-a)^{\alpha\mu_j}, \mu_j \geq 0, a < x, \quad (33)$$

$${}^{m}_{a}T^{\alpha}y(x)|_{x=a} = d_m, m = 0,1,...,n-1 \quad (34)$$

where d_k are real numbers. First, let us consider the homogeneous problem (33), (34).

Theorem 10. *Let* $\lambda_j = 0, j = 1,2,...,p$. *Then, the solution to the Cauchy problem* (33), (34) *exists, is unique, and can be represented as*

$$y(x) = \sum_{j=0}^{n-1}\frac{d_j}{\alpha^j\Gamma(j+1)}(x-a)^{\alpha j}\sum_{i=0}^{\infty}\frac{\lambda^i}{\alpha^{i(\beta+\nu)}}C_{\beta,\nu}(\beta+\nu+\gamma,j,i)(x-a)^{\alpha(\beta+\nu+\gamma)i}. \quad (35)$$

Proof. Let $\lambda_j = 0, j = 1,2,...,p$. According to Theorem 7, function $y(x)$ in (35) is a solution to Equation (33). Let us show that $y(x)$ satisfies the initial conditions (34). For function $(x-a)^{\alpha j}$we have

$${}^{m}_{a}T^{\alpha}(x-a)^{\alpha j} = 0, m > j,\ {}^{m}_{a}T^{\alpha}(x-a)^{\alpha j} = \alpha j(\alpha j-\alpha)...(\alpha j-(m-1)\alpha)(x-a)^{\alpha(j-m)}$$

$$= \alpha^m j(j-1)...(j-(m-1))(x-a)^{\alpha(j-m)} = \frac{\alpha^m\Gamma(j+1)}{\Gamma(j+1-m)}(x-a)^{\alpha(j-m)}, m = 1,2...,j.$$

Then

$$\lim_{x\to a}{}^{m}_{a}T^{\alpha}(x-a)^{\alpha j} = \begin{cases}\alpha^m\Gamma(m+1), m = j \\ 0, m < j\end{cases}.$$

Hence, for function $y(x)$ we get

$${}^{m}_{a}T^{\alpha}y(x)|_{x=a} = \lim_{x\to a}{}^{m}_{a}T^{\alpha}y(x) = \frac{d_m}{\alpha^m\Gamma(k+1)}\alpha^m\Gamma(k+1) = d_m. \quad (36)$$

The theorem is proved. □

From Theorem 10 the following theorem can be derived.

Theorem 11. *Functions $y_j(t), j = 0, 1, ..., n-1$ from (28) are linearly independent.*

Proof. For functions $y_0(t), y_1(t), ..., y_{n-1}(t)$ we introduce an analogue of Wronskian: $W_\alpha(x) = \det\left({}_a^m T^\alpha y_j(x)\right)_{m,j=1}^{n-1}, a \leq x \leq b$.

As in the case of the theorem for linear differential equations of order n, the following statement can be proved. □

Lemma 8. *For solutions $y_1(t), y_2(t), ..., y_m(t)$ to Equation (33) be linearly independent, it is necessary and sufficient that $W_\alpha(x_0) \neq 0$ at a point $x_0 \in [a, b]$.*

According to (36) we get $W_\alpha(a) \neq 0$ and, hence, according to the lemma, the solutions $y_1(t), y_2(t), ..., y_m(t)$ to Equation (33) are linearly independent. The theorem is proved.

Theorems 9 and 10 imply the following assertion.

Theorem 12. *If $\mu_j \geq 0, j = 1, 2, ..., p,$, then the solution to the Cauchy problem (33), (34) exists, is unique, and can be represented as*

$$
\begin{aligned}
y(x) = &\sum_{j=0}^{n-1} \frac{d_j}{\alpha^j \Gamma(j+1)} (x-a)^{\alpha j} \sum_{i=0}^{\infty} \frac{\lambda^i}{\alpha^{i(\beta+\nu)}} C_{\beta,\nu}(\beta+\nu+\gamma, j, i)(x-a)^{\alpha(\beta+\nu+\gamma)i} \\
&+ \sum_{j=1}^{p} \frac{\lambda_j \Gamma(\mu_j+1)(x-a)^{\alpha(\beta+\mu_j)}}{\alpha^\beta \Gamma(\mu_j+1+\beta)} \sum_{i=0}^{\infty} \frac{\lambda^i}{\alpha^{i(\beta+\nu)}} C_{\beta,\nu}(\beta+\gamma+\nu, \mu_j+\beta, i) \\
&\times (x-a)^{i\alpha(\beta+\gamma+\nu)}. \qquad (37)
\end{aligned}
$$

