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Article

On Discrete Delta Caputo–Fabrizio Fractional Operators and Monotonicity Analysis

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Abstract: The discrete delta Caputo-Fabrizio fractional differences and sums are proposed to distinguish their monotonicity analysis from the sense of Riemann and Caputo operators on the time scale $\mathcal{Z}$. Moreover, the action of $\mathcal{Q}-$ operator and discrete delta Laplace transform method are also reported. Furthermore, a relationship between the discrete delta Caputo-Fabrizio-Caputo and Caputo-Fabrizio-Riemann fractional differences is also studied in detail. To better understand the dynamic behavior of the obtained monotonicity results, the fractional difference mean value theorem is derived. The idea used in this article is readily applicable to obtain monotonicity analysis of other discrete fractional operators in discrete fractional calculus.

Keywords: delta caputo-fabrizio fractional operators; υ-monotonicity analysis; fractional difference mean value theorem

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1. Introduction

In recent years increasing research attention has been devoted to the study of discrete fractional calculus (DFC) and its various models. Moreover, DFC has been suitably characterized the term "memory" especially in physics, economics, mathematics, biology, engineering, control etc., and its study is not only interesting from a purely mathematical point of view, but has been found extremely useful for modeling super-diffusion processes, which naturally appear in many applications in biology, probability, physics, economics, medicine and ecology (see [1–4]). Additionally, there are some recent works on variable-order fractional difference equations such as [5–10] in discrete fractional calculus.

Recently, several authors have begun to study monotonicity analysis in the context of discrete fractional calculus, especially in fractional difference equations. They have often obtained some υ-increasing and υ-decreasing results for the discrete nabla and delta operators. Additionally, there are many discrete models that have been studied in their research articles. Atici and Uyanik [11] obtained several monotonicity analysis results for the discrete nabla Riemann–Liouville fractional operators on the time scale $\mathcal{Z}$. Moreover, Suwan et al. [12] obtained some new results for the discrete delta Riemann–Liouville fractional operators on the time scale $h\,\mathcal{Z}$. For the discrete nabla Attangana–Baleanu fractional operators, several monotonicity analysis results were obtained by Abdeljawad and Baleanu [13] on the time scale $\mathcal{Z}$, and Suwan et al. [14] on the time scale $h\,\mathcal{Z}$. Abdeljawad and Abdallaa [15] used the dual identities to obtain some monotonicity results for the discrete nabla and delta Riemann–Liouville and Caputo fractional operators on the time scale $\mathcal{Z}$. Goodrich et al. [16] obtained some analysis results including monotonicity for the discrete fractional operators with exponential kernels. Recently, Mohammed et al. [17] established new monotonicity results for discrete generalized nabla Attangana–Baleanu fractional operators with discrete generalized Mittag–Leffler kernels on the time scale $\mathcal{Z}$.

Motivated by the article [18], this article is devoted to a detailed study of discrete delta Caputo–Fabrizio (CF) fractional operators, the associated monotonicity analysis of the operators, related concepts such as the discrete delta Laplace transform method, the relationship between the Riemann and Caputo operators, and fractional difference mean value theorem. However, Abdeljawad and Baleanu [18] have established different results for discrete nabla CF fractional operators due to the difference in the kernels of discrete delta and nabla CF fractional operators; however, their results are not in detail.

Specifically, the structure of this article is as follows. We discuss the discrete delta CF fractional differences in Section 2. We derive the discrete delta CF fractional sums and we present some related properties in Section 3, then pass to monotonicity analysis of the discrete delta operators in Section 4, first discrete delta Caputo-Fabrizio-Caputo (CFC) operators and then discrete delta Caputo-Fabrizio-Riemann (CFR) operators. In Section 5, we investigate the discrete fractional difference Mean Value Theorem (MVT) based on the monotonicity results. Section 6 is devoted to discussion and conclusion of our article.

2. Preliminaries and Basic Concepts

We first indicate the definitions of the discrete delta CF fractional operators that we will consider in this article.

Definition 1 (see [19])**.** *Let $\lambda = -\frac{\nu}{1-\nu}$, $\nu \in [0,1)$ and $a, b \in \mathsf{R}$. Let $\Delta\rho(\mathtt{w}) = \rho(\mathtt{w}+1) - \rho(\mathtt{w})$ be the forward difference operator and $\nabla\rho(\mathtt{w}) = \rho(\mathtt{w}) - \rho(\mathtt{w}-1)$ be the backward difference operator. Then, for any function ρ defined on $\mathsf{G}_a := \{a, a+1, \ldots\}$, the left discrete delta CFC and CFR fractional differences are, respectively, defined by*

$$\begin{aligned}\left({}^{CFC}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) &= \frac{\mathcal{N}(\nu)}{1-\nu}\sum_{\ell=a}^{\mathtt{w}-1}(\Delta_{\ell}\rho)(\ell)(1+\lambda)^{\mathtt{w}-\sigma(\ell)} \\ &= \frac{\mathcal{N}(\nu)}{1-2\nu}\sum_{\ell=a}^{\mathtt{w}-1}(\Delta_{\ell}\rho)(\ell)(1+\lambda)^{\mathtt{w}-\ell}, \end{aligned} \tag{1}$$

and

$$\begin{aligned}\left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) &= \frac{\mathcal{N}(\nu)}{1-\nu}\Delta_{\mathtt{w}}\sum_{\ell=a}^{\mathtt{w}-1}\rho(\ell)(1+\lambda)^{\mathtt{w}-\sigma(\ell)} \\ &= \frac{\mathcal{N}(\nu)}{1-2\nu}\Delta_{\mathtt{w}}\sum_{\ell=a}^{\mathtt{w}-1}\rho(\ell)(1+\lambda)^{\mathtt{w}-\ell}. \end{aligned} \tag{2}$$

Additionally, for any function ρ defined on ${}_b\mathsf{G} := \{\ldots, b-1, b\}$, the right discrete delta CFC and CFR fractional differences are, respectively, defined by

$$\begin{aligned}\left({}^{CFC}\Delta^{\nu}_{b}\rho\right)(\mathtt{w}) &= \frac{\mathcal{N}(\nu)}{1-\nu}\sum_{\ell=\mathtt{w}+1}^{b}(-\nabla_{\ell}\rho)(\ell)(1+\lambda)^{\ell-\sigma(\mathtt{w})} \\ &= \frac{\mathcal{N}(\nu)}{1-2\nu}\sum_{\ell=\mathtt{w}+1}^{b}(-\nabla_{\ell}\rho)(\ell)(1+\lambda)^{\ell-\mathtt{w}}, \end{aligned} \tag{3}$$

and

$$\begin{aligned}\left({}^{CFR}\Delta^{\nu}_{b}\rho\right)(\mathtt{w}) &= \frac{\mathcal{N}(\nu)}{1-\nu}(-\nabla_{\mathtt{w}})\sum_{\ell=\mathtt{w}+1}^{b}\rho(\ell)(1+\lambda)^{\ell-\sigma(\mathtt{w})} \\ &= \frac{\mathcal{N}(\nu)}{1-2\nu}(-\nabla_{\mathtt{w}})\sum_{\ell=\mathtt{w}+1}^{b}\rho(\ell)(1+\lambda)^{\ell-\mathtt{w}}, \end{aligned} \tag{4}$$

where a normalizing positive constant $\mathcal{N}(\nu)$ satisfying $\mathcal{N}(0) = \mathcal{N}(1) = 1$.

