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Certain Coefficient Estimate Problems for Three-Leaf-Type Starlike Functions

Lei Shi [1], Muhammad Ghaffar Khan [2], Bakhtiar Ahmad [3], Wali Khan Mashwani [2], Praveen Agarwal [4,5,6,*] and Shaher Momani [5,7]

1 School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, China; shimath@aynu.edu.cn
2 Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat 26000, Pakistan; ghaffarkhan020@gmail.com (M.G.K.); mashwanigr8@gmail.com (W.K.M.)
3 Government Degree College Mardan, Mardan 23200, Pakistan; pirbakhtiarbacha@gmail.com
4 Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India
5 Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman AE 346, United Arab Emirates; shahermm@yahoo.com
6 International Center for Basic and Applied Sciences, Jaipur 302029, India
7 Department of Mathematics, Faculty of Science, The University of Jordon, Amman 11942, Jordan
* Correspondence: praveen.agarwal@anandice.ac.in

Abstract: In our present investigation, some coefficient functionals for a subclass relating to starlike functions connected with three-leaf mappings were considered. Sharp coefficient estimates for the first four initial coefficients of the functions of this class are addressed. Furthermore, we obtain the Fekete–Szegö inequality, sharp upper bounds for second and third Hankel determinants, bounds for logarithmic coefficients, and third-order Hankel determinants for two-fold and three-fold symmetric functions.

Keywords: starlike functions; subordinations; three-leaf function; coefficient bounds; logarithmic coefficients; Hankel determinant; two- and three-fold symmetric functions

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1. Introduction and Preliminary Results

Let the family of all the functions f that are analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be represented by $\mathcal{A}$ and have the series form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ z \in \mathbb{U}. \tag{1}$$

By convention, $\mathcal{S}$ represents a subfamily of class $\mathcal{A}$ containing all the functions that are univalent in $\mathbb{U}$ and satisfy the normalization property $f(0) = 0 = f'(0) - 1$. In geometric function theory, a key problem of analytic functions is their connection with coefficient estimates for these functions. In 1916, Bieberbach conjectured that $|a_n| \leq n$, $n = 2, 3, \ldots$ This famous coefficient problem, the "Bieberbach conjecture" played an important role in research in this field for decades until, in 1984, Louis de Branges proved this result; see [1]. During 1916–1984, researchers used different techniques and established a lot of coefficient results for various subclasses of $\mathcal{S}$. The subclasses worth mentioning here are the class $\mathcal{S}^*$, of starlike functions; the class $\mathcal{K}$, of convex functions; and $\mathcal{R}$, known as the functions of bounded turning. They are defined as below:

$$
\begin{aligned}
\mathcal{S}^* &= \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U} \right\}, \\
\mathcal{K} &= \left\{ f \in \mathcal{S} : \Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, \quad z \in \mathbb{U} \right\}, \\
\mathcal{R} &= \left\{ f \in \mathcal{S} : \Re\left[f'(z)\right] > 0, \quad z \in \mathbb{U} \right\},
\end{aligned}
$$

respectively. These classes can also be defined with the help of the subordination relation. We say that, for analytic functions, $f_1(z)$ is to be subordinated to $f_2(z)$ in the region $\mathbb{U}$ and denoted mathematically as $f_1(z) \prec f_2(z)$ if a function $u(z)$, known as the Schwarz function, satisfies the conditions $|u(z)| \leq 1$ and $u(0) = 1$, such that $f_1(z) = f_2(u(z))$. Moreover, if $f_2(z)$, belongs to $\mathcal{S}$, then due to [2,3], the following equivalent conditions will be true

$$f_1(\mathbb{U}) \subseteq f_2(\mathbb{U}) \text{ and } f_1(0) = f_2(0).$$

Thus, one can define $\mathcal{S}^*(\psi)$, $\mathcal{K}(\psi)$ and $\mathcal{R}(\psi)$ as:

$$
\begin{aligned}
\mathcal{S}^*(\psi) &= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \psi = \frac{1+z}{1-z}, \quad z \in \mathbb{U} \right\}, \\
\mathcal{K}(\psi) &= \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec \psi = \frac{1+z}{1-z}, \quad z \in \mathbb{U} \right\}, \\
\mathcal{R}(\psi) &= \left\{ f \in \mathcal{S} : f'(z) \prec \psi = \frac{1+z}{1-z}, \quad z \in \mathbb{U} \right\}.
\end{aligned}
\tag{2}
$$