Corollary 5. *Let the conditions of Theorem 12 be satisfied and $\nu = 0$. Then, the solution to the Cauchy problem (33), (34) is represented as*

$$
\begin{aligned}
y(x) = &\sum_{j=0}^{n-1} \frac{d_j}{\alpha^j \Gamma(j+1)} (x-a)^{\alpha j} E_{\beta, 1+\frac{\gamma}{\beta}, \frac{\gamma+j}{\beta}}\left(\lambda \frac{(x-a)^{\alpha(\beta+\gamma)}}{\alpha^\beta}\right) \\
&+ \sum_{j=1}^{p} \frac{\lambda_j \Gamma(\mu_j+1)}{\alpha^\beta \Gamma(\mu_j+1+\beta)} (x-a)^{\alpha(\beta+\mu_j)} E_{\beta, 1+\frac{\gamma}{\beta}, \frac{\gamma+\mu_j}{\beta}}\left(\lambda \frac{(x-a)^{\alpha(\beta+\gamma)}}{\alpha^\beta}\right). \qquad (38)
\end{aligned}
$$

In case $\alpha = 1, a = 0$ the obtained representation of the solution coincides with Formula (65) in the work of A.A. Kilbas and M. Saigo [39].

Remark 5. *If $\gamma = 0$, then it is not difficult to find an explicit form of the system $f_i(x)$ from (29).*

Indeed, in this case, by virtue of Formula (31) from [28], the equality ${}_a^\beta J^\alpha \cdot {}_a^\nu J^\alpha = {}_a^{\beta+\nu} J^\alpha$ is valid and thus

$$
f_i(x) = \left({}_a^\beta J^\alpha \cdot \lambda \cdot {}_a^\nu J^\alpha\right)^i f_0(x) \equiv \lambda^i {}_a^{(\beta+\nu)i+\beta} f_0(x) = \lambda^i {}_a^{(\beta+\nu)i+\beta} J^\alpha f(x), i = 1, 2, \ldots.
$$

Then

$$
y_f(x) = \sum_{i=0}^{\infty} f_i(x) = \sum_{i=0}^{\infty} \lambda^i \cdot {}_a^{(\beta+\nu)i+\beta} J^\alpha f(x)
$$

$$
= \sum_{i=0}^{\infty} \frac{\lambda^i}{\Gamma[(\beta+\nu)i+\beta]} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{(\beta+\nu)i+\beta-1} f(t) \frac{dt}{(t-a)^{1-\alpha}}
$$

$$= \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1}$$

$$\cdot \left(\sum_{i=0}^{\infty} \frac{\lambda^i}{\Gamma[(\beta+\nu)i+\beta]} \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{(\beta+\nu)i} \right) f(t) \frac{dt}{(t-a)^{1-\alpha}}$$

$$= \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} E_{\beta+\nu,\beta}\left(\lambda \frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right) f(t) \frac{dt}{(t-a)^{1-\alpha}}.$$

In the simple case when $\gamma = 0$ from the last formula and from the statement of Theorem 10 the following assertion follows.

Corollary 6. *Let $f(t)$ be a smooth function. Then, the solution to the Cauchy problem*

$$ {}_a^{C\beta}D^\alpha y(x) = \lambda {}_a^\nu J^\alpha y(x) + f(x), a < x, \tag{39}$$

$$ {}_a^m T^\alpha y(x)|_{x=a} = d_m, m = 0, 1, ..., n-1 \tag{40}$$

is the function

$$y(x) = \sum_{k=0}^{n-1} \frac{d_k (x-a)^{\alpha k}}{\alpha^k} E_{\beta+\nu,k+1}\left(\lambda \frac{(x-a)^{\alpha(\beta+\nu)}}{\alpha^{(\beta+\nu)}} \right)$$
$$+ \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} E_{\beta+\nu,\beta}\left(\lambda \frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right) f(t) \frac{dt}{(t-a)^{1-\alpha}}. \tag{41}$$

In particular, for $n = 1, \nu = 0$ we get

$$y(x) = d_0 E_{\beta,1}\left(\lambda \frac{(x-a)^{\alpha\beta}}{\alpha^\beta} \right) +$$

$$\int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} E_{\beta,\beta}\left(\lambda \frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right) f(t) \frac{dt}{(t-a)^{1-\alpha}}.$$

This formula for $a = 0$ and $0 < \alpha \le 1$ was obtained in [30].

