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Reduction Formulas for Generalized Hypergeometric Series Associated with New Sequences and Applications

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Abstract: In this paper, by introducing two sequences of *new* numbers and their derivatives, which are closely related to the Stirling numbers of the first kind, and choosing to employ six known generalized Kummer's summation formulas for ${}_2F_1(-1)$ and ${}_2F_1(1/2)$, we establish six classes of generalized summation formulas for ${}_{p+2}F_{p+1}$ with arguments -1 and $1/2$ for any positive integer p. Next, by differentiating both sides of six chosen formulas presented here with respect to a specific parameter, among numerous ones, we demonstrate six identities in connection with finite sums of ${}_4F_3(-1)$ and ${}_4F_3(1/2)$. Further, we choose to give simple particular identities of some formulas presented here. We conclude this paper by highlighting a potential use of the newly presented numbers and posing some problems.

Keywords: Gamma function; Psi function; Pochhammer symbol; hypergeometric function ${}_2F_1$; generalized hypergeometric functions ${}_tF_u$; Gauss's summation theorem for ${}_2F_1(1)$; Kummer's summation theorem for ${}_2F_1(-1)$; generalized Kummer's summation theorem for ${}_2F_1(-1)$; Stirling numbers of the first kind

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1. Introduction and Preliminaries

The Pochhammer symbol $(\xi)_\eta$ $(\xi, \eta \in \mathbb{C})$ is defined, in terms of Gamma function Γ (see, e.g., [1], p. 2 and p. 5), by

$$\begin{aligned}(\xi)_\eta &= \frac{\Gamma(\xi+\eta)}{\Gamma(\xi)} \quad \left(\xi+\eta \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ \eta \in \mathbb{C}\setminus\{0\};\ \xi \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ \eta = 0\right) \\ &= \begin{cases} 1 & (\eta = 0), \\ \xi(\xi+1)\cdots(\xi+n-1) & (\eta = n \in \mathbb{N}), \end{cases}\end{aligned} \tag{1}$$

it accepted that $(0)_0 = 1$. Here and throughout, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ represent, respectively, the sets of complex numbers, positive real numbers, integers, and positive integers. Furthermore, let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, $\mathbb{Z}^- := \mathbb{Z}\setminus\mathbb{N}_0$ and $\mathbb{Z}_0^- := \mathbb{Z}\setminus\mathbb{N}$. Further, throughout this article, it is assumed that an empty sum and an empty product are read as 0 and 1, respectively. The generalized hypergeometric series (or function) ${}_pF_q$ $(p, q \in \mathbb{N}_0)$, which is a parametric and logical extension of the Gaussian hypergeometric series ${}_2F_1$, is defined by (see, e.g., [1–9])

$$\begin{aligned}{}_pF_q\left[\begin{matrix}\mu_1, \ldots, \mu_p\,; \\ \nu_1, \ldots, \nu_q\,;\end{matrix}\ w\right] &= \sum_{\ell=0}^{\infty} \frac{\prod_{j=1}^{p}(\mu_j)_\ell}{\prod_{j=1}^{q}(\nu_j)_\ell}\,\frac{w^\ell}{\ell!} \\ &= {}_pF_q(\mu_1, \ldots, \mu_p; \nu_1, \ldots, \nu_q; w).\end{aligned} \tag{2}$$

Here it is supposed that the variable w, the numerator parameters $\mu_1, \ldots, \mu_p$, and the denominator parameters $\nu_1, \ldots, \nu_q$ take on complex values, provided that

$$(\nu_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1, \ldots, q). \tag{3}$$

Then, if a numerator parameter is in $\mathbb{Z}_0^-$, the series ${}_pF_q$ is found to terminate and becomes a polynomial in w.

With none of the numerator and denominator parameters being zero or a negative integer, the series ${}_pF_q$ in (2)

(i) diverges for all $w \in \mathbb{C} \setminus \{0\}$, if $p > q + 1$;
(ii) converges for all $w \in \mathbb{C}$, if $p \leq q$;
(iii) converges for $|w| < 1$ and diverges for $|w| > 1$ if $p = q + 1$;
(iv) converges absolutely for $|w| = 1$, if $p = q + 1$ and $\Re(\varpi) > 0$;
(v) converges conditionally for $|w| = 1$ $(w \neq 1)$, if $p = q + 1$ and $-1 < \Re(\varpi) \leqq 0$;
(vi) diverges for $|w| = 1$, if $p = q + 1$ and $\Re(\varpi) \leqq -1$.

where

$$\varpi := \sum_{j=1}^{q} \nu_j - \sum_{j=1}^{p} \mu_j \tag{4}$$

which is called the parametric excess of the series.

Gauss's famous summation formula [10]:

$$\begin{gathered} {}_2F_1(\kappa, \lambda; \mu; 1) = \frac{\Gamma(\mu)\,\Gamma(\mu - \kappa - \lambda)}{\Gamma(\mu - \kappa)\,\Gamma(\mu - \lambda)} \\ \left(\Re(\mu - \kappa - \lambda) > 0,\ \mu \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) \end{gathered} \tag{5}$$

has been a significant, pioneering, and essential identity, especially in the theories of hypergeometric and generalized hypergeometric functions, as well as related special functions. Formula (5) can be proved by using Euler's integral representation for ${}_2F_1(z)$ (see, e.g., [6], pp. 44–49) or telescoping (see, e.g., [11], pp. 181–182). Since (5) appeared, a number of researchers have devoted their arduous, intrigued and penetrated endeavors to getting summation formulas for the generalized hypergeometric series in (2). As a result, the generalized hypergeometric series in (2) of the case $p = q + 1$ have been found to be classified as follows: ${}_{q+1}F_q$ in (2) is said to be ϖ-balanced if the parametric excess equals ϖ and balanced if $\varpi = 1$. Further, if $\varpi = 1$ and one of the numerator parameters is a negative integer, it is called Saalschützian. It is well-poised if the parameters μ_j, ν_j can be separately permuted so that

$$1 + \mu_1 = \mu_2 + \nu_1 = \cdots = \mu_{q+1} + \nu_q$$

and very well-poised if the condition $\mu_2 = 1 + \frac{\mu_1}{2}$ holds true, along with the above condition for the well-poised nature. Consequently, a large number of summation and transformation formulas for ${}_pF_q$ have been established by means of diverse techniques. In fact, usually, certain mixed techniques are used in getting a summation formula or a transformation formula for ${}_pF_q$. Here we recall only several representative techniques which are employed in deriving some summation and transformation formulas for ${}_pF_q$:

(i) Contiguous function relations (and computer programs) [12–25].
(ii) The idea of partition of the set of nonnegative integers into its terms modulo N applied to a series involving functions Ψ_n $(n \in \mathbb{N}_0)$ displayed by

$$\sum_{n=0}^{\infty} \Psi_n = \sum_{r=0}^{N-1} \sum_{n=0}^{\infty} \Psi_{nN+r} \tag{6}$$

is ubiquitously employed (see, e.g., [26,27]). In particular, partition of the series into even and odd terms gives

$$\sum_{n=0}^{\infty} \Psi_n = \sum_{n=0}^{\infty} \Psi_{2n} + \sum_{n=0}^{\infty} \Psi_{2n+1}. \tag{7}$$