In (2), if the right hand side is changed, the several well-known subfamilies are originated. For example, if we put $\psi = \frac{1+Az}{1+Bz}$, we obtain the Janowski-type class of starlike functions; see [4] for details. Meanwhile, if we change the parameters A and B by $1 - 2\alpha$ ans -1, respectively, then we obtain a family of starlike mappings of order α; these were defined and discussed in [5]. Additionally, for the choice of $\psi = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, we obtained a corresponding class of starlike functions, introduced by Ronning; see [6]. Furthermore, if $\psi = \sqrt{1+z}$, we obtain the class starlike function related to the lemniscate of the Bernoulli domain, defined by Cho et al. [7,8]. Goel and Kumar, in [9], defined the class $\mathcal{S}^*_{SG}$, the family of starlike mappings connected with a type of mapping known as modified sigmoid functions. Moreover, if we use $\psi = 1 + \sin(z)$, we obtain a subclass of starlike mappings in relation to the sine function; for details, see [10]. Mendiratta et al. The authors of [11] obtained a subfamily of strongly starlike mappings connected with the exponential function for the choice of $\psi = e^z$. Sharma et al. [12] derived a subfamily of starlike mappings associated with a cardoid domain.

In a similar way, one can find various important subclasses of these functions in [13–21] for some specific value of ψ. Of these, some well-known ones are the mappings associated with and related to Bell numbers, curves that are shell-like in association with Fibonacci numbers, and mappings associated with the conic domains.

Lately, utilizing the techniques of Ma and Minda [22], Gandhi [23] defined a family of starlike functions associated with a three-leaf function, i.e.,

$$\mathcal{S}^*_{3\mathcal{L}} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}, \ (z \in \mathbb{U}), \tag{3}$$

and characterized it with some important properties.

For the function f that has the form (1), Pommerenke [24,25] defined the Hankel determinant $H_{q,n}(f)$ with the parameter q, and $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{4}$$

For some subclasses related to the class $\mathcal{A}$, the bounds of $H_{q,n}(f)$, for any fixed integer q and n, are evaluated. Almost all the subclasses related to the class $\mathcal{S}$ were investigated for the sharp estimates of $H_{2,2}(f) = |a_2a_4 - a_3^2|$ by Janteng et al. [8,26]. However, for the family of close-to-convex functions, the sharp estimates are still not known (see [27]). On the other hand, Krishna et al. [28] proved the better estimate of $|H_{2,2}(f)|$ for a subfamily of Bazilevič functions. More detailed work on $H_{2,2}(f)$ can be seen in [29–33] and also the references cited therein.

The determinant

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \tag{5}$$

is known as the third-order Hankel determinant, and an estimate of this Hankel determinant $|H_{3,1}(f)|$ is more difficult than the second Hankel determinant; that is why a lot of researchers have focused on this field. In 1966–1967, Pommerenke defined the Hankel determinant, but it was not evaluated till the year 2010. In 2010, Babalola [34] was the first researcher who worked on $H_{3,1}(f)$ and successfully obtained the upper bounds of $|H_{3,1}(f)|$ related to the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$. Following this result, a few researchers extended this work for the various subcollections of univalent and holomorphic functions; see [35–47]. In the year 2017, Zaprawa [48] developed their work by proving

$$|H_{3,1}(f)| \leq \begin{cases} 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{49}{540}, & \text{for} \quad f \in \mathcal{K}, \\ \frac{41}{60}, & \text{for} \quad f \in \mathcal{R}. \end{cases}$$

Additionally, he asserted that the inequality above is not sharp. For sharpness, he considered the subfamily of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ functions to define them with m-fold symmetry, acquiring a sharp bound. In 2018, Kowalczyk et al. [49] and Lecko et al. [50] obtained sharp inequalities, which are

$$|H_{3,1}(f)| \leq 4/135, \quad \text{and} \quad |H_{3,1}(f)| \leq 1/9,$$

for the classes $\mathcal{K}$ and $\mathcal{S}^*(1/2)$, where the symbol $\mathcal{S}^*(1/2)$ represents the subcollection of starlike functions of order $1/2$. In [51], an improved bound $|H_{3,1}(f)| \leq 8/9$ for $f \in \mathcal{S}^*$ was given, which is not the best possible.