In conclusion, we will consider an example of applying the results obtained to the equation in the theory of electrical circuits.

Example 1. *Let $0 < \beta \le 1, \alpha, \gamma \ge 0$. Consider the following Cauchy problem*

$$\left({}_a^{C\beta}D^\alpha V \right)(t) + \rho (t-a)^{\gamma\alpha} \cdot {}_a^\nu J^\alpha V(t) = A, t > a, V(0) = V_0. \tag{42}$$

where A, ρ, V_0 are given as real numbers, $V(t)$ is an unknown function. By virtue of Formula (37), the solution to problem (42) is the function

$$V(t) = V_0 \cdot \sum_{i=0}^{\infty} \frac{(-\rho)^i}{\alpha^{i(\beta+\nu)}} C_{\beta,\nu}(\beta+\nu+\gamma, 0, i)(t-a)^{\alpha(\beta+\nu+\gamma)i}$$

$$+ A \frac{(t-a)^{\alpha\beta}}{\alpha^\beta \Gamma(1+\beta)} \sum_{i=0}^{\infty} \frac{(-\rho)^i}{\alpha^{i(\beta+\nu)}} C_{\beta,\nu}(\beta+\gamma+\nu, \beta, i)(t-a)^{i\alpha(\beta+\gamma+\nu)}.$$

If $\nu = 0$, *the solution to problem* (42) *is represented as*

$$V(t) = V_0 \cdot E_{\beta,\frac{\gamma}{\beta}+1,\frac{\gamma}{\beta}}\left(-\frac{\rho}{\alpha^\beta}(t-a)^{\alpha(\beta+\gamma)}\right) + A\frac{(t-a)^{\alpha\beta}}{\alpha^\beta\Gamma(\beta+1)}E_{\beta,\frac{\gamma}{\beta}+1,\frac{\gamma+\beta}{\beta}}\left(-\frac{\rho}{\alpha^\beta}(t-a)^{\alpha(\beta+\gamma)}\right). \quad (43)$$

If $\nu = 0, \gamma = 0$, *then Equation* (42) *coincides with the differential equation of motion of electrons in metals (the Drude model), considered in* [29]. *In this case, function* (43) *will be written as*

$$y(t) = V_0 \cdot E_{\beta,1}\left(-\frac{\rho}{\alpha^\beta}(t-a)^{\alpha(\beta+\gamma)}\right) + A\frac{(t-a)^{\alpha\beta}}{\alpha^\beta}E_{\beta,\beta+1}\left(-\frac{\rho}{\alpha^\beta}(t-a)^{\alpha(\beta+\gamma)}\right). \quad (44)$$

As

$$E_{\beta,\beta+1}\left(-\frac{\rho}{\alpha^\beta}(t-a)^{\alpha(\beta+\gamma)}\right) = \sum_{i=0}^{\infty}\frac{(-\rho)^i}{\alpha^{i\beta}}\frac{(t-a)^{i\alpha\beta}}{\Gamma(\beta k+\beta+1)} = \sum_{i=0}^{\infty}\frac{(-\rho)^i}{\alpha^{i\beta}}\frac{(t-a)^{i\alpha\beta}}{\beta(k+1)\Gamma(\beta k+1)},$$

function (44) *coincides with Formula (40) obtained in* [29].

Remark 6. *Similar investigations can be carried out for the equation with the operator* ${}_a^\beta D^\alpha$.

Author Contributions: B.K.T., K.I.U. and K.Z.N. contributed equally to the conceptualization and the formal analysis of the problem discussed in the paper. All authors have read and agreed to the published version of the manuscript.