Remark 1. *By comparing Definition 1 and Definition 1 of [18], we can notice a few differences between the discrete delta and nabla CF fractional operators, e.g., their kernels. This and further differences in their properties confirm that our results in this article are quite different from those obtained in [18]. To see further differences, we advise readers to read both articles.*

Remark 2 (see [19]). *From Definition 1, we can note the following limiting cases:*

(i) *As $\nu \to 0$, we have*

$$\left({}^{CFC}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) \to \rho(\mathtt{w}) - \rho(a), \quad \text{and} \quad \left({}^{CFC}\Delta\Delta^{\nu}_{b}\rho\right)(\mathtt{w}) \to \rho(\mathtt{w}) - \rho(b),$$
$$\left({}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) \to \rho(\mathtt{w}), \quad \text{and} \quad \left({}^{CFR}\Delta\Delta^{\nu}_{b}\rho\right)(\mathtt{w}) \to \rho(\mathtt{w}).$$

(ii) *As $\nu \to 1$, we have*

$$\left({}^{CFC}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) \to \Delta\rho(\mathtt{w}), \quad \text{and} \quad \left({}^{CFC}\Delta\Delta^{\nu}_{b}\rho\right)(\mathtt{w}) \to -\nabla\rho(\mathtt{w}),$$
$$\left({}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) \to \Delta\rho(\mathtt{w}), \quad \text{and} \quad \left({}^{CFR}\Delta\Delta^{\nu}_{b}\rho\right)(\mathtt{w}) \to -\nabla\rho(\mathtt{w}).$$

Definition 2 (see [20] [$\mathcal{Q}-$operator action]). *Let ρ be defined on $\mathsf{G}_a \cap {}_b\mathsf{G} := \{a, a+1, a+2, \ldots, b-2, b-1, b\}$ with $a < b$ and $a \equiv b \pmod 1$, then the $\mathcal{Q}-$operator action of ρ is used to connect the left and right fractional differences and sums, and it is defined by $(\mathcal{Q}f)(\mathtt{w}) = \rho(a + b - \mathtt{w})$.*

In the following we apply the $\mathcal{Q}-$operator for the new fractional differences stated in Definition 1.

Proposition 1. *For the fractional differences in Definition 1, one can obtain the following transformation:*

(i) $\left(\mathcal{Q}\,{}^{CFC}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) = \left({}^{CFC}\Delta\Delta^{\nu}_{b}\mathcal{Q}\rho\right)(\mathtt{w}),$

(ii) $\left(\mathcal{Q}\,{}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) = \left({}^{CFR}\Delta\Delta^{\nu}_{b}\mathcal{Q}\rho\right)(\mathtt{w}).$

Proof. These follow immediately by applying the $\mathcal{Q}-$operator on the fractional differences in Definition 1 and using the fact that $-\nabla\big(\mathcal{Q}\rho(\mathtt{w})\big) = \mathcal{Q}\big(\Delta\rho(\mathtt{w})\big)$. □

It is of interest to recall the discrete delta Laplace transform.

Definition 3 (see [1]). *The discrete delta Laplace transform for a function ρ defined on G_a is defined by*

$$\mathbf{L}_a\{\rho(\mathtt{w})\}(s) = \sum_{k=0}^{\infty} \frac{\rho(a+k)}{(s+1)^{k+1}}.$$

Definition 4 (see [1]). *Let $\rho, g : \mathsf{G}_a \to \mathsf{R}$ be two functions and $s \in \mathsf{R}, \nu \in (0,1)$. The discrete delta convolution transform of a function ρ with g is defined by*

$$(\rho * g)(\mathtt{w}) = \sum_{\ell=a}^{\mathtt{w}-1} \rho(\mathtt{w} - \sigma(\ell) + a) g(\ell). \tag{5}$$

Moreover, the discrete delta Laplace transform of $(\rho * g)(\mathtt{w})$ is given by

$$\mathbf{L}_a\{(\rho * g)(\mathtt{w})\}(s) = \mathbf{L}_a\{\rho(\mathtt{w})\}(s) \cdot \mathbf{L}_a\{g(\mathtt{w})\}(s). \tag{6}$$

Lemma 1 (see [1]). *For $\ell \neq -1$ and $\mathtt{w} \in \mathsf{G}_a$, we have*

$$\mathbf{L}_a\Big\{(1+\ell)^{\mathtt{w}-\ell}\Big\}(s) = \frac{1}{s-\ell} \quad (\text{for } |s+1| > |\ell+1|),$$

In particular, when $\ell = 0$, we have

$$\mathbf{L}_a\{1\}(s) = \frac{1}{\ell} \quad (\text{for } |s+1| > 1).$$

Lemma 2 (see [1]). *For any function ρ defined on G_a, we have*

$$\mathbf{L}_a\{\Delta\rho\}(s) = s\,\mathbf{L}_a\{\rho\}(s) - \rho(a).$$

3. Discrete Delta Caputo–Fabrizio Fractional Sums

To derive the discrete delta CF fractional sums corresponding to the discrete delta CF fractional differences in Definition 1, we consider the delta fractional difference equation:

$$\Big({}^{CFR}_{\ a}\Delta^{\nu}\rho\Big)(\mathtt{w}) = u(\mathtt{w}). \tag{7}$$

In the light of (2) and (4), we can rewrite (7) as follows:

$$\frac{\mathcal{N}(\nu)}{1-\nu}\Delta_{\mathtt{w}}\Big\{\rho(\mathtt{w}) * (1+\lambda)^{\mathtt{w}-a}\Big\} = \Big({}^{CFR}_{\ a}\Delta^{\nu}\rho\Big)(\mathtt{w}) = u(\mathtt{w}). \tag{8}$$

Taking discrete Laplace transform on both sides of (8), and then using (6) and Lemma 2, we obtain

$$\begin{aligned}\mathbf{L}_a\Big\{\Big({}^{CFR}_{\ a}\Delta^{\nu}\rho\Big)(\mathtt{w})\Big\}(s) &= \frac{\mathcal{N}(\nu)}{1-\nu}\Big(s\,\mathbf{L}_a\{\rho(\mathtt{w}) * (1+\lambda)^{\mathtt{w}-a}\}(s) - \rho(\mathtt{w}) * (1+\lambda)^{\mathtt{w}-a}\Big|_{\mathtt{w}=a}\Big) \\ &= \frac{\mathcal{N}(\nu)}{1-\nu}\frac{sF(s)}{s-\lambda} = U(s),\end{aligned} \tag{9}$$

where $F(s) = \mathbf{L}_a\{\rho\}(s)$ and $U(s) = \mathbf{L}_a\{u\}(s)$. It follows that

$$F(s) = \frac{1-\nu}{\mathcal{N}(\nu)}U(s) + \frac{\nu}{\mathcal{N}(\nu)}s\,U(s).$$

Taking inverse Laplace transforms on both sides, we obtain

$$\rho(\mathtt{w}) = \frac{1-\nu}{\mathcal{N}(\nu)}u(\mathtt{w}) + \frac{\nu}{\mathcal{N}(\nu)}\sum_{r=a}^{\mathtt{w}-1}u(r).$$

This allows us to define the following discrete delta CF fractional sums.