Our main purpose in this article is to first study four sharp coefficient estimates, the Fekete–Szegö inequality and sharp second Hankel determinant, the third-order Hankel determinant, the bounds for logarithmic coefficients, and the two- and three-fold symmetric functions.

2. The Sets of Lemmas

Let $\mathcal{P}$ be the subclass of mappings p that are analytic in $\mathbb{D}$ with $\Re p(z) > 0$ and its series form, as follows:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}). \tag{6}$$

Lemma 1. *If $p(z) \in \mathcal{P}$ and it is of the form (6), then*

$$|c_n| \leq 2 \text{ for } n \geq 1, \tag{7}$$
$$|c_{n+k} - \delta c_n c_k| \leq 2 \text{ for } 0 \leq \delta \leq 1, \tag{8}$$

and for $\xi \in \mathbb{C}$

$$\left|c_2 - \xi c_1^2\right| \leq 2 \max\{1; |2\xi - 1|\}, \tag{9}$$

and for real λ

$$\left|c_3 - \lambda c_2^2\right| \leq \begin{cases} -4\lambda + 2, \text{ if } \lambda \leq 0, \\ 2, \quad \text{ if } 0 \leq \lambda \leq 1, \\ 4\lambda - 2, \text{ if } \lambda \geq 1. \end{cases} \tag{10}$$

For the results in (7) and (8), see [52]. Additionall, see [53] for (9) and [22] for (10).

Lemma 2 ([54])**.** *Let $p \in \mathcal{P}$ have the representation of the form (6); then, for any real numbers α, β and γ*

$$\left|\alpha c_1^3 - \beta c_1 c_2 + \gamma c_3\right| \leq 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma| \tag{11}$$

Lemma 3 ([55])**.** *Let m, n, l and r satisfy the inequalities $0 < m < 1, 0 < r < 1$ and*

$$8r(1-r)\left[(mn - 2l)^2 + (m(r+m) - n)^2\right] + m(1-m)(n - 2rm)^2 \leq 4m^2(1-m)^2 r(1-r).$$

If $p \in \mathcal{P}$ and has power series (6), then

$$\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3}{2}nc_1^2c_2 - c_4\right| \leq 2.$$

Lemma 4 ([53])**.** *Let $h \in \mathcal{P}$ have the series expansion of the form (6) . Then, for $x, z \in \bar{\mathbb{D}} = \mathbb{D} \cup \{1\}$,*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{12}$$
$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z. \tag{13}$$

3. Upper Bound $H_{3,1}(f)$ for Set $\mathcal{S}^*_{3\mathcal{L}}$

Theorem 1. *Let $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ be of the form (1); then:*

$$|a_2| \leq \frac{4}{5}, \tag{14}$$
$$|a_3| \leq \frac{2}{5}, \tag{15}$$
$$|a_4| \leq \frac{4}{15}, \tag{16}$$
$$|a_5| \leq \frac{1}{5}. \tag{17}$$

All these bounds are sharp for the functions defined below, respectively.

$$f_1(z) = z\exp\left(\int_0^z \left(\frac{4}{5}+\frac{1}{5}t^3\right)dt\right) = z+\frac{4}{5}z^2+\cdots, \quad (18)$$

$$f_2(z) = z\exp\left(\int_0^z \left(\frac{4}{5}t+\frac{1}{5}t^7\right)dt\right) = z+\frac{2}{5}z^3+\cdots, \quad (19)$$

$$f_3(z) = z\exp\left(\int_0^z \left(\frac{4}{5}t^2+\frac{1}{5}t^{11}\right)dt\right) = z+\frac{4}{15}z^4+\cdots, \quad (20)$$

$$f_4(z) = z\exp\left(\int_0^z \left(\frac{4}{5}t^3+\frac{1}{5}t^{15}\right)dt\right) = z+\frac{1}{5}z^5+\cdots. \quad (21)$$