Funding: The work was supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (grant no. AP08956307).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All the data is present within the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hilfer, R. *Applications of Fractional Calculus in Physics*, 1st ed.; World Scientific: Singapore, 2000; ISBN 9810234570.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Application of Fractional Differential Equations*, 1st ed.; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; ISBN 0444518320/9780444518323.
3. Magin, R.L. *Fractional Calculus in Bioengineering*; Begell House Publishers: Redding, CA, USA, 2006; ISBN 1567002153/978-1567002157.
4. Podlubny, I. *Fractional Differential Equations*, 1st ed.; Academic Press: San Diego, CA, USA, 1999; ISBN 558840-2.
5. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*, 1st ed.; CRC Press: Boca Raton, FL, USA, 1993; ISBN 2881248640/978-2881248641.
6. Baleanu, D.; Fernandez, A. On Fractional Operators and Their Classifications. *Mathematics* **2019**, *7*, 830. [CrossRef]
7. Bargamadi, E.; Torkzadeh, L.; Nouri, K.; Jajarmi, A. Solving a system of fractional-order volterra-fredholm integro-differential equations with weakly singular kernels via the second Chebyshev wavelets method. *Fractal Fract.* **2021**, *5*, 70.
8. Jarad, F.; Ugurlu, E.; Abdeljawad, T.; Baleanu, D. On a new class of fractional operators. *Adv. Differ. Equ.* **2017**, *247*, 1–16. [CrossRef]
9. Katugampola, U.N. New approach to generalized fractional integral. *Appl. Math. Comput.* **2011**, *218*, 860–865. [CrossRef]
10. Hilfer, R.; Luchko, Y.; Tomovski, Z. Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives. *Fract. Calc. Appl. Anal.* **2009**, *12*, 299–318.
11. Luchko, Y. Operational method in fractional calculus. *Fract. Calc. Appl. Anal.* **1999**, *2*, 463–488.
12. Luchko, Y.; Gorenflo, R. An operational method for solving fractional differential equations. *Acta Math. Vietnam.* **1999**, *24*, 207–234.
13. Luchko, Y.; Srivastava, H.M. The exact solution of certain differential equations of fractional order by using operational calculus. *Comput. Math. Appl.* **1995**, *29*, 73–85. [CrossRef]