The (6) and (7) have been used to get certain identities involving generalized hypergeometric series and their extensions (see, e.g., [28–37]). Exton [30] considered the following two combinations

$$\begin{aligned}
&{}_{q+1}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_{q+1}; \\ b_1, b_2, \ldots, b_q; \end{matrix}\ 1\right] + {}_{q+1}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_{q+1}; \\ b_1, b_2, \ldots, b_q; \end{matrix}\ -1\right] \\
&= 2\, {}_{2q+2}F_{2q+1}\left[\begin{matrix} \frac{a_1}{2}, \frac{a_1+1}{2}, \ldots, \frac{a_{q+1}}{2}, \frac{a_{q+1}+1}{2}; \\ \frac{1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}, \ldots, \frac{b_q}{2}, \frac{b_q+1}{2}; \end{matrix}\ 1\right]
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&{}_{q+1}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_{q+1}; \\ b_1, b_2, \ldots, b_q; \end{matrix}\ 1\right] - {}_{q+1}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_{q+1}; \\ b_1, b_2, \ldots, b_q; \end{matrix}\ -1\right] \\
&= 2\,\frac{\prod_{j=1}^{q+1} a_j}{\prod_{j=1}^{q} b_j}\, {}_{2q+2}F_{2q+1}\left[\begin{matrix} \frac{a_1+1}{2}, \frac{a_1}{2}+1, \ldots, \frac{a_{q+1}+1}{2}, \frac{a_{q+1}}{2}+1; \\ \frac{3}{2}, \frac{b_1+1}{2}, \frac{b_1}{2}+1, \ldots, \frac{b_q+1}{2}, \frac{b_q}{2}+1; \end{matrix}\ 1\right].
\end{aligned} \tag{9}$$

If the summation formulas for ${}_{q+1}F_q(1)$ and ${}_{q+1}F_q(-1)$ are known, then summation formulas for ${}_{2q+2}F_{2q+1}(1)$ in (8) and (9) can be derived. Obviously, the reverse process can work.

(iii) The method in (ii) is to obtain summation formulas for certain generalized hypergeometric functions of higher order from those of lower order. Conversely, reduction formulas of generalized hypergeometric and their extended special functions are to reduce those of higher order to some other ones of lower order (see, e.g., [14,38–45]).

In connection with the method (iii), for a generalized hypergeometric function ${}_pF_q(z)$ with positive integral differences between certain numerator and denominator parameters, Karlsson [39] provided a formula expressing the ${}_pF_q(z)$ as a finite sum of lower-order functions as follows (see also [42,46,47]):

$$\begin{aligned}
&{}_pF_q\left[\begin{matrix} b_1+\ell_1, \ldots, b_n+\ell_n, a_{n+1}, \ldots, a_p; \\ b_1, \ldots, b_n, b_{n+1}, \ldots, b_q; \end{matrix}\ z\right] \\
&= \sum_{j_1=0}^{\ell_1} \cdots \sum_{j_n=0}^{\ell_n} A(j_1, \ldots, j_n)\, z^{J_n} \\
&\qquad \times {}_{p-n}F_{q-n}\left[\begin{matrix} a_{n+1}+J_n, \ldots, \ldots, a_p+J_n; \\ b_{n+1}+J_n, \ldots, b_q+J_n; \end{matrix}\ z\right],
\end{aligned} \tag{10}$$

where $J_n = j_1 + \cdots + j_n$ and

$$A(j_1, \ldots, j_n) = \prod_{r=1}^{n} \binom{\ell_r}{j_r} \cdot \frac{\prod_{r=2}^{n} (b_r+\ell_r)_{J_{r-1}} \cdot \prod_{r=n+1}^{p} (a_r)_{J_n}}{\prod_{r=1}^{n} (b_r)_{J_r} \cdot \prod_{r=n+1}^{q} (b_r)_{J_n}}. \tag{11}$$

Here the following constraints are assumed that, with suitable permutation of parameters, $a_r = b_r + \ell_r$, $\ell_r \in \mathbb{N}$ $(r = 1, \ldots, n)$, $n \leq \min\{p,q\}$, $p \leq q+1$, $b_r \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(r = 1, \ldots, q)$; if $a_r \in \mathbb{Z}_0^-$ for some $r \in \{1, \ldots, p\}$, the condition $p \leq q+1$ is cancelled.

Using (10), Minton's two summation theorems in [43] for $p = q+1$, $z = 1$ are derived. Srivastava [45] gave a simpler proof of (10). Gottschalk and Maslen [38] provided a good account of reduction formulas for the generalized hypergeometric functions of one variable with some useful comments on (10) and listed certain transformation formulas for generalized hypergeometric functions in [38].

The content of this paper would be derived from the reduction formula (10). Yet, in this paper, by introducing two sequences of *new* numbers and their derivatives as in Section 2 and using the six generalized summation formulas (15)–(20), we aim to establish families of generalized summation formulas for ${}_{t+2}F_{t+1}$ $(t \in \mathbb{N})$ with their arguments -1 and $1/2$ as in Sections 4 and 5. Furthermore, we select to give simple particular identities of some formulas presented here. By differentiating both sides of two chosen formulas presented here with respect to a specific parameter, among numerous ones, further, we demonstrate two identities associated with finite sums of ${}_4F_3(-1)$. We close this article by emphasizing some of the possible applications for the newly introduced numbers and presenting certain problems.

For our purpose, we also recall three basic and useful summation formulas for ${}_2F_1$ due to Kummer [48], p. 134, Entries 1, 2 and 3 (see also [49], Equations (1.3), (1.4) and (1.5); see further [50]) (the interested reader may refer to [49], p. 853 for clarifications on the first and true contributors to the following three summation formulae):

Summation Formula 1 due to Kummer:

$$
{}_2F_1\left[\begin{matrix} \kappa,\ \lambda\,; \\ 1+\kappa-\lambda\,; \end{matrix}\ -1\right] = \frac{\Gamma(1+\kappa-\lambda)\Gamma\left(1+\frac{\kappa}{2}\right)}{\Gamma\left(1+\frac{\kappa}{2}-\lambda\right)\Gamma(1+\kappa)}
\quad \big(\kappa-\lambda \in \mathbb{C}\setminus\mathbb{Z}^-,\ \Re(\lambda) < 1\big). \tag{12}
$$

Summation Formula 2 due to Kummer:

$$
{}_2F_1\left[\begin{matrix} \kappa,\ \lambda\ ; \\ \frac{1}{2}(\kappa+\lambda+1)\ ; \end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\kappa+\frac{1}{2}\lambda+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\kappa+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\lambda+\frac{1}{2}\right)}
\quad \left(\frac{\kappa+\lambda+1}{2} \in \mathbb{C}\setminus\mathbb{Z}_0^-\right). \tag{13}
$$

Summation Formula 3 due to Kummer:

$$
{}_2F_1\left[\begin{matrix} \kappa,\ 1-\kappa\ ; \\ \lambda\ ; \end{matrix}\ \frac{1}{2}\right] = \frac{2^{1-\lambda}\,\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\lambda+\frac{1}{2}\kappa\right)\Gamma\left(\frac{1}{2}\lambda-\frac{1}{2}\kappa+\frac{1}{2}\right)} \quad \big(\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-\big). \tag{14}
$$

Further a number of generalizations and contiguous extensions of the above-mentioned Kummer's summation theorems have been given (see, e.g., [16,49,51–54] and the references therein). Amid this trend, Choi et al. [51], Equations (2.2) and (2.3) presented the following extensions of (12) (see also [53], Theorems 3 and 4):

$$
\begin{aligned}
{}_2F_1&\left[\begin{matrix} \kappa,\ \lambda\,; \\ 1+\kappa-\lambda+p\,; \end{matrix}\ -1\right] \\
&= \frac{\Gamma(1+\kappa-\lambda+p)}{2\,\Gamma(\kappa)\,(1-\lambda)_p}\sum_{r=0}^{p}\binom{p}{r}\frac{(-1)^r\,\Gamma\left(\frac{r+\kappa}{2}\right)}{\Gamma\left(\frac{r+\kappa}{2}+1-\lambda\right)} \\
&\left(p \in \mathbb{N}_0,\ \kappa-\lambda+p \in \mathbb{C}\setminus\mathbb{Z}^-,\ \Re(\lambda) < 1+\frac{p}{2}\right)
\end{aligned} \tag{15}
$$

and

$$
{}_2F_1\left[\begin{matrix}\kappa,\ \lambda\,; \\ 1+\kappa-\lambda-p\,;\end{matrix}\ -1\right]
= \frac{\Gamma(1+\kappa-\lambda-p)}{2\,\Gamma(\kappa)} \sum_{r=0}^{p} \binom{p}{r} \frac{\Gamma\left(\frac{r+\kappa}{2}\right)}{\Gamma\left(\frac{r+\kappa}{2}+1-\lambda-p\right)}
\tag{16}
$$

$$
\left(p \in \mathbb{N}_0,\ \kappa-\lambda-p \in \mathbb{C}\setminus\mathbb{Z}^-,\ \Re(\lambda) < 1-\frac{p}{2}\right).
$$

Rakha and Rathie [53], Theorem 1 gave the following generalization of (13):

$$
{}_2F_1\left[\begin{matrix}\kappa,\ \lambda\,; \\ \dfrac{1+\kappa+\lambda+p}{2}\,;\end{matrix}\ \frac{1}{2}\right]
= \frac{2^{\kappa-1}\Gamma\left(\frac{1+\kappa+\lambda+p}{2}\right)\Gamma\left(\frac{1-\kappa+\lambda-p}{2}\right)}{\Gamma(\kappa)\Gamma\left(\frac{1-\kappa+\lambda+p}{2}\right)}
\times \sum_{r=0}^{p} \binom{p}{r} \frac{(-1)^r\,\Gamma\left(\frac{\kappa+r}{2}\right)}{\Gamma\left(\frac{1+\lambda+r-p}{2}\right)}
\tag{17}
$$

$$
\left(p \in \mathbb{N}_0,\ \frac{1+\kappa+\lambda+p}{2} \in \mathbb{C}\setminus\mathbb{Z}_0^-\right).
$$

The following extension of (13) is recorded in [9], p. 491, Entry 7.3.7-2 (see also [53], Theorem 2):

$$
{}_2F_1\left[\begin{matrix}\kappa,\ \lambda\,; \\ \dfrac{1+\kappa+\lambda-p}{2}\,;\end{matrix}\ \frac{1}{2}\right]
= \frac{2^{\lambda-1}\Gamma\left(\frac{1+\kappa+\lambda-p}{2}\right)}{\Gamma(\lambda)} \sum_{r=0}^{p} \binom{p}{r} \frac{\Gamma\left(\frac{\lambda+r}{2}\right)}{\Gamma\left(\frac{1+\kappa+r-p}{2}\right)}
\tag{18}
$$

$$
\left(p \in \mathbb{N}_0,\ \frac{1+\kappa+\lambda-p}{2} \in \mathbb{C}\setminus\mathbb{Z}_0^-\right).
$$

Rakha and Rathie ([53], Theorems 5 and 6) provided two generalizations of (14) which, with the aid of Legendre's duplication formula for the Gamma function (e.g., [1], p. 6, Equation (29)), are slightly modified as follows:

$$
{}_2F_1\left[\begin{matrix}\kappa,\ 1-\kappa+p\,; \\ \lambda\,;\end{matrix}\ \frac{1}{2}\right]
= \frac{2^{p-\kappa}\,\Gamma(\kappa-p)\,\Gamma(\lambda)}{\Gamma(\kappa)\,\Gamma(\lambda-\kappa)}
\times \sum_{r=0}^{p} (-1)^r \binom{p}{r} \frac{\Gamma\left(\frac{\lambda-\kappa+r}{2}\right)}{\Gamma\left(\frac{\lambda+\kappa+r}{2}-p\right)}
\tag{19}
$$

$$
\left(\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ p \in \mathbb{N}_0\right)
$$

and

$$
{}_2F_1\left[\begin{matrix}\kappa,\ 1-\kappa-p\,; \\ \lambda\,;\end{matrix}\ \frac{1}{2}\right]
= \frac{2^{-p-\kappa}\,\Gamma(\lambda)}{\Gamma(\lambda-\kappa)} \sum_{r=0}^{p} \binom{p}{r} \frac{\Gamma\left(\frac{\lambda-\kappa+r}{2}\right)}{\Gamma\left(\frac{\lambda+\kappa+r}{2}\right)}
\tag{20}
$$

$$
\left(\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ p \in \mathbb{N}_0\right),
$$

which is a corrected version of [53], Theorem 6.

In addition, we recall the Psi (or digamma) function $\psi(z)$ (see, e.g., [1], pp. 24–33) defined by

$$
\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \left(z \in \mathbb{C}\setminus\mathbb{Z}_0^-\right).
\tag{21}
$$

We recall one of the many identities involving the Psi function

$$
\psi(z+n) - \psi(z) = \sum_{j=1}^{n} \frac{1}{z+j-1} \quad (n \in \mathbb{N}).
\tag{22}
$$

Remark 1. *Magnus Gösta Mittag–Leffler (1846–1927), a Swedish mathematician (see [55]; see also [56,57]), invented the function $E_\alpha(z)$ (23) in conjunction with the summation technique for divergent series, which is eponymously referred to as the Mittag–Leffler function and represented by the following convergent power series across the whole complex plane:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (\Re(\alpha) > 0,\, z \in \mathbb{C}). \tag{23}$$

The two parameterized Mittag–Leffler function $E_{\alpha,\beta}(z)$ is defined by (see, e.g., [58,59])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (\Re(\alpha) > 0,\, \beta \in \mathbb{C}). \tag{24}$$

There have been a variety of extensions of the Mittag–Leffler functions (23) and (24), most of which belong to certain special cases of the following Fox-Wright function (see [60–63], [64], p. 21):

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p)\,; \\ (\beta_1, B_1), \ldots, (\beta_q, B_q)\,;\end{matrix}\; z\right] = \sum_{k=0}^{\infty} \frac{\prod_{\ell=1}^{p} \Gamma(\alpha_\ell + A_\ell\, k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j\, k)} \frac{z^k}{k!}, \tag{25}$$

where $z \in \mathbb{C}$, $\alpha_\ell, \beta_j \in \mathbb{C}$ $(\ell = 1, \ldots, p,\ j = 1, \ldots, q)$, the coefficients $A_1, \ldots, A_p \in \mathbb{R}^+$ and $B_1, \ldots, B_q \in \mathbb{R}^+$ such that $\alpha_\ell + A_\ell k \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(k \in \mathbb{N}_0)$ and

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0. \tag{26}$$

A particular case of (25) is

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, 1), \ldots, (\alpha_p, 1)\,; \\ (\beta_1, 1), \ldots, (\beta_q, 1)\,;\end{matrix}\; z\right] = \frac{\prod_{\ell=1}^{p} \Gamma(\alpha_\ell)}{\prod_{j=1}^{q} \Gamma(\beta_j)}\, {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,;\end{matrix}\; z\right]. \tag{27}$$

In light of (27), the topic of this article may be regarded to be Mittag–Leffler type functions.

Indeed, owing to the range of its applications in fractional calculus, some scholars have nicknamed the Mittag–Leffler function the "Queen Function of the Fractional Calculus" in the past (see, e.g., [65]).

2. Sequences of New Numbers

Numerous polynomials, numbers, their extensions, degenerations, and new polynomials and new numbers have been developed and studied, owing primarily to their potential applications and use in a diverse variety of research fields (see, e.g., [66–71] and the references therein). For example, Bernoulli polynomials and numbers are among most important and useful ones (see, e.g., [5], pp. 35–40, [1], Sections 1.7 and 1.8). As with Definitions 1 and 2, this section introduces two sequences of new numbers and their derivatives that are and will be useful (at the very least) for our current and related study topics.

Definition 1. *A sequence of new numbers* $\{\mathcal{A}_j(\alpha,\ell)\}_{j=0}^{\ell}$ *is defined by*

$$\begin{aligned}(\alpha+k)_\ell &= (\alpha+k)(\alpha+k+1)\cdots(\alpha+k+\ell-1)\\ &:= \sum_{j=0}^{\ell}\mathcal{A}_j(\alpha,\ell)\,k(k-1)\cdots(k-j+1)\\ &\quad (k\in\mathbb{N}_0,\ \ell\in\mathbb{N},\ \alpha\in\mathbb{C})\end{aligned} \tag{28}$$

and

$$\mathcal{A}_0(\alpha,0) := 1 \quad (\alpha\in\mathbb{C}). \tag{29}$$

Definition 2. *A sequence of new numbers* $\{\mathcal{B}_j(\alpha,\ell)\}_{j=0}^{\ell}$ *is defined by*

$$\mathcal{B}_j(\alpha,\ell) := \frac{d}{d\alpha}\mathcal{A}_j(\alpha,\ell) \quad (\ell\in\mathbb{N},\ \alpha\in\mathbb{C}) \tag{30}$$

and

$$\mathcal{B}_0(\alpha,0) := 0 \quad (\alpha\in\mathbb{C}). \tag{31}$$

Both of the following lemmas may be used to represent the numbers in Definitions 1 and 2 explicitly.

Lemma 1. *Let* $\alpha\in\mathbb{C}$ *and* $\ell\in\mathbb{N}_0$. *Then*

$$\sum_{j=\nu}^{\ell}\mathcal{A}_j(\alpha,\ell)\,s(j,\nu) = \sum_{j=\nu}^{\ell}(-1)^{\ell+j}s(\ell,j)\binom{j}{\nu}\alpha^{j-\nu} \quad (\nu=0,1,\ldots,\ell). \tag{32}$$

Also

$$\mathcal{A}_j(\alpha,\ell) = \binom{\ell}{j}\frac{(\alpha)_\ell}{(\alpha)_j} = \binom{\ell}{j}(\alpha+j)_{\ell-j} \quad (j=0,1,\ldots,\ell). \tag{33}$$

Lemma 2. *Let* $\alpha\in\mathbb{C}$ *and* $\ell\in\mathbb{N}_0$. *Then*

$$\begin{aligned}\sum_{j=\nu}^{\ell}\mathcal{B}_j(\alpha,\ell)\,s(j,\nu) &= \sum_{j=\nu}^{\ell}(-1)^{\ell+j}j\,s(\ell,j)\binom{j-1}{\nu}\alpha^{j-1-\nu}\\ &\quad (\nu=0,1,\ldots,\ell).\end{aligned} \tag{34}$$

Also

$$\mathcal{B}_j(\alpha,\ell) = \binom{\ell}{j}(\alpha+j)_{\ell-j}\sum_{k=j}^{\ell-1}\frac{1}{\alpha+k} \quad (j=0,1,\ldots,\ell). \tag{35}$$

Proof of Lemma 1. The Stirling numbers $s(m,r)$ of the first kind are recalled and defined by the generating function (see, e.g., [1], Section 1.6)

$$\omega(\omega-1)\cdots(\omega-m+1) = \sum_{r=0}^{m}s(m,r)\,\omega^r. \tag{36}$$

We use (36) to expand the Pochhammer symbol (1) as follows:

$$(\omega)_m = \omega(\omega+1)\cdots(\omega+m-1) = \sum_{r=0}^{m}(-1)^{m+r}s(m,r)\,\omega^r, \tag{37}$$

where $(-1)^{m+r}s(m,r)$ indicates the number of permutations of m symbols, which possesses exactly r cycles.

Applying (36) and (37) to (28), we obtain

$$\sum_{j=0}^{\ell}(-1)^{\ell+j}s(\ell,j)\sum_{\nu=0}^{j}\binom{j}{\nu}\alpha^{j-\nu}\,k^{\nu}=\sum_{j=0}^{\ell}\mathcal{A}_j(\alpha,\ell)\sum_{\nu=0}^{j}s(j,\nu)\,k^{\nu}. \tag{38}$$

Using a series rearrangement technique (see, e.g., [72], Equation (1.24))

$$\sum_{j=0}^{\ell}\sum_{\nu=0}^{j}f(j,\nu)=\sum_{\nu=0}^{\ell}\sum_{j=\nu}^{\ell}f(j,\nu) \tag{39}$$

in (38), we get

$$\sum_{\nu=0}^{\ell}\sum_{j=\nu}^{\ell}(-1)^{\ell+j}s(\ell,j)\binom{j}{\nu}\alpha^{j-\nu}\,k^{\nu}=\sum_{\nu=0}^{\ell}\sum_{j=\nu}^{\ell}\mathcal{A}_j(\alpha,\ell)\,s(j,\nu)\,k^{\nu}. \tag{40}$$

Now the desired identity (32) follows from (40).

The identity (33) can be obtained by matching the right-handed members of (10) and (50) when $n=1$. □

Proof of Lemma 2. Differentiating both sides of (32) and (33) yields (44) and (35), respectively. □

We recall the following identities (see, e.g., [1], Section 1.6):

$$\begin{aligned}&s(m,0)=\begin{cases}1 & (m=0)\\ 0 & (m\in\mathbb{N}),\end{cases}\qquad s(m,m)=1,\\ &s(m,1)=(-1)^{m+1}\,(m-1)!,\quad s(m,m-1)=-\binom{m}{2}\end{aligned} \tag{41}$$

and

$$\begin{aligned}&\sum_{r=1}^{m}s(m,r)=0\quad(m\in\mathbb{N}\setminus\{1\});\quad\sum_{r=0}^{m}(-1)^{m+r}\,s(m,r)=m!;\\ &\sum_{j=r}^{m}s(m+1,j+1)\,m^{j-r}=s(m,r).\end{aligned} \tag{42}$$

The identity (32), with the aid of (41) and (42), or the identity (33) can give explicit expressions for any $\ell\in\mathbb{N}$ with $0\le j\le\ell$ and $\alpha\in\mathbb{C}$. For example,

$$\mathcal{A}_\ell(\alpha,\ell)=1\quad(\ell\in\mathbb{N}). \tag{43}$$

$$\begin{gathered}\mathcal{A}_0(\alpha,1)=\alpha,\quad\mathcal{A}_0(\alpha,2)=\alpha+\alpha^2,\quad\mathcal{A}_1(\alpha,2)=2+2\,\alpha,\\ \mathcal{A}_0(\alpha,3)=2\alpha+3\alpha^2+\alpha^3,\quad\mathcal{A}_1(\alpha,3)=6+9\alpha+3\alpha^2,\quad\mathcal{A}_2(\alpha,3)=6+3\alpha,\\ \mathcal{A}_0(\alpha,4)=6\,\alpha+11\,\alpha^2+6\,\alpha^3+\alpha^4,\quad\mathcal{A}_1(\alpha,4)=24+44\,\alpha+24\,\alpha^2+4\,\alpha^3,\\ \mathcal{A}_2(\alpha,4)=36+30\,\alpha+6\,\alpha^2,\quad\mathcal{A}_3(\alpha,4)=12+4\,\alpha.\end{gathered}$$

Differentiating both sides of (32) with respect to α, we get

$$\begin{gathered}\sum_{j=\nu}^{\ell}\mathcal{B}_j(\alpha,\ell)\,s(j,\nu)=\sum_{j=\nu}^{\ell}(-1)^{\ell+j}j\,s(\ell,j)\binom{j-1}{\nu}\alpha^{j-1-\nu}\\ (\nu=0,\,1,\,\ldots,\,\ell).\end{gathered} \tag{44}$$

Likewise, the relation (44), with the aid of (41) and (42), or the identity (35) can give explicit expressions for any $\ell \in \mathbb{N}$ with $0 \le j \le \ell$ and $\alpha \in \mathbb{C}$. For example,

$$\mathcal{B}_\ell(\alpha, \ell) = 0 \quad (\ell \in \mathbb{N}). \tag{45}$$

$$\mathcal{B}_0(\alpha, 1) = 1, \quad \mathcal{B}_0(\alpha, 2) = 1 + 2\alpha, \quad \mathcal{B}_1(\alpha, 2) = 2,$$
$$\mathcal{B}_0(\alpha, 3) = 2 + 6\alpha + 3\alpha^2, \quad \mathcal{B}_1(\alpha, 3) = 9 + 6\alpha, \quad \mathcal{B}_2(\alpha, 3) = 3,$$
$$\mathcal{B}_0(\alpha, 4) = 6 + 22\,\alpha + 18\,\alpha^2 + 4\alpha^3, \quad \mathcal{B}_1(\alpha, 4) = 44 + 48\,\alpha + 12\,\alpha^2,$$
$$\mathcal{B}_2(\alpha, 4) = 30 + 12\,\alpha, \quad \mathcal{B}_3(\alpha, 4) = 4.$$

Remark 2.

(i) $(\alpha + k)_\ell$ *is a polynomial in both* α *and* k *of the same degree* ℓ.
(ii) $\mathcal{A}_j(\alpha, \ell)$ *is a polynomial in* α *of degree* $\ell - j$.
(iii) $\mathcal{B}_j(\alpha, \ell)$ *is a polynomial in* α *of degree* $\ell - j - 1$.
(iv) *The generalized harmonic numbers* $H_n^{(s)}(\alpha)$ *are defined by (see, e.g., [73], Equation (1.3))*

$$H_n^{(s)}(\alpha) := \sum_{k=1}^{n} \frac{1}{(k+\alpha)^s} \quad \left(n \in \mathbb{N},\, s \in \mathbb{C},\, \alpha \in \mathbb{C} \setminus \mathbb{Z}^-\right), \tag{46}$$

where $H_n^{(1)}(\alpha) := H_n(\alpha)$ *and* $H_n^{(s)}(0) := H_n^{(s)}$ *are the harmonic numbers of order s (see, e.g., [73], Equation (1.2))*

$$H_n^{(s)} := \sum_{k=1}^{n} \frac{1}{k^s} \quad (n \in \mathbb{N},\, s \in \mathbb{C}) \tag{47}$$

and $H_n^{(1)} := H_n$ *are the harmonic numbers (see, e.g., [73], Equation (1.1))*

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n \in \mathbb{N}). \tag{48}$$

It follows from (35) and (46) that

$$\mathcal{B}_j(\alpha, \ell) = \binom{\ell}{j} (\alpha + j)_{\ell - j} \left(H_\ell(\alpha - 1) - H_j(\alpha - 1)\right). \tag{49}$$

3. Reduction Theorems in Terms of the Sequence in Definition 1

In this section, using the sequence in Definition 1, we present certain reduction formulas for ${}_pF_q$.

Theorem 1. *Let* $\ell \in \mathbb{N}$, $1 \le \min\{p, q\}$, $p \le q + 1$, $b, b_r \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(r = 2, \ldots, q)$*; if* $b + \ell$, $a_r \in \mathbb{Z}_0^-$ *for some* $r \in \{2, \ldots, p\}$*, the condition* $p \le q + 1$ *is cancelled. Then*

$$\begin{aligned} {}_pF_q\left[\begin{matrix} b+\ell,\, a_2,\, \ldots,\, a_p\,; \\ b,\, b_2,\, b_3,\, \ldots,\, b_q\,; \end{matrix}\; z\right] &= \frac{1}{(b)_\ell} \sum_{j=0}^{\ell} \mathcal{A}_j(b, \ell)\, z^j \\ &\times \frac{\prod_{r=2}^{p} (a_r)_j}{\prod_{r=2}^{q} (b_r)_j}\, {}_{p-1}F_{q-1}\left[\begin{matrix} a_2 + j,\, \ldots,\, a_p + j\,; \\ b_2 + j,\, b_3 + j,\, \ldots,\, b_q + j\,; \end{matrix}\; z\right]. \end{aligned} \tag{50}$$

Proof. Let $\mathcal{L}_1$ be the left member of (50). Then using the identity

$$\frac{(b+\ell)_k}{(b)_k} = \frac{(b)_{k+\ell}}{(b)_\ell\,(b)_k} = \frac{(b+k)_\ell}{(b)_\ell} \tag{51}$$

to expand $\mathcal{L}_1$ gives

$$\mathcal{L}_1 = \frac{1}{(b)_\ell}\sum_{k=0}^{\infty}\frac{(b+k)_\ell\,(a_2)_k\cdots(a_p)_k}{k!\,(b_2)_k\cdots(b_q)_k}\,z^k. \tag{52}$$

Employing (28) in (52), we obtain

$$\mathcal{L}_1 = \frac{1}{(b)_\ell}\sum_{j=0}^{\ell}\mathcal{A}_j(b,\ell)\sum_{k=j}^{\infty}\frac{(a_2)_k\cdots(a_p)_k}{(k-j)!\,(b_2)_k\cdots(b_q)_k}\,z^k.$$

Setting $k-j=k'$ and dropping the prime on k yields

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{(b)_\ell}\sum_{j=0}^{\ell}\mathcal{A}_j(b,\ell)\sum_{k=0}^{\infty}\frac{(a_2)_{k+j}\cdots(a_p)_{k+j}}{k!\,(b_2)_{k+j}\cdots(b_q)_{k+j}}\,z^{k+j}\\ &= \frac{1}{(b)_\ell}\sum_{j=0}^{\ell}\mathcal{A}_j(b,\ell)\,z^j\,\frac{\prod_{r=2}^{p}(a_r)_j}{\prod_{r=2}^{q}(b_r)_j}\sum_{k=0}^{\infty}\frac{(a_2+j)_k\cdots(a_p+j)_k}{k!\,(b_2+j)_k\cdots(b_q+j)_k}\,z^k,\end{aligned}$$

which is instantly apparent to be equivalent to the right-handed component of (50). □

Theorem 2. *Let $a_r = b_r + \ell_r$, $\ell_r \in \mathbb{N}$ $(r=1,\ldots,n)$, $n \le \min\{p,q\}$, $p \le q+1$, $b_r \in \mathbb{C}\setminus\mathbb{Z}_0^-$ $(r=1,\ldots,q)$; if $a_r \in \mathbb{Z}_0^-$ for some $r \in \{1,\ldots,p\}$, the condition $p \le q+1$ is cancelled. Then*

$$\begin{aligned}&{}_pF_q\left[\begin{matrix}b_1+\ell_1,\ldots,b_n+\ell_n,\,a_{n+1},\ldots,a_p\,;\\ b_1,\ldots,b_n,\,b_{n+1},\ldots,b_q\,;\end{matrix}\;z\right]\\ &= \frac{1}{\prod_{r=1}^{n}(b_r)_{\ell_r}}\sum_{j_1=0}^{\ell_1}\cdots\sum_{j_n=0}^{\ell_n}\mathcal{A}(j_1,\ldots,j_n)\,z^{J_n}\\ &\quad\times{}_{p-n}F_{q-n}\left[\begin{matrix}a_{n+1}+J_n,\ldots,\ldots,a_p+J_n\,;\\ b_{n+1}+J_n,\ldots,b_q+J_n\,;\end{matrix}\;z\right],\end{aligned} \tag{53}$$

where $J_n = j_1+\cdots+j_n$ and $J_0 = 0$ and

$$\mathcal{A}(j_1,\ldots,j_n) = \prod_{r=1}^{n}\mathcal{A}_{j_r}(b_r+J_{r-1},\ell_r)\,\frac{\prod_{r=n+1}^{p}(a_r)_{J_n}}{\prod_{r=n+1}^{q}(b_r)_{J_n}}. \tag{54}$$

Proof. We may proceed with induction on n in order to demonstrate (53). This may be accomplished by applying the proof of Theorem 1 repeatedly. We omit specifics. □

Remark 3. *we have*

$$\prod_{r=1}^{n}\mathcal{A}_{j_r}(b_r+J_{r-1},\ell_r) = \prod_{r=1}^{n}\binom{\ell_r}{j_r}\frac{(b_r)_{\ell_r+J_{r-1}}}{(b_r)_{J_r}}. \tag{55}$$

The case $n=1$ of (55) is easily found to yield the equivalent relation (33).

4. Generalized Summation Theorems 5 mm Based on (16), (18) and (20)

The following theorems provide generalized summation formulae for the ${}_{t+2}F_{t+1}$ $(t \in \mathbb{N})$ with its arguments -1 and $\frac{1}{2}$.

4.1. Generalized Summation Formulas Based on (16)

Theorem 3. *Let ℓ, $m \in \mathbb{N}_0$ with $\ell \leq m$, and α, $\beta \in \mathbb{C}$. Furthermore, let $\alpha - \beta - m \in \mathbb{C} \setminus \mathbb{Z}^-$ and $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Further let $\Re(\beta) < \frac{2-m-\ell}{2}$. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} \alpha,\, \beta,\, c+\ell\,; \\ 1+\alpha-\beta-m,\, c\,; \end{matrix} \; -1\right] &= \frac{\Gamma(1+\alpha-\beta-m)}{2\,(c)_\ell\,\Gamma(\alpha)} \\
&\times \sum_{j=0}^{\ell} (-1)^j (\beta)_j\, \mathcal{A}_j(c,\ell) \sum_{r=0}^{m-j} \binom{m-j}{r} \frac{\Gamma\left(\frac{r+j+\alpha}{2}\right)}{\Gamma\left(\frac{r+j+\alpha}{2}+1-\beta-m\right)}.
\end{aligned} \tag{56}
$$

Proof. In view of (29), the case $\ell = 0$ of (56) is found to become the identity (16). Without loss of generality, assume that ℓ is a positive integer. Let $\mathcal{L}_1$ be the left-handed member of (56). Then using the identity

$$
\frac{(c+\ell)_k}{(c)_k} = \frac{(c)_{k+\ell}}{(c)_\ell\,(c)_k} = \frac{(c+k)_\ell}{(c)_\ell} \tag{57}
$$

to expand $\mathcal{L}_1$ gives

$$
\mathcal{L}_1 = \frac{1}{(c)_\ell} \sum_{k=0}^{\infty} \frac{(\alpha)_k\,(\beta)_k\,(c+k)_\ell}{k!\,(1+\alpha-\beta-m)_k} (-1)^k. \tag{58}
$$

Employing (28) in (58), we obtain

$$
\mathcal{L}_1 = \frac{1}{(c)_\ell} \sum_{j=0}^{\ell} \mathcal{A}_j(c,\ell) \sum_{k=j}^{\infty} \frac{(\alpha)_k\,(\beta)_k\,(-1)^k}{(k-j)!\,(1+\alpha-\beta-m)_k}.
$$

Setting $k - j = k'$ and dropping the prime on k yields

$$
\begin{aligned}
\mathcal{L}_1 &= \frac{1}{(c)_\ell} \sum_{j=0}^{\ell} \mathcal{A}_j(c,\ell) \sum_{k=0}^{\infty} \frac{(\alpha)_{j+k}\,(\beta)_{j+k}\,(-1)^{j+k}}{k!\,(1+\alpha-\beta-m)_{j+k}} \\
&= \frac{1}{(c)_\ell} \sum_{j=0}^{\ell} \mathcal{A}_j(c,\ell) \frac{(-1)^j (\alpha)_j\,(\beta)_j}{(1+\alpha-\beta-m)_j} \sum_{k=0}^{\infty} \frac{(\alpha+j)_k\,(\beta+j)_k\,(-1)^k}{k!\,(j+1+\alpha-\beta-m)_k} \\
&= \frac{1}{(c)_\ell} \sum_{j=0}^{\ell} \mathcal{A}_j(c,\ell) \frac{(-1)^j (\alpha)_j\,(\beta)_j}{(1+\alpha-\beta-m)_j}\, {}_2F_1\left[\begin{matrix} \alpha+j,\, \beta+j\,; \\ j+1+\alpha-\beta-m\,; \end{matrix} \; -1\right].
\end{aligned}
$$

For the last ${}_2F_1(-1)$, replacing α, λ, and p by $\alpha + j$, $\beta + j$, and $m - j$, respectively, in (16), we obtain the desired summation formula (56). □

Theorem 4. *Let ℓ, ρ, $m \in \mathbb{N}_0$ with $\ell + \rho \le m$, and α, $\beta \in \mathbb{C}$. Furthermore, let $\alpha - \beta - m \in \mathbb{C} \setminus \mathbb{Z}^-$ and c, $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Further let $\Re(\beta) < \frac{2-m-\ell-\rho}{2}$. Then*

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix} \alpha,\, \beta,\, c+\ell,\, d+\rho\,; \\ 1+\alpha-\beta-m,\, c,\, d\,; \end{matrix}\; -1\right] &= \frac{\Gamma(1+\alpha-\beta-m)}{2\,(c)_\ell\,(d)_\rho\,\Gamma(\alpha)} \\
&\times \sum_{\nu=0}^{\rho}\sum_{j=0}^{\ell}(-1)^{\nu+j}(\beta)_{\nu+j}\,\mathcal{A}_j(c+\nu,\ell)\mathcal{A}_\nu(d,\rho) \\
&\times \sum_{r=0}^{m-\nu-j}\binom{m-\nu-j}{r}\frac{\Gamma\left(\frac{r+\nu+j+\alpha}{2}\right)}{\Gamma\left(\frac{r+\nu+j+\alpha}{2}+1-\beta-m\right)}.
\end{aligned}
\tag{59}
$$

Proof. As in the beginning of the proof of Theorem 3, here also let assume that ℓ and ρ are positive integers. Let $\mathcal{L}_2$ be the left member of (59). Then, using (51), we have

$$
\begin{aligned}
\mathcal{L}_2 &= \sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k\,(c+\ell)_k\,(d+\rho)_k}{k!\,(1+\alpha-\beta-m)_k\,(c)_k\,(d)_k}(-1)^k \\
&= \frac{1}{(d)_\rho}\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k\,(c+\ell)_k\,(d+k)_\rho}{k!\,(1+\alpha-\beta-m)_k\,(c)_k}(-1)^k.
\end{aligned}
$$

Employing (28) in the last sum, here, with the aid of (56), as in the similar process of proof of Theorem 3, we can prove the identity (59). We omit the details. □

Theorem 5. *Let t, $\ell_1, \ldots, \ell_t$, $m \in \mathbb{N}_0$ with $\boldsymbol{j}_t \le m$. Furthermore, let α, $\beta \in \mathbb{C}$, and $\alpha - \beta - m \in \mathbb{C} \setminus \mathbb{Z}^-$, and $c_1, \ldots, c_t \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Further let $\Re(\beta) < \frac{2-m-\boldsymbol{l}_t}{2}$. Then*

$$
\begin{aligned}
{}_{t+2}F_{t+1}\left[\begin{matrix} \alpha,\, \beta,\, c_1+\ell_1,\, \ldots,\, c_t+\ell_t\,; \\ 1+\alpha-\beta-m,\, c_1,\, \ldots,\, c_t\,; \end{matrix}\; -1\right] &= \frac{\Gamma(1+\alpha-\beta-m)}{2\,\Gamma(\alpha)\prod_{j=1}^{t}(c_j)_{\ell_j}} \\
&\times \sum_{j_t=0}^{\ell_t}\cdots\sum_{j_1=0}^{\ell_1}(-1)^{\boldsymbol{j}_t}(\beta)_{\boldsymbol{j}_t}\prod_{k=1}^{t}\mathcal{A}_{j_k}(c_k+\boldsymbol{j}_t-\boldsymbol{j}_k,\ell_k) \\
&\times \sum_{r=0}^{m-\boldsymbol{j}_t}\binom{m-\boldsymbol{j}_t}{r}\frac{\Gamma\left(\frac{r+\boldsymbol{j}_t+\alpha}{2}\right)}{\Gamma\left(\frac{r+\boldsymbol{j}_t+\alpha}{2}+1-\beta-m\right)},
\end{aligned}
\tag{60}
$$

where

$$
\boldsymbol{l}_t := \sum_{\eta=1}^{t}\ell_\eta \quad (t \in \mathbb{N}) \qquad \textit{and} \qquad \boldsymbol{j}_k := \sum_{\eta=1}^{k} j_\eta \quad (k \in \mathbb{N}). \tag{61}
$$

Proof. By using mathematical induction on $t \in \mathbb{N}$, we may replicate the procedure used to establish Theorem 4 and therefore show the conclusion here. The specifics are avoided. □

4.2. Generalized Summation Formulas Based on (18)

Theorem 6. *Let ℓ, $m \in \mathbb{N}_0$, and α, $\beta \in \mathbb{C}$. Furthermore, let c, $\frac{1+\alpha+\beta-m}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} \alpha,\, \beta,\, c+\ell\,; \\ \frac{1+\alpha+\beta-m}{2},\, c\,; \end{matrix}\; \frac{1}{2}\right] &= \frac{2^{\beta-1}\,\Gamma\left(\frac{1+\alpha+\beta-m}{2}\right)}{\Gamma(\beta)\,(c)_\ell} \\
&\times \sum_{j=0}^{\ell}(\alpha)_j\,\mathcal{A}_j(c,\ell)\sum_{r=0}^{m}\binom{m}{r}\frac{\Gamma\left(\frac{\beta+j+r}{2}\right)}{\Gamma\left(\frac{1+\alpha-m+j+r}{2}\right)}.
\end{aligned}
\tag{62}
$$

Proof. The proof would run in parallel with that of Theorem 3 with the aid of (18). The details are omitted. □

Theorem 7. *Let ρ, ℓ, $m \in \mathbb{N}_0$, and α, $\beta \in \mathbb{C}$. Furthermore, let c, d, $\frac{1+\alpha+\beta-m}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ c+\ell,\ d+\rho\,; \\ \dfrac{1+\alpha+\beta-m}{2},\ c,\ d\,;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{\beta-1}\,\Gamma\left(\frac{1+\alpha+\beta-m}{2}\right)}{(c)_\ell\,(d)_\rho\,\Gamma(\beta)} \\
&\times \sum_{\nu=0}^{\rho}\sum_{j=0}^{\ell}(\alpha)_{\nu+j}\,\mathcal{A}_\nu(d,\rho)\,\mathcal{A}_j(c+\nu,\ell)\sum_{r=0}^{m}\binom{m}{r}\frac{\Gamma\left(\frac{\beta+\nu+j+r}{2}\right)}{\Gamma\left(\frac{1+\alpha-m+\nu+j+r}{2}\right)}.
\end{aligned}
\tag{63}
$$

Proof. The proof would continue in the same manner as that of Theorem 4, aided by (62). We omit specifics. □

Theorem 8. *Let t, $\ell_1, \dots, \ell_t$, $m \in \mathbb{N}_0$, and α, $\beta \in \mathbb{C}$. Furthermore, let c, d, $\frac{1+\alpha+\beta-m}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
{}_{t+2}F_{t+1}\left[\begin{matrix}\alpha,\ \beta,\ c_1+\ell_1,\ \dots,\ c_t+\ell_t\,; \\ \dfrac{1+\alpha+\beta-m}{2},\ c_1,\ \dots,\ c_t\,;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{\beta-1}\Gamma\left(\frac{1+\alpha+\beta-m}{2}\right)}{\Gamma(\beta)\ \prod_{k=1}^{t}(c_k)_{\ell_k}} \\
&\times \sum_{j_t=0}^{\ell_t}\cdots\sum_{j_1=0}^{\ell_1}(\alpha)_{\boldsymbol{j}_t}\,\mathcal{A}_{j_k}(c_k+\boldsymbol{j}_t-\boldsymbol{j}_k,\ell_k)\sum_{r=0}^{m}\binom{m}{r}\frac{\Gamma\left(\frac{\beta+r+\boldsymbol{j}_t}{2}\right)}{\Gamma\left(\frac{1+\alpha-m+r+\boldsymbol{j}_t}{2}\right)},
\end{aligned}
\tag{64}
$$

where $\boldsymbol{j}_k$ is the same as in (61).

Proof. The proof would be accomplished by following the lines of that of Theorem 5. The involved details are omitted. □

4.3. Generalized Summation Formulas Based on (20)

Theorem 9. *Let ℓ, $m \in \mathbb{N}_0$ with $2\ell \le m$. Furthermore, let $\alpha \in \mathbb{C}$, and β, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha,\ 1-\alpha-m,\ c+\ell\,; \\ \beta,\ c\,;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{-\alpha-m}\Gamma(\beta)}{(c)_\ell\,\Gamma(\beta-\alpha)} \\
&\times \sum_{j=0}^{\ell}(\alpha)_j\,(1-\alpha-m)_j\,\mathcal{A}_j(c,\ell)\sum_{r=0}^{m-2j}\binom{m-2j}{r}\frac{\Gamma\left(\frac{\beta-\alpha+r}{2}\right)}{\Gamma\left(\frac{\beta+\alpha+r}{2}+j\right)}.
\end{aligned}
\tag{65}
$$

Proof. The proof would run in parallel with that of Theorem 3 with the aid of (20). The details are omitted. □

Theorem 10. *Let ρ, ℓ, $m \in \mathbb{N}_0$ with $2(\ell+\rho) \le m$. Furthermore, let $\alpha \in \mathbb{C}$, and β, c, $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}\alpha,\ 1-\alpha-m,\ c+\ell,\ d+\rho\,; \\ \beta,\ c,\ d\,;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{-\alpha-m}\Gamma(\beta)}{(c)_\ell\,(d)_\rho\,\Gamma(\beta-\alpha)} \\
&\times \sum_{\nu=0}^{\rho}\sum_{j=0}^{\ell}(\alpha)_{\nu+j}\,(1-\alpha-m)_{\nu+j}\,\mathcal{A}_\nu(d,\rho)\,\mathcal{A}_j(c+\nu,\ell) \\
&\times \sum_{r=0}^{m-2\nu-2j}\binom{m-2\nu-2j}{r}\frac{\Gamma\left(\frac{\beta-\alpha+r}{2}\right)}{\Gamma\left(\frac{\beta+\alpha+r}{2}+\nu+j\right)}.
\end{aligned}
\tag{66}
$$

Proof. The proof would run in line with that of Theorem 4 with the help of (65). We omit the details. □

Theorem 11. *Let $t,\ \ell_1,\ \ldots,\ \ell_t,\ m \in \mathbb{N}_0$ with $2\boldsymbol{j}_t \leq m$. Furthermore, let $\alpha \in \mathbb{C}$ and $\beta,\ c_1,\ \ldots,\ c_t \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
{}_{t+2}F_{t+1}\left[\begin{matrix}\alpha,\ 1-\alpha-m,\ c_1+\ell_1,\ \ldots,\ c_t+\ell_t\ ; \\ \beta,\ c_1,\ \ldots,\ c_t\ ;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{-\alpha-m}\,\Gamma(\beta)}{\Gamma(\beta-\alpha)\ \prod\limits_{k=1}^{t}(c_k)_{\ell_k}} \\
&\times \sum_{j_t=0}^{\ell_t}\cdots\sum_{j_1=0}^{\ell_1}(\alpha)_{\boldsymbol{j}_t}\,(1-\alpha-m)_{\boldsymbol{j}_t}\prod_{k=1}^{t}\mathcal{A}_{j_k}(c_k+\boldsymbol{j}_t-\boldsymbol{j}_k,\ell_k) \\
&\times \sum_{r=0}^{m-2\boldsymbol{j}_t}\binom{m-2\boldsymbol{j}_t}{r}\frac{\Gamma\left(\frac{\beta-\alpha+r}{2}\right)}{\Gamma\left(\frac{\beta+\alpha+r}{2}+\boldsymbol{j}_t\right)},
\end{aligned} \tag{67}
$$

where $\boldsymbol{j}_k$ is the same as in (61).

Proof. The proof would flow along the lines of that of Theorem 5. The involved details are omitted. □

5. Generalized Summation Theorems 5 mm Based on (15), (17) and (19)

The following theorems offer generalized summation formulae for the ${}_{t+2}F_{t+1}$ $(t \in \mathbb{N})$ and its arguments -1 and $\frac{1}{2}$. The proofs of each theorem are skipped here, principally because they can be checked in the same manner as the preceding section's counterpart.

5.1. Generalized Summation Formulas Based on (15)

Theorem 12. *Let $\ell,\ m \in \mathbb{N}_0$, and $\alpha,\ \beta \in \mathbb{C}$. Furthermore, let $\alpha-\beta+m \in \mathbb{C} \setminus \mathbb{Z}^-$ and $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Further let $\Re(\beta) < \frac{2+m-\ell}{2}$. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha,\ \beta,\ c+\ell\ ; \\ 1+\alpha-\beta+m,\ c\ ;\end{matrix}\ -1\right] &= \frac{\Gamma(1+\alpha-\beta+m)}{2\,(c)_\ell\,\Gamma(\alpha)\,(1-\beta)_m} \\
&\times \sum_{j=0}^{\ell}\mathcal{A}_j(c,\ell)\sum_{