Proof. Since $f \in \mathcal{S}^*_{3\mathcal{L}}$, there exists an analytic function $w(z)$, $|w(z)| \le 1$ and $w(0) = 0$, such that

$$\frac{zf'(z)}{f(z)} = 1+\frac{4}{5}w(z)+\frac{1}{5}(w(z))^4.$$

Denote

$$\Psi(w(z)) = 1+\frac{4}{5}w(z)+\frac{1}{5}(w(z))^4,$$

and

$$k(z) = 1+c_1z+c_2z^2+\cdots = \frac{1+w(z)}{1-w(z)}.$$

Obviously, the function $k(z) \in \mathcal{P}$ and $w(z) = \frac{k(z)-1}{k(z)+1}$. This gives

$$w(z) = \frac{k(z)-1}{k(z)+1} = \frac{c_1z+c_2z^2+c_3z^3+\cdots}{2+c_1z+c_2z^2+c_3z^3+\cdots},$$

and

$$1+\frac{4}{5}w(z)+\frac{1}{5}(w(z))^4$$

$$\begin{aligned} &= 1+\frac{2}{5}c_1z+\left(\frac{2}{5}c_2-\frac{1}{5}c_1^2\right)z^2+\left(\frac{1}{10}c_1^3-\frac{2}{5}c_2c_1+\frac{2}{5}c_3\right)z^3 \\ &\quad+\left(-\frac{3}{80}c_1^4+\frac{3}{10}c_1^2c_2-\frac{2}{5}c_3c_1-\frac{1}{5}c_2^2+\frac{2}{5}c_4\right)z^4+\cdots. \end{aligned} \quad (22)$$

while

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1+a_2z+\left(2a_3-a_2^2\right)z^2+\left(a_2^3-3a_2a_3+3a_4\right)z^3 \\ &\quad+\left(-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2+4a_5\right)z^4+\cdots. \end{aligned} \quad (23)$$

Upon equating the coefficients of (22) and (23), we obtain

$$a_2 = \frac{2}{5}c_1, \quad (24)$$

$$a_3 = \frac{1}{5}c_2-\frac{1}{50}c_1^2, \quad (25)$$

$$a_4 = \frac{1}{250}c_1^3-\frac{4}{75}c_2c_1+\frac{2}{15}c_3, \quad (26)$$

$$a_5 = \frac{81}{40,000}c_1^4+\frac{643}{37,000}c_1^2c_2-\frac{7}{150}c_3c_1-\frac{3}{100}c_2^2+\frac{1}{10}c_4. \quad (27)$$

Now, applying (7), to Equation (24), we obtain

$$|a_2| \leq \frac{4}{5}.$$

Applying (8) with $n = k = 1$, to Equation (25), we obtain

$$|a_3| \leq \frac{2}{5}.$$

From the application of Lemma 2 to Equation (26), we obtain

$$|a_4| \leq \frac{4}{15}.$$

Now,

$$a_5 = \frac{-1}{10}\left(-\frac{81}{4000}c_1^4 - \frac{643}{3700}c_1^2c_2 + \frac{7}{15}c_3c_1 + \frac{3}{10}c_2^2 - c_4\right),$$

applying Lemma 3, with $l = \frac{-81}{4000}$, $r = \frac{3}{10}$, $m = \frac{7}{30}$ and $n = \frac{643}{5550}$; all the conditions of Lemma 3 are satisfied, so

$$\begin{aligned} |a_5| &= \frac{1}{10}\left|-\frac{81}{4000}c_1^4 - \frac{643}{3700}c_1^2c_2 + \frac{7}{15}c_3c_1 + \frac{3}{10}c_2^2 - c_4\right| \\ &\leq \frac{1}{10} \times 2 = \frac{1}{5}. \end{aligned}$$

Hence, complete the proof. □

Theorem 2. *Let $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ be of the form (1). Then,*

$$\left|a_3 - \zeta a_2^2\right| \leq \frac{2}{5}\max\left\{1, \frac{4}{5}|2\zeta - 1|\right\} \text{ for } \zeta \in \mathbb{C}. \tag{28}$$

The result is sharp for the function defined in Equation (19).

Proof. Since from (24) and (25), we have

$$\left|a_3 - \zeta a_2^2\right| = \frac{1}{5}\left|c_2 - \frac{8\zeta + 1}{10}c_1^2\right|,$$

by applying (9) to the above equation, we obtain the desired result. □

For $\zeta = 1$, we obtain the corollary stated below:

Corollary 1. *Let $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ be of the form (1). Then*

$$\left|a_3 - a_2^2\right| \leq \frac{2}{5}. \tag{29}$$

The bound is sharp for the function defined in Equation (19).