Definition 5 (Left discrete delta Caputo–Fabrizio fractional sums). *Let $\nu \in (0,1)$ and ρ be defined on G_a, then we define the left discrete delta CF fractional sum as follows:*

$$\Big({}^{CF}_{\ a}\Delta^{-\nu}\rho\Big)(\mathtt{w}) = \frac{1-\nu}{\mathcal{N}(\nu)}\rho(\mathtt{w}) + \frac{\nu}{\mathcal{N}(\nu)}\sum_{\ell=a}^{\mathtt{w}-1}\rho(\ell) \quad (\forall\, \mathtt{w} \in \mathsf{G}_a). \tag{10}$$

By applying the $\mathcal{Q}-$operator on both sides of the fractional sum (10), we obtain

$$\mathcal{Q}\Big({}^{CF}_{\ a}\Delta^{-\nu}\rho\Big)(\mathtt{w}) = \frac{1-\nu}{\mathcal{N}(\nu)}\rho(a+b-\mathtt{w}) + \frac{\nu}{\mathcal{N}(\nu)}\sum_{\ell=a}^{a+b-\mathtt{w}-1}\rho(\ell).$$

Changing the variable $s := a+b-r$, it follows that

$$\begin{aligned}\mathcal{Q}\left({}^{CF}_{a}\Delta^{-\nu}\rho\right)(\mathtt{w}) &= \frac{1-\nu}{\mathcal{N}(\nu)}\rho(a+b-\mathtt{w}) + \frac{\nu}{\mathcal{N}(\nu)}\sum_{r=\mathtt{w}+1}^{b}\rho(a+b-r) \\ &= \left({}^{CF}_{a}\Delta^{-\nu}\mathcal{Q}\rho\right)(\mathtt{w}).\end{aligned}$$

Thus, the following definition is valid.

Definition 6 (Right discrete delta Caputo–Fabrizio fractional sums)**.** *Let $\nu \in (0,1)$ and ρ be defined on ${}_{b}\mathsf{G}$, then we define the right discrete delta CF fractional sum as follows:*

$$\left({}^{CF}\Delta_{b}^{-\nu}\rho\right)(\mathtt{w}) = \frac{1-\nu}{\mathcal{N}(\nu)}\rho(\mathtt{w}) + \frac{\nu}{\mathcal{N}(\nu)}\sum_{\ell=\mathtt{w}+1}^{b}\rho(\ell) \quad \left(\forall\, \mathtt{w} \in {}_{b}\mathsf{G}\right). \tag{11}$$

Proposition 2. *For $\nu \in (0,1)$, we have*

(i) $\left({}^{CF}_{a}\Delta^{-\nu}\,{}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) = \rho(\mathtt{w})$ *and* $\left({}^{CFR}_{a}\Delta^{\nu}\,{}^{CF}_{a}\Delta^{-\nu}\rho\right)(\mathtt{w}) = \rho(\mathtt{w}) \quad \left(\forall\, \mathtt{w} \in \mathsf{G}_{a}\right)$,

(ii) $\left({}^{CF}\Delta_{b}^{-\nu}\,{}^{CFR}\Delta_{b}^{\nu}\rho\right)(\mathtt{w}) = \rho(\mathtt{w})$ *and* $\left({}^{CFR}\Delta_{b}^{\nu}\,{}^{CF}\Delta_{b}^{-\nu}\rho\right)(\mathtt{w}) = \rho(\mathtt{w}) \quad \left(\forall\, \mathtt{w} \in {}_{b}\mathsf{G}\right)$.

Proof. The proofs follow directly from Definitions 1, 5 and 6. □

The relationship between discrete delta CFC and CFR fractional differences are given in the following proposition:

Proposition 3. *For $\lambda = -\frac{\nu}{1-\nu}$, $\nu \in (0,1)$, we have*

(i) $\left({}^{CFC}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) = \left({}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) - \frac{\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a} \quad \left(\forall\, \mathtt{w} \in \mathsf{G}_{a}\right)$,

(ii) $\left({}^{CFC}\Delta_{b}^{\nu}\rho\right)(\mathtt{w}) = \left({}^{CFR}\Delta_{b}^{\nu}\rho\right)(\mathtt{w}) - \frac{\mathcal{N}(\nu)}{1-\nu}\rho(b)(1+\lambda)^{b-\mathtt{w}} \quad \left(\forall\, \mathtt{w} \in {}_{b}\mathsf{G}\right)$.

Proof. By taking the Laplace transform to (1), we obtain

$$\begin{aligned}&\mathbf{L}_{a}\left\{(\Delta\rho(\mathtt{w})) * (1+\lambda)^{\mathtt{w}-a}\right\}(s) \\ &\qquad = \frac{\mathcal{N}(\nu)}{1-\nu}\mathbf{L}_{a}\{\Delta\rho(\mathtt{w})\}(s)\cdot\mathbf{L}_{a}\left\{(1+\lambda)^{\mathtt{w}-a}\right\}(s) \\ &\qquad = \frac{\mathcal{N}(\nu)}{1-\nu}\frac{sF(s)}{s-\lambda} - \frac{\mathcal{N}(\nu)}{1-\nu}\frac{\rho(a)}{s-\lambda} \\ &\qquad = \underbrace{\mathbf{L}_{a}\left\{\left({}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w})\right\}(s)}_{\text{using (9)}} - \frac{\mathcal{N}(\nu)}{1-\nu}\frac{\rho(a)}{s-\lambda}.\end{aligned} \tag{12}$$

Taking inverse Laplace transforms and using Lemma 1, we obtain the result for the first item. The second item can be proved by applying the action of the $\mathcal{Q}-$operator on the first item. □

The following lemmas are essential in order to proceed.

Lemma 3. *Let $\lambda = -\frac{\nu}{1-\nu}$, $\nu \in \left(0, \frac{1}{2}\right)$ and $\mathtt{w} \in \mathsf{G}_{a}$, we have*

1. $\left({}^{CF}_{a}\Delta^{-\nu}(1+\lambda)^{\mathtt{w}}\right)(\mathtt{w}) = \frac{(1-\nu)(1+\lambda)^{a}}{\mathcal{N}(\nu)}$;
2. $\Delta_{\ell}(1+\lambda)^{\mathtt{w}-\ell} = -\lambda(1+\lambda)^{\mathtt{w}-\ell-1}$;
3. $\left({}^{CF}_{a}\Delta^{-\nu}\Delta\rho\right)(\mathtt{w}) = \left(\Delta\,{}^{CF}_{a}\Delta^{-\nu}\rho\right)(\mathtt{w}) - \frac{\nu}{\mathcal{N}(\nu)}\rho(a)$;
4. $\Delta(1+\lambda)^{\mathtt{w}} = \lambda(1+\lambda)^{\mathtt{w}}$;

5. $\left({}^{CFR}_{a}\Delta^{\nu}(1+\lambda)^{\mathtt{w}}\right)(\mathtt{w}) = \frac{\mathcal{N}(\nu)}{1-\nu}(1+\lambda)^{\mathtt{w}-1}\left[1+\lambda(\mathtt{w}+1-a)\right]$;
6. $\left({}^{CFR}_{a}\Delta^{\nu}1\right)(\mathtt{w}) = \frac{\mathcal{N}(\nu)}{1-\nu}(1+\lambda)^{\mathtt{w}-a}$.

Proof. Since the proof of items (2) and (4) are easy and direct, we only prove only prove the items (1), (3), (5), and (6) as follows.