14. Luchko, Y. Operational calculus for the general fractional derivative and its applications. *Fract. Calc. Appl. Anal.* **2021**, *24*, 338–375. [CrossRef]
15. Luchko, Y. General Fractional Integrals and Derivatives with the Sonine Kernels. *Mathematics* **2021**, *9*, 594. [CrossRef]
16. Kim, M.H.; Ri, G.C.; Hyong-Chol, O. Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives. *Fract. Calc. Appl. Anal.* **2014**, *17*, 79–95. [CrossRef]
17. Pskhu, A.V. Initial-value problem for a linear ordinary differentialequation of noninteger order. *Sb. Math.* **2011**, *202*, 571–582. [CrossRef]
18. Cheng, J.F.; Chu, Y.M. Solution to the linear fractional differentialequation using Adomian decomposition method. *Math. Probl. Eng.* **2011**, *1*, 1–14. [CrossRef]
19. Hu, Y.; Luo, Y.; Lu, Z. Analytical solution of the linear fractional differential equation by Adomian decomposition method. *J. Comput. Appl. Math.* **2008**, *215*, 220–229. [CrossRef]
20. Karachik, V.V. Normalized system of functions with respect to the Laplace operator and its applications. *J. Math. Anal. Appl.* **2003**, *287*, 577–592. [CrossRef]
21. Karachik, V.V. Method for constructing solutions of linear ordinary differential equations with constant coefficients. *Comput. Math. Math. Phys.* **2012**, *52*, 219–234. [CrossRef]
22. Liu, L.; Ren, G.B. Normalized system for wave and Dunkl operators. *Taiwan. J. Math.* **2007**, *14*, 675–683. [CrossRef]
23. Yuan, H.F.; Karachik, V.V. Dunkl–Poisson Equation and Related Equations in Superspace. *Math. Model. Anal.* **2015**, *20*, 768–781. [CrossRef]
24. Ashurov, R.R.; Cabada, A.; Turmetov, B.K. Operator method for construction of solutions of linear fractional differential equations with constant coefficients. *Fract. Calc. Appl. Anal.* **2016**, *19*, 229–251. [CrossRef]
25. Turmetov, B. On Certain Operator Method for Solving Differential Equations. *Filomat* **2017**, *31*, 4275–4286. [CrossRef]
26. Turmetov, B. Construction of explicit solutions of fractional integral-differential equations of Volterra type. *AIP Conf. Proc.* **2017**, *1880*, 1–6.
27. Turmetov, B.K. On a method for constructing a solution of integral-differential equations of fractional order. *Electron. J. Qual. Theory Differ. Equ.* **2018**, *25*, 1–14. [CrossRef]
28. Gambo, Y.Y.; Ameen, R.; Jarad, F.; Abdeljawad, T. Existence and uniqueness of solutions to fractional differential equations in the frame of generalized Caputo fractional derivatives. *Adv. Differ. Equ.* **2018**, *134*, 1–13. [CrossRef]
29. Gómez-Aguilar, J.F. Novel analytical solutions of the fractional Drude model. *Optik* **2018**, *168*, 728–740. [CrossRef]
30. Jarad, F.; Abdeljawad, T. A modified Laplace transform for certain generalized fractional operators. *Results Nonlinear Anal.* **2018**, *1*, 88–98.
31. Morales-Delgado, V.F.; Gómez-Aguilar, J.F.; Taneco-Hernández, M.A. Analytical solutions of electrical circuits described by fractional conformable derivatives in Liouville-Caputo sense. *Aeu-Int. J. Electron. Commun.* **2018**, *85*, 108–117. [CrossRef]
32. Sene, N.; Gómez-Aguilar, J.F. Analytical solutions and numerical schemes of certain generalized fractional diffusion models. *Eur. Phys. J. Plus.* **2019**, *134*, 1–16. [CrossRef]
33. Abdeljawad, T.; Al-Madallal, Q.M.; Jarad, F. Fractional logistic models in the frame of fractional operators generated by conformable derivatives. *Chaos Solitons Fractals* **2019**, *119*, 94–101. [CrossRef]
34. Gómez-Aguilar, J.F. Fundamental solutions to electrical circuits of non-integer order via fractional derivatives with and without singular kernels. *Eur. Phys. J. Plus* **2018**, *197*, 1–20. [CrossRef]
35. Morales-Delgado, V.F.; Gómez-Aguilar, J.F.; Escobar-Jiménez, R.F.; Taneco-Hernández, M.A. Fractional conformable derivatives of Liouville–Caputo type with low-fractionality. *Phys. A* **2018**, *503*, 424–438. [CrossRef]
36. Morales-Delgado, V.F.; Gómez-Aguilar, J.F.; Taneco-Hernández, M.A. Mathematical modeling approach to the fractional Bergman's model. *Discret. Contin. Dyn. Syst. Ser. S* **2020**, *13*, 805–821. [CrossRef]
37. Péres, J.E.S.; Gómez-Aguilar, J.F.; Baleanu, D.; Tchier, F. Chaotic Attractors with Fractional Conformable Derivatives in the Liouville–Caputo Sense and Its Dynamical Behaviors. *Entropy* **2018**, *20*, 384. [CrossRef]
38. Kilbas, A.A.; Saigo, M. On solution of integral equation of Abel-Volterra type. *Differ. Integral Equ.* **1995**, *8*, 993–1011.
39. Kilbas, A.A.; Saigo, M. Solution in closed form of a class of linear differential equations of fractional order. *Differ. Uravneniya* **1997**, *33*, 95–204. (In Russian)
40. Kilbas, A.A.; Saigo, M. Fractional integral and derivatives of Mittag-Leffler type function. *Dokl. Akad. Nauk Belarus* **1995**, *39*, 22–26. (In Russian)
41. Kilbas, A.A.; Saigo, M. On Mittag-Leffler type function, fractional calculus operators and solutions of integral equations. *Integral Transform. Spec. Funct.* **1996**, *4*, 355–370. [CrossRef]
42. Boudabsa, L.; Simon, T. Some Properties of the Kilbas-Saigo Function. *Mathematics* **2021**, *9*, 217. [CrossRef]