Theorem 3. *Let $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ be of the form (1). Then,*

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} \frac{-16\lambda + 8}{25}, \text{ if } \lambda \leq -\frac{1}{8}, \\ \frac{2}{5}, \quad \text{ if } -\frac{1}{8} \leq \lambda \leq \frac{9}{8}, \\ \frac{-16\lambda + 8}{25}, \text{ if } \lambda \geq \frac{9}{8}. \end{cases} \tag{30}$$

Proof. Since from (24) and (25), we have

$$\left|a_3 - \lambda a_2^2\right| = \frac{1}{5}\left|c_2 - \frac{8\lambda + 1}{10}c_1^2\right|,$$

by applying (10) to the above equation, we obtain the desired result. □

Theorem 4. *Let $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ be of the form (1). Then,*

$$|a_4 - a_2 a_3| \leq \frac{4}{15}. \tag{31}$$

The estimate is sharp for the function defined in Equation (20).

Proof. Since from (24)–(26), we have

$$|a_4 - a_2 a_3| = \left|\frac{3}{250}c_1^3 - \frac{2}{15}c_1 c_2 + \frac{2}{15}c_3\right|,$$

now, the implementation of Lemma 2 to above equation leads us to the desired result. □

Theorem 5. *Let $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ be of the form (1). Then,*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{4}{25}. \tag{32}$$

The result is sharp for the function defined in Equation (19).

Proof. Since from (24)–(26), we have

$$\left|a_2 a_4 - a_3^2\right| = \left|\frac{3}{2500}c_1^4 - \frac{1}{75}c_1^2 c_2 + \frac{4}{75}c_3 c_1 - \frac{1}{25}c_2^2\right|,$$

using (12) and (13) to put c_2 and c_3 in terms of c_1 and directly state that $c_1 = c$ with $c \in [0,2]$, we have

$$\left|a_2 a_4 - a_3^2\right| = \left|-\frac{1}{100}\left(4 - c^2\right)^2 x^2 - \frac{1}{75}\left(4 - c^2\right)x^2 c^2 + \frac{2}{75}\left(4 - c^2\right)\left(1 - |x|^2\right)cz - \frac{4}{1875}c^4\right|.$$

Applying a triangular inequality along with $|z| \leq 1$ and $|x| = b$ with $b \in [0,1]$, we have

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{1}{100}\left(4 - c^2\right)^2 b^2 + \frac{1}{75}\left(4 - c^2\right)b^2 c^2 + \frac{2}{75}\left(4 - c^2\right)\left(1 - b^2\right)c + \frac{4}{1875}c^4 = H(c,b).$$

Since $H(c,b)$ is an increasing function with respect to b so $H(c,b) \leq H(c,1)$, putting $b = 1$ in the above, we obtain

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{1}{100}\left(4 - c^2\right)^2 + \frac{1}{75}\left(4 - c^2\right)c^2 + \frac{4}{1875}c^4 = G(c).$$

Now,

$$\begin{aligned} G'(c) &= -\frac{3}{625}c^3 - \frac{4}{75}c, \\ G''(c) &= -\frac{9}{625}c^2 - \frac{4}{75}. \end{aligned}$$

Clearly, $G''(c) < 0$ for $c = 0$, so the maximum is attained at $c = 0$; hence,

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{25}.$$

□

Now, one comes to the third Hankel determinant:

Theorem 6. *Let $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ be of the form (1). Then,*

$$|H_{3,1}(f)| \leq \frac{242}{1125} \simeq 0.215.$$

Proof. From (5), we have

$$|H_{3,1}(f)| \leq |a_3|\left|a_2a_4 - a_3^2\right| + |a_4||a_4 - a_2a_3| + |a_3|\left|a_3 - a_2^2\right|,$$

and using (15)–(17), (29), (31) and (32), we obtain the required result. □

For function f of class $\mathcal{S}$, we denote the logarithmic coefficients with $\gamma_n = \gamma_n(f)$, and they are defined by the following series expansion:

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n z^n.$$

The logarithmic coefficients of function f given in (1) are as follows:

$$\gamma_1 = \frac{1}{2}a_2, \tag{33}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{34}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{35}$$

Theorem 7. *Let $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ be of the form (1); then,*

$$|\gamma_1| \leq \frac{2}{5},$$

$$|\gamma_2| \leq \frac{1}{5}, \tag{36}$$

$$|\gamma_3| \leq \frac{6}{25}. \tag{37}$$

The first two bounds are sharp.