(1) We use the definition (10) for $\rho(\mathtt{w}) := (1+\lambda)^{\mathtt{w}}$ to obtain

$$\begin{aligned}
\left({}^{CF}_{a}\Delta^{-\nu}(1+\lambda)^{\mathtt{w}}\right)(\mathtt{w}) &= \frac{1-\nu}{\mathcal{N}(\nu)}(1+\lambda)^{\mathtt{w}} + \frac{\nu}{\mathcal{N}(\nu)}\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\ell} \\
&= \frac{1-\nu}{\mathcal{N}(\nu)}(1+\lambda)^{\mathtt{w}} + \frac{\nu}{\mathcal{N}(\nu)}\left[(1+\lambda)^{a}\frac{1-(1+\lambda)^{\mathtt{w}-a}}{1-(1+\lambda)}\right] \\
&= \frac{1-\nu}{\mathcal{N}(\nu)}(1+\lambda)^{\mathtt{w}} + \frac{\nu}{\mathcal{N}(\nu)}\frac{(1+\lambda)^{a}-(1+\lambda)^{\mathtt{w}}}{-\lambda} \\
&= \frac{1}{\mathcal{N}(\nu)}\left[(1-\nu)(1+\lambda)^{\mathtt{w}} + (1-\nu)(1+\lambda)^{a} - (1-\nu)(1+\lambda)^{\mathtt{w}}\right] \\
&= \frac{(1-\nu)(1+\lambda)^{a}}{\mathcal{N}(\nu)},
\end{aligned}$$

which is the desired result.

(3) Again, we use the definition (10) for $\rho(\mathtt{w}) := \Delta\rho(\mathtt{w})$ to obtain

$$\begin{aligned}
\left({}^{CF}_{a}\Delta^{-\nu}\Delta\rho\right)(\mathtt{w}) &= \frac{1-\nu}{\mathcal{N}(\nu)}\Delta\rho(\mathtt{w}) + \frac{\nu}{\mathcal{N}(\nu)}\sum_{\ell=a}^{\mathtt{w}-1}\Delta\rho(\ell) \\
&= \frac{1-\nu}{\mathcal{N}(\nu)}\Delta\rho(\mathtt{w}) + \frac{\nu}{\mathcal{N}(\nu)}[\rho(\mathtt{w})-\rho(a)] \\
&= \Delta\left[\frac{1-\nu}{\mathcal{N}(\nu)}\rho(\mathtt{w}) + \frac{\nu}{\mathcal{N}(\nu)}\sum_{\ell=a}^{\mathtt{w}-1}\rho(\ell)\right] - \frac{\nu}{\mathcal{N}(\nu)}\rho(a) \\
&= \left(\Delta\ {}^{CF}_{a}\Delta^{-\nu}\rho\right)(\mathtt{w}) - \frac{\nu}{\mathcal{N}(\nu)}\rho(a),
\end{aligned}$$

which is the required result.

(5) Here we use the definition (2) for $\rho(\mathtt{w}) := (1+\lambda)^{\mathtt{w}}$ and we obtain

$$\begin{aligned}
\left({}^{CFR}_{a}\Delta^{\nu}(1+\lambda)^{\mathtt{w}}\right)(\mathtt{w}) &= \frac{\mathcal{N}(\nu)}{1-\nu}\Delta_{\mathtt{w}}\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\ell}(1+\lambda)^{\mathtt{w}-\ell-1} = \frac{\mathcal{N}(\nu)}{1-\nu}\Delta_{\mathtt{w}}\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-1} \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}\Delta_{\mathtt{w}}\left[(\mathtt{w}-a)(1+\lambda)^{\mathtt{w}-1}\right] \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}\left[(\mathtt{w}+1-a)(1+\lambda)^{\mathtt{w}} - (\mathtt{w}-a)(1+\lambda)^{\mathtt{w}-1}\right] \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}(1+\lambda)^{\mathtt{w}-1}[1+\lambda(\mathtt{w}+1-a)],
\end{aligned}$$

which is the stated result.

(6) Definition (2) with $\rho(\mathtt{w}) := 1$ leads to

$$\begin{aligned}
\left({}^{CFR}_{\ a}\Delta^{\nu}1\right)(\mathtt{w}) &= \frac{\mathcal{N}(\nu)}{1-\nu}\Delta_{\mathtt{w}}\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-\ell-1} \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}\left(1+\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-\ell}-\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-\ell-1}\right) \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}\left(1+\lambda\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-\ell-1}\right) \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}\left(1+\lambda(1+\lambda)^{\mathtt{w}-a-1}+\lambda\sum_{\ell=a+1}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-\ell-1}\right) \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}\left(1+\lambda(1+\lambda)^{\mathtt{w}-a-1}+\lambda\sum_{j=0}^{\mathtt{w}-a-2}(1+\lambda)^{j}\right) \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}\left(1+\lambda(1+\lambda)^{\mathtt{w}-a-1}+\lambda\frac{1-(1+\lambda)^{\mathtt{w}-a-1}}{1-(1+\lambda)}\right) \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}(1+\lambda)^{\mathtt{w}-a},
\end{aligned}$$

which is the end of the proof. □

4. Results on Discrete Monotonicity Analysis

In this part, we focus on implementing monotonicity analysis for the discrete delta CF fractional operators.

Definition 7 (see [11,14,15,21])**.** *Let $0 < \nu \le 1$ and $\rho : \mathsf{G}_a \to \mathsf{R}$ be a function satisfying $\rho(a) \ge 0$. Then, ρ is called an ν-increasing function on G_a, if*

$$\rho(\mathtt{w}+1) \ge \nu\,\rho(\mathtt{w}) \quad (\forall \mathtt{w} \in \mathsf{G}_a),$$

and ρ is called an ν-decreasing function on G_a, if

$$\rho(\mathtt{w}+1) \le \nu\,\rho(\mathtt{w}) \quad (\forall \mathtt{w} \in \mathsf{G}_a).$$

Remark 3. *Note that*

- *If $\nu = 1$ in Definition 7, then increasing and ν-increasing concepts coincide, and decreasing and ν-decreasing concepts coincide.*
- *If $\rho(\mathtt{w})$ is increasing on G_a, then $\rho(\mathtt{w}+1) \ge \rho(\mathtt{w})$ for all $\mathtt{w} \in \mathsf{G}_a$, and thus $\rho(\mathtt{w})$ is ν-increasing on G_a. Moreover, if $\rho(\mathtt{w})$ is decreasing on G_a, then $\rho(\mathtt{w}+1) \le \rho(\mathtt{w})$ for all $\mathtt{w} \in \mathsf{G}_a$, and thus $\rho(\mathtt{w})$ is ν-decreasing on G_a.*

Remark 4. *It should be noted that $1+\lambda = \frac{1-2\nu}{1-\nu} > 0$ for each $\nu \in \left(0, \frac{1}{2}\right)$.*

Theorem 1. *If a function $\rho : \mathsf{G}_a \to \mathsf{R}$ satisfies $\rho(a) \ge 0$ and $\left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) \ge 0$ for each $\mathtt{w} \in \mathsf{G}_a$ and $\nu \in \left(0, \frac{1}{2}\right)$, then $\rho(\mathtt{w})$ is $\left(\frac{\nu}{1-\nu}\right)$–increasing.*

Proof. Rewriting $\left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w})$ as follows:

$$\left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) := \frac{\mathcal{N}(\nu)}{1-\nu}\Delta\chi(\mathtt{w}), \quad \text{where} \quad \chi(\mathtt{w}) := \sum_{\ell=a}^{\mathtt{w}-1}\rho(\ell)(1+\lambda)^{\mathtt{w}-\ell-1}.$$

Now, based on the assumption, we have

$$\chi(\mathtt{w}+1)-\chi(\mathtt{w})=\rho(\mathtt{w})+\lambda\sum_{\ell=a}^{\mathtt{w}-1}\rho(\ell)(1+\lambda)^{\mathtt{w}-\ell-1}$$
$$=\rho(\mathtt{w})-\frac{\nu}{1-\nu}\sum_{\ell=a}^{\mathtt{w}-1}\rho(\ell)(1+\lambda)^{\mathtt{w}-\ell-1}\geq 0. \tag{13}$$

We shall proceed by induction on $\mathtt{w}\in\mathsf{G}_a$. First, by substituting $\mathtt{w}=a$ into (13), we see that $\rho(a)\geq 0$. Again, by substituting $\mathtt{w}=a+1$ into (13) yields

$$\rho(a+1)-\frac{\nu}{1-\nu}\rho(a)\geq 0,$$

this implies that $\rho(a+1)\geq\frac{\nu}{1-\nu}\rho(a)\geq 0$. Assume that

$$\rho(k+1)\geq\frac{\nu}{1-\nu}\rho(k)\geq 0\quad\forall\, k,\mathtt{w}\in\mathsf{G}_a \text{ such that } k<\mathtt{w}. \tag{14}$$

Then, we will try to show that $\rho(\mathtt{w}+1)\geq\frac{\nu}{1-\nu}\rho(\mathtt{w})$. Use identity (13) by replacing $\mathtt{w}$ by $\mathtt{w}+1$ to obtain

$$\begin{aligned}
&\rho(\mathtt{w}+1)\\
&=\frac{\nu}{1-\nu}\sum_{\ell=a}^{\mathtt{w}}\rho(\ell)(1+\lambda)^{\mathtt{w}-\ell}\\
&=\frac{\nu}{1-\nu}\Big[(1+\lambda)^{\mathtt{w}-a}\rho(a)+(1+\lambda)^{\mathtt{w}-a-1}\rho(a+1)+\cdots+(1+\lambda)\rho(\mathtt{w}-1)+\rho(\mathtt{w})\Big]\\
&=\underbrace{\frac{\nu}{1-\nu}\Big[(1+\lambda)^{\mathtt{w}-a}\rho(a)+(1+\lambda)^{\mathtt{w}-a-1}\rho(a+1)+\cdots+(1+\lambda)\rho(\mathtt{w}-1)\Big]}_{\geq 0 \text{ by (14) and Remark 4}}+\frac{\nu}{1-\nu}\rho(\mathtt{w})\\
&\geq\frac{\nu}{1-\nu}\rho(\mathtt{w}).
\end{aligned}$$

Thus, the result is proved. □

Theorem 2. *If all notations given in Theorem 1 are satisfied and*

$$\left({}^{CFC}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w})\geq\frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a}\quad(\forall\,\mathtt{w}\in\mathsf{G}_a \text{ and } \nu\in(0,0.5)),$$

then $\rho(\mathtt{w})$ is $(\frac{\nu}{1-\nu})$–increasing.

Proof. This follows immediately from Proposition 2 and Theorem 1. □

The following results are the decreasing analogues of Theorem 1 and Theorem 2, respectively.

Proposition 4. *If a function $\rho:\mathsf{G}_a\to\mathsf{R}$ satisfies $\rho(a)\leq 0$ and $\left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w})\leq 0$ for each $\mathtt{w}\in\mathsf{G}_a$ and $\nu\in\left(0,\frac{1}{2}\right)$, then $\rho(\mathtt{w})$ is $(\frac{\nu}{1-\nu})$–decreasing.*

Proposition 5. *If a function $\rho:\mathsf{G}_a\to\mathsf{R}$ satisfies $\rho(a)\leq 0$ and*

$$\left({}^{CFC}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w})\leq\frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a}\qquad(\forall\,\mathtt{w}\in\mathsf{G}_a \text{ and } \nu\in(0,0.5)),$$

then $\rho(\mathtt{w})$ is $(\frac{\nu}{1-\nu})$–decreasing.

Moreover, the following results are the strictly increasing (or strictly decreasing) analogues of Theorem 1 (or Proposition 4) and Theorem 2 (or Proposition 5), respectively.

Proposition 6. *If a function $\rho : \mathsf{G}_a \to \mathsf{R}$ satisfies $\rho(a) > 0$ (or $\rho(a) < 0$) and $\left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) > 0$ $\left(\text{or } \left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) < 0\right)$ for each $\mathtt{w} \in \mathsf{G}_a$ and $\nu \in \left(0, \frac{1}{2}\right)$, then $\rho(\mathtt{w})$ is $\left(\frac{\nu}{1-\nu}\right)$–strictly increasing $\left(\text{or } \left(\frac{\nu}{1-\nu}\right) - \text{strictly decreasing}\right)$, respectively.*

Proposition 7. *If a function $\rho : \mathsf{G}_a \to \mathsf{R}$ satisfies $\rho(a) > 0$ (or $\rho(a) < 0$) and*

$$\left({}^{CFC}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) > \frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a} \qquad (\forall\ \mathtt{w} \in \mathsf{G}_a \text{ and } \nu \in (0, 0.5)),$$

or

$$\left({}^{CFC}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) < \frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a} \qquad (\forall\ \mathtt{w} \in \mathsf{G}_a \text{ and } \nu \in (0, 0.5)),$$

then $\rho(\mathtt{w})$ is $\left(\frac{\nu}{1-\nu}\right)$–strictly increasing $\left(\text{or } \left(\frac{\nu}{1-\nu}\right) - \text{strictly decreasing}\right)$, respectively.

Theorem 3. *Let $\rho : \mathsf{G}_a \to \mathsf{R}$ be a function with $\rho(a) \geq 0$ and increasing on G_{a+1}. Then, we have*

$$\left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) \geq 0 \qquad (\forall\ \mathtt{w} \in \mathsf{G}_a \text{ and } \nu \in (0, 0.5)).$$

Proof. The result will be proved if we can show that $\chi(\mathtt{w})$ is increasing on G_a, where $\chi(\mathtt{w})$ is as before. We proceed by induction on $\mathtt{w} \in \mathsf{G}_a$. The $\mathtt{w} = a$ case of (13) leads to $\chi(a+1) - \chi(a) = \rho(a) \geq 0$ by assumption. Now let us assume the result is true for $k < \mathtt{w}$, i.e., $\chi(k+1) - \chi(k) \geq 0$ for each $k, \mathtt{w} \in \mathsf{G}_a$ such that $k < \mathtt{w}$, and prove it for $k = \mathtt{w}$, i.e., $\chi(\mathtt{w}+1) - \chi(\mathtt{w}) \geq 0$.

Considering the assumption, $\rho(\mathtt{w})$ is increasing on G_a, it follows that

$$\rho(\mathtt{w}+1) \geq \rho(\mathtt{w}) \geq \rho(a) \geq 0 \qquad (\forall\ \mathtt{w} \in \mathsf{G}_a). \tag{15}$$

Then, by using (13) yields

$$\begin{aligned}
\chi(\mathtt{w}+1) - \chi(\mathtt{w}) &= \rho(\mathtt{w}) - \frac{\nu}{1-\nu}\sum_{\ell=a}^{\mathtt{w}-1}\rho(\ell)(1+\lambda)^{\mathtt{w}-\ell-1} \\
&= \rho(\mathtt{w}) - \frac{\nu}{1-\nu}\rho(\mathtt{w}-1) - \frac{\nu}{1-\nu}\sum_{\ell=a}^{\mathtt{w}-2}\rho(\ell)(1+\lambda)^{\mathtt{w}-\ell-1} \\
&= \rho(\mathtt{w}) - \frac{\nu}{1-\nu}\rho(\mathtt{w}-1) \\
&\quad + \frac{\nu}{1-\nu}\left(\sum_{\ell=a}^{\mathtt{w}-2}(\rho(\mathtt{w}-1) - \rho(\ell))(1+\lambda)^{\mathtt{w}-\ell-1} - \sum_{\ell=a}^{\mathtt{w}-2}\rho(\mathtt{w}-1)(1+\lambda)^{\mathtt{w}-\ell-1}\right).
\end{aligned}$$

It should be remarked that $\rho(\mathtt{w}-1)-\rho(\ell)\geq 0$ since $\rho(\mathtt{w})$ is increasing on G_{a+1}. For each $\ell=a,a+1,\ldots,t-1$, it follows that

$$\begin{aligned}
\chi(\mathtt{w}+1)-\chi(\mathtt{w}) &\geq \rho(\mathtt{w})-\frac{\nu}{1-\nu}\rho(\mathtt{w}-1)-\frac{\nu}{1-\nu}\sum_{\ell=a}^{\mathtt{w}-2}\rho(\mathtt{w}-1)(1+\lambda)^{\mathtt{w}-\ell-1} \\
&= \rho(\mathtt{w})-\frac{\nu}{1-\nu}\rho(\mathtt{w}-1)\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-\ell-1} \\
&= \rho(\mathtt{w})+\lambda\rho(\mathtt{w}-1)\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-\ell-1} \\
&= \underbrace{\rho(\mathtt{w})-\rho(\mathtt{w}-1)}_{\geq 0 \text{ by } (15)}+\rho(\mathtt{w}-1)+\lambda\rho(\mathtt{w}-1)\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-\ell-1} \\
&\geq \rho(\mathtt{w}-1)\left[1+\lambda\sum_{\ell=a}^{\mathtt{w}-1}(1+\lambda)^{\mathtt{w}-\ell-1}\right] \\
&= \rho(\mathtt{w}-1)\left[1+\lambda\sum_{k=0}^{(\mathtt{w}-a)-1}(1+\lambda)^{k}\right] \\
&= \rho(\mathtt{w}-1)\left[1+\lambda\frac{1-(1+\lambda)^{\mathtt{w}-a}}{1-(1+\lambda)}\right] \\
&= \rho(\mathtt{w}-1)(1+\lambda)^{\mathtt{w}-a}\geq 0 \quad (\forall\, \mathtt{w}\in\mathsf{G}_a),
\end{aligned}$$

where we use (15), Remark 4 and the geometric series sum formula. Thus, we have shown that $\chi(\mathtt{w})$ is increasing as required. □

Theorem 4. *With all notation given in Theorem 3, we have*

$$\left({}^{CFC}_{a}\Delta^{\nu}\rho\right)(\mathtt{w})\geq\frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a}\quad\big(\forall\,\mathtt{w}\in\mathsf{G}_a \text{ and } \nu\in(0,0.5)\big).$$

Proof. This is immediate from Proposition 2 and Theorem 3. □

The decreasing analogues of Theorem 3 and Theorem 4 are given in the following propositions, respectively.

Proposition 8. *Let $\rho:\mathsf{G}_a\to\mathsf{R}$ be a function with $\rho(a)\leq 0$ and decreasing on G_{a+1}. Then, we have*

$$\left({}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w})\leq 0\qquad\big(\forall\,\mathtt{w}\in\mathsf{G}_a \text{ and } \nu\in(0,0.5)\big).$$

Proposition 9. *Let $\rho:\mathsf{G}_a\to\mathsf{R}$ be a function with $\rho(a)\leq 0$ and decreasing on G_{a+1}. Then, we have*

$$\left({}^{CFC}_{a}\Delta^{\nu}\rho\right)(\mathtt{w})\leq\frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a}\quad\big(\forall\,\mathtt{w}\in\mathsf{G}_a \text{ and } \nu\in(0,0.5)\big).$$

Furthermore, the strictly increasing (or strictly decreasing) analogues of Theorem 3 (or Proposition 8) and Theorem 4 (or Proposition 9) are given in the following propositions, respectively.

Proposition 10. *Let $\rho : \mathsf{G}_a \to \mathsf{R}$ be a function with $\rho(a) > 0$ (or $\rho(a) < 0$) and strictly increasing (or strictly decreasing) on G_{a+1}. Then, we have*

$$\left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) > 0 \qquad (\forall\ \mathtt{w} \in \mathsf{G}_a \text{ and } \nu \in (0, 0.5)),$$

or

$$\left({}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) < 0 \qquad (\forall\ \mathtt{w} \in \mathsf{G}_a \text{ and } \nu \in (0, 0.5)),$$

respectively.

Proposition 11. *Let $\rho : \mathsf{G}_a \to \mathsf{R}$ be a function with $\rho(a) > 0$ (or $\rho(a) < 0$) and strictly increasing (or strictly decreasing) on G_{a+1}. Then, we have*

$$\left({}^{CFC}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) > \frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a} \quad (\forall\ \mathtt{w} \in \mathsf{G}_a \text{ and } \nu \in (0, 0.5)),$$

or

$$\left({}^{CFC}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) < \frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a} \quad (\forall\ \mathtt{w} \in \mathsf{G}_a \text{ and } \nu \in (0, 0.5)),$$

respectively.

5. Fractional Difference Mean Value Theorem

The monotonicity results of the previous section can be applied to reformulate the discrete fractional MVT. First, we recall that $\left({}^{CF}_{a+1}\Delta^{-\nu}\, {}^{CFR}_{a+1}\Delta^{\nu}\rho\right)(\mathtt{w}) = \rho(\mathtt{w})$. However, the next result contains an initial condition $\rho(a)$ and it will be a useful tool to obtain the discrete fractional difference MVT.

Theorem 5. *For $\nu \in (0, 1)$, we have*

$$\left({}^{CF}_{a+1}\Delta^{-\nu}\, {}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}) = \rho(\mathtt{w}) + \lambda\rho(a) \qquad (\forall\ \mathtt{w} \in \mathsf{G}_a). \tag{16}$$

Proof. Using Definition 1 and Lemma 3, we have

$$\begin{aligned}
\left({}^{CF}_{a+1}\Delta^{-\nu}\, {}^{CFR}_{\ a}\Delta^{\nu}\rho\right)(\mathtt{w}+1) &= {}^{CF}_{a+1}\Delta^{-\nu}\left(\frac{\mathcal{N}(\nu)}{1-\nu}\Delta_{\mathtt{w}}\sum_{\ell=a}^{\mathtt{w}-1}\rho(\ell)(1+\lambda)^{\mathtt{w}-\ell-1}\right) \\
&= \frac{\mathcal{N}(\nu)}{1-\nu}\, {}^{CF}_{\ a}\Delta^{-\nu}\Delta_{\mathtt{w}}\left(\sum_{\ell=a+1}^{\mathtt{w}}\rho(\ell)(1+\lambda)^{\mathtt{w}-\ell-1} + \rho(a)(1+\lambda)^{\mathtt{w}-a-1}\right) \\
&= \left({}^{CF}_{a+1}\Delta^{-\nu}\, {}^{CFR}_{a+1}\Delta^{\nu}\rho\right)(\mathtt{w}) + \lambda\frac{\mathcal{N}(\nu)}{1-\nu}\rho(a)\, {}^{CF}_{\ a}\Delta^{-\nu}(1+\lambda)^{\mathtt{w}-a-1} \\
&= \rho(\mathtt{w}) + \lambda\frac{\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{-a-1}\, {}^{CF}_{a+1}\Delta^{-\nu}(1+\lambda)^{\mathtt{w}} \\
&= \rho(\mathtt{w}) + \lambda\rho(a).
\end{aligned}$$

Hence, the result. □

Corollary 1. *Let g be a function defined on G_a with $g(a) > 0$ and strictly increasing on G_{a+1}. Then, for each $\nu \in \left(0, \frac{1}{2}\right)$, we have*

$$g(\mathtt{w}) + \lambda g(a) > 0 \quad (\forall\ \mathtt{w} \in \mathsf{G}_a).$$

Proof. Since g is strictly increasing and $g(a) > 0$, we know from Proposition 10 that

$$\left({}^{CFR}_{a}\Delta^{\nu} g\right)(\mathtt{w}) > 0.$$

Applying ${}^{CF}_{a+1}\Delta^{-\nu}$ to the above inequality and using Theorem 5, we see that

$$g(\mathtt{w}) + \lambda g(a) > 0.$$

Thus, we have proved the result as required. □

Theorem 6 (Fractional difference MVT for the CFR case)**.** *Let ρ and g be two functions defined on $\mathsf{G}_a \cap {}_b\mathsf{G}$ such that g is strictly increasing and $g(a) > 0$, and let $\nu \in \left(0, \frac{1}{2}\right)$ and $a < b$ with $a \equiv b \pmod 1$, then, there exist $x_1, x_2 \in \mathsf{G}_a \cap {}_b\mathsf{G}$ with*

$$\frac{\left({}^{CFR}_{a}\Delta^{\nu} \rho\right)(x_1)}{\left({}^{CFR}_{a}\Delta^{\nu} g\right)(x_1)} \le \frac{\rho(b) + \lambda\rho(a)}{g(b) + \lambda g(a)} \le \frac{\left({}^{CFR}_{a}\Delta^{\nu} \rho\right)(x_2)}{\left({}^{CFR}_{a}\Delta^{\nu} g\right)(x_2)}. \tag{17}$$

Proof. First, from Proposition 10 and Corollary 1, we can see that the denominators in (17) are all positive. We proceed by contradiction. Suppose that (17) is not true. Then, either

$$\frac{\rho(b+1) - \lambda\rho(a)}{g(b) + \lambda g(a)} > \frac{\left({}^{CFR}_{a}\Delta^{\nu} \rho\right)(\mathtt{w})}{\left({}^{CFR}_{a}\Delta^{\nu} g\right)(\mathtt{w})} \quad \left(\forall \mathtt{w} \in \mathsf{G}_a \cap {}_b\mathsf{G}\right), \tag{18}$$

or

$$\frac{\rho(b) + \lambda\rho(a)}{g(b) + \lambda g(a)} < \frac{\left({}^{CFR}_{a}\Delta^{\nu} \rho\right)(\mathtt{w})}{\left({}^{CFR}_{a}\Delta^{\nu} g\right)(\mathtt{w})} \quad \left(\forall \mathtt{w} \in \mathsf{G}_a \cap {}_b\mathsf{G}\right). \tag{19}$$

Considering Proposition 10, the inequality (18) can be rewritten as

$$\frac{\rho(b) + \lambda\rho(a)}{g(b) + \lambda g(a)} \left({}^{CFR}_{a}\Delta^{\nu} g\right)(\mathtt{w}) > \left({}^{CFR}_{a}\Delta^{\nu} \rho\right)(\mathtt{w}).$$

We apply the fractional sum operator ${}^{CF}_{a+1}\Delta^{-\nu} g$ to both sides of the above inequality at $\mathtt{w} = b$ and use the (16) to obtain

$$\rho(b) + \lambda\rho(a) > \rho(b) + \lambda\rho(a),$$

which is a contradiction. Analogously, inequality (19) will lead to contradiction. Thus, the result is proved. □

In the next theorem, we give the result of $\left({}^{CF}_{a+1}\Delta^{-\nu}\, {}^{CFC}_{a}\Delta^{\nu} \rho\right)(\mathtt{w})$ as we did for the CFC case in Theorem 5.

Theorem 7. *For $\nu \in \left(0, \frac{1}{2}\right)$, we have*

$$\left({}^{CF}_{a+1}\Delta^{-\nu}\, {}^{CFC}_{a}\Delta^{\nu} \rho\right)(\mathtt{w}) = \rho(\mathtt{w}) - \rho(a) \qquad \left(\forall\ \mathtt{w} \in \mathsf{G}_a\right). \tag{20}$$

Proof. We begin using the relationship in Proposition 3:

$$\left({}^{CFC}_{a}\Delta^{\nu} \rho\right)(\mathtt{w}) = \left({}^{CFR}_{a}\Delta^{\nu} \rho\right)(\mathtt{w}) - \frac{\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a} \quad \left(\forall\ \mathtt{w} \in \mathsf{G}_a\right). \tag{21}$$

By applying ${}^{CF}_{a+1}\Delta^{-\nu}$ to both sides of (21), we have:

$$\left({}^{CF}_{a+1}\Delta^{-\nu}\, {}^{CFC}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) = \left({}^{CF}_{a+1}\Delta^{-\nu}\, {}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) - \frac{\mathcal{N}(\nu)}{1-\nu}\rho(a)\left({}^{CF}_{a+1}\Delta^{-\nu}(1+\lambda)^{\mathtt{w}-a}\right)(\mathtt{w})$$

Using Theorem 5 and Lemma 3 (1), we have:

$$\begin{aligned}\left({}^{CF}_{a+1}\Delta^{-\nu}\, {}^{CFC}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) &= \rho(\mathtt{w}) + \lambda\rho(a) - \frac{\mathcal{N}(\nu)}{1-\nu}\rho(a)\cdot(1+\lambda)^{-a}\frac{1-\nu}{\mathcal{N}(\nu)}(1+\lambda)^{a+1} \\ &= \rho(\mathtt{w}) - \rho(a),\end{aligned}$$

which is the end of the proof. □

Remark 5. *As we have discussed in our recently published article [22], it is impossible to obtain fractional difference MVT for the CFC case, i.e., the following inequalities*

$$\frac{\left({}^{CFC}_{a}\Delta^{\nu}\rho\right)(x_1)}{\left({}^{CFC}_{a}\Delta^{\nu}g\right)(x_1)} \le \frac{\rho(b)-\rho(a)}{g(b)-g(a)} \le \frac{\left({}^{CFC}_{a}\Delta^{\nu}\rho\right)(x_2)}{\left({}^{CFC}_{a}\Delta^{\nu}g\right)(x_2)} \tag{22}$$

do not hold true. The main reason for this is that the discrete delta CFC fractional difference $\left({}^{CFC}_{a}\Delta^{\nu}g\right)(\mathtt{w})$ *is not clear whether it is greater than zero or not by means of Proposition 11.*

6. Conclusions

To obtain the mean value theorem with discrete fractional difference terms, monotonicity analysis is considered for the discrete delta Caputo–Fabrizio fractional operators. Before analysing the monotonicity results, the discrete delta Caputo–Fabrizio fractional differences are introduced and the discrete delta CF fractional sums are investigated on the time scale $\mathcal{Z}$. Additionally, the discrete right operators are found by applying action of $\mathcal{Q}-$operator to the corresponding discrete left operators. Besides that, the discrete delta Laplace transform technique is applied to find a correlation between the discrete delta CFC and CFR fractional differences. We see that a function $\rho : \mathsf{G}_a \to \mathsf{R}$ is $\left(\frac{\nu}{1-\nu}\right)$-increasing, when $\rho(a) \ge 0$ and $\left({}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) \ge 0$ $\left(\text{or by the correlation } \left({}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) \ge \frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a}\right)$ for $\mathtt{w} \in \mathsf{G}_a$ and $\nu \in \left(0, \frac{1}{2}\right)$. Conversely, we see that $\left({}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) \ge 0$ $\left(\text{or } \left({}^{CFR}_{a}\Delta^{\nu}\rho\right)(\mathtt{w}) \ge \frac{-\mathcal{N}(\nu)}{1-\nu}\rho(a)(1+\lambda)^{\mathtt{w}-a}\right)$ for $\mathtt{w} \in \mathsf{G}_a$ and $\nu \in \left(0, \frac{1}{2}\right)$ for $\mathtt{w} \in \mathsf{G}_a$ and $\nu \in \left(0, \frac{1}{2}\right)$, when the function ρ is increasing on G_{a+1} and $\rho(a) \ge 0$. These results finally lead to obtaining the fractional difference mean value theorem.

Some of the ideas used in the current paper are similar to those for previously existing models of discrete fractional calculus with various singular- and nonsingular-type kernels; however, this is the first time that the discrete delta Caputo–Fabrizio fractional operators have been used in this way to construct fractional difference mean value theorem and monotonicity analysis. Previous contributions in this direction have included using discrete nabla Caputo–Fabrizio fractional operators to construct difference mean value theorem [18], using discrete nabla Atangana-Baleanu fractional operators to construct difference mean value theorem [13], and using discrete nabla Caputo and Riemann fractional operators to construct difference mean value theorem [15].

In the future, researchers can extend the results of this paper by considering other types of discrete fractional calculus. The work here is set within the discrete delta Caputo–Fabrizio model, but it may be possible to extend it, applying the same arguments in some general class of discrete fractional operators, to obtain further results which would be useful in the direction of monotonicity analysis.

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References

1. Goodrich, C.S.; Peterson, A. *Discrete Fractional Calculus*; Springer: Berlin/Heidelberg, Germany, 2015.
2. Goodrich, C.S.; Muellner, M. An analysis of the sharpness of monotonicity results via homotopy for sequential fractional operators. *Appl. Math. Lett.* **2019**, *98*, 446–452. [CrossRef]
3. Silem, A.; Wu, H.; Zhang, D. An Discrete rogue waves and blow-up from solitons of a nonisospectral semi-discrete nonlinear Schrödinger equation. *Appl. Math. Lett.* **2021**, *116*, 107049. [CrossRef]
4. Mohammed, P.O.; Abdeljawad, T. Discrete generalized fractional operators defined using h-discrete Mittag-Leffler kernels and applications to AB fractional difference systems. *Math. Methods Appl. Sci.* **2020**, 1–26. [CrossRef]
5. Xu, J.; Goodrich, C.S; Cui, Y. Positive solutions for a system of first-order discrete fractional boundary value problems with semipositone nonlinearities. *RACSAM* **2021**, *113*, 1343–1358 . [CrossRef]
6. Srivastava, H.M.; Mohammed, P.O.; Ryoo, C.S.; Hamed, Y.S. Existence and uniqueness of a class of uncertain Liouville-Caputo fractional difference equations. *J. King Saud Univ. Sci.* **2021**, *33*, 101497. [CrossRef]
7. Srivastava, H.M.; Mohammed, P.O.; Guirao, J.L.G.; Hamed, Y.S. Link theorem and distributions of solutions to uncertain Liouville-Caputo difference equations. *DCDS-S* **2021**, *14*, 1–14.
8. Srivastava, H.M.; Mohammed, P.O. A correlation between solutions of uncertain fractional forward difference equations and their paths. *Front. Phys.* **2020**, *8*, 280. [CrossRef]
9. Mohammed, P.O.; Abdeljawad, T.; Jarad, F.; Chu, Y.-M. Existence and uniqueness of uncertain fractional backward difference equations of Riemann–Liouville type. *Math. Probl. Eng.* **2020**, *2020*, 6598682. [CrossRef]
10. Abdeljawad, T.; Jarad, F.; Atangana, A.; Mohammed, P.O. On a new type of fractional difference operators on h-step isolated time scales. *J. Fract. Calc. Nonlinear Syst.* **2021**, *1*, 46–74.
11. Atici, F.; Uyanik, M. Analysis of discrete fractional operators. *Appl. Anal. Discret. Math.* **2015**, *9*, 139–149. [CrossRef]
12. Suwan, I.; Owies, S.; Abdeljawad, T. Monotonicity results for *h*-discrete fractional operators and application. *Adv. Differ. Equ.* **2018**, *2018*, 207. [CrossRef]
13. Abdeljawad, T.; Baleanu, D. Monotonicity analysis of a nabla discrete fractional operator with discrete Mittag-Leffler kernel. *Chaos Solit. Fract.* **2017**, *116*, 1–5. [CrossRef]
14. Suwan, I.; Abdeljawad, T.; Jarad, F. Monotonicity analysis for nabla *h*-discrete fractional Atangana-Baleanu differences. *Chaos Solit. Fract.* **2018**, *117*, 50–59. [CrossRef]
15. Abdeljawad, T.; Abdallaa, B. Monotonicity results for delta and nabla Caputo and Riemann fractional differences via dual identities. *Filomat* **2017**, *31*, 3671–3683. [CrossRef]
16. Goodrich, C.S.; Jonnalagadda, J.M.; Lyons, B. Convexity, monotonicity, and positivity results for sequential fractional nabla difference operators with discrete exponential kernels. *Math. Methods Appl. Sci.* **2021**, *44*, 7099–7120. [CrossRef]
17. Mohammed, P.O.; Hamasalh, F.K.; Abdeljawad, T. Difference monotonicity analysis on discrete fractional operators with discrete generalized Mittag-Leffler kernels. *Adv. Differ. Equ.* **2021**, *2021*, 213. [CrossRef]
18. Abdeljawad, T.; Baleanu, D. Monotonicity results for fractional difference operators with discrete exponential kernels. *Adv. Differ. Equ.* **2017**, *2017*, 78. [CrossRef]
19. Mozyrska, D.; Torres, D.F.M.; Wyrwas, M. Solutions of systems with the Caputo-Fabrizio fractional delta derivative on time scales. *Nonlinear Anal. Hybrid Syst.* **2019**, *32*, 168–176. [CrossRef]
20. Abdeljawad, T. On Riemann and Caputo fractional differences. *Commput. Math. Appl.* **2011**, *62*, 1602–1611. [CrossRef]
21. Abdeljawad, T. Different type kernel *h*–fractional differences and their fractional *h*–sums. *Chaos Solit. Fract.* **2018**, *116*, 146–156. [CrossRef]
22. Mohammed, P.O.; Abdeljawad, T.; Hamasalh, F.K. On Riemann-Liouville and Caputo fractional forward difference monotonicity analysis. *Mathematics* **2021**, *9*, 1303. [CrossRef]