Proof. From Equations (33) to (35), we obtain

$$\begin{aligned}
|\gamma_1| &= \frac{1}{5}c_1, \\
|\gamma_2| &= \frac{1}{10}\left(c_2 - \frac{1}{2}c_1^2\right), \\
|\gamma_3| &= \frac{1}{2}\left[\left(\frac{3}{250}c_1^3 - \frac{2}{15}c_1c_2 + \frac{2}{15}c_3\right) + \frac{4}{75}c_1^2\right].
\end{aligned}$$

The bounds of $|\gamma_1|$, $|\gamma_2|$ follow from Lemma 1, and $|\gamma_3|$ follows from Lemmas 1 and 2. □

4. Bounds of $H_{3,1}(f)$ for Two-Fold and Three-Fold Symmetric Functions

Let $m \in \mathbb{N} = \{1, 2, 3, \cdots\}$; if a rotation of domain $\mathbb{D}$ about the origin through an angle $\frac{2\pi}{m}$ carries itself on the domain, $\mathbb{D}$ is called m-fold symmetric. It is very clear to see that an analytic function f is m-fold symmetric in $\mathbb{D}$, if

$$f\left(e^{\frac{2\pi}{m}} z\right) = e^{\frac{2\pi}{m}} f(z), z \in \mathbb{D}.$$

By $\mathcal{S}^{(m)}$, we mean the set of m-fold symmetric univalent functions having the following series form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1},\ z \in \mathbb{D}. \tag{38}$$

The subclass $\mathcal{S}^{*(m)}_{3\mathcal{L}}$ is a set of m-fold symmetric starlike functions associated with a modified sigmoid function. More precisely, an analytic function f of the form (38) belongs to class $\mathcal{S}^{*(m)}_{3\mathcal{L}}$ if and only if

$$\frac{zf'(z)}{f(z)} = 1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4,\ p \in \mathcal{P}^{(m)}, \tag{39}$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk},\ z \in \mathbb{D} \right\}. \tag{40}$$

Theorem 8. *If $f \in \mathcal{S}^{*(2)}_{3\mathcal{L}}$ is of the form (38), then*

$$|H_{3,1}(f)| \leq \frac{2}{25}. \tag{41}$$

Proof. Since $f \in \mathcal{S}^{*(2)}_{3\mathcal{L}}$, there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4.$$

Using the series form (38) and (40), when $m = 2$ in the above relation, we have

$$a_3 = \frac{1}{5} c_2, \tag{42}$$

$$a_5 = \frac{1}{10} c_4 - \frac{3}{100} c_2^2. \tag{43}$$

Now, using (42) and (43), we obtain

$$\begin{aligned} |H_{3,1}(f)| &= |a_3| \left| a_5 - a_3^2 \right| \\ &= \frac{1}{50} |c_2| \left| c_4 - \frac{7}{10} c_2^2 \right| \end{aligned}$$

Now, using (7) and (8) with the above, we obtain the required result. □

Theorem 9. *If $f \in \mathcal{S}^{*(3)}_{3\mathcal{L}}$ is of the form (38), then*

$$|H_{3,1}(f)| \leq \frac{16}{225}.$$

The result is sharp for the function defined in (20).

Proof. Since $f \in \mathcal{S}^{*(3)}_{3\mathcal{L}}$, there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{4}{5}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{5}\left(\frac{p(z)-1}{p(z)+1}\right)^4.$$

Using the series form (38) and (40), when $m = 3$ in above relation, we have

$$a_4 = \frac{2}{15}c_3,$$

Now,

$$H_{3,1}(f) = -a_4^2.$$

Therefore,

$$\begin{aligned} |H_{3,1}(f)| &= \left|-\frac{4}{225}c_3^2\right| \\ &= \frac{4}{225}|c_3|^2 \end{aligned}$$

Using (7), we obtain the desired result. □

5. Conclusions

In the present article, we find four initial sharp coefficient bounds, the sharp Fekete–Szegö inequality, the sharp second Hankel determinant, the third Hankel determinant, and the bounds for logarithmic coefficients, and at last, we find out the bounds of $H_{3,1}(f)$ for two-fold and three-fold symmetric functions for the class $\mathcal{S}^*_{3\mathcal{L}}$. Obtaining a sharp estimate for the third Hankel determinant is still an open problem for a considered class. Additionally, there is an opportunity for researchers to investigate the generalized Zalcman conjecture, Krushkal inequality and fourth-order Hankel determinant for this class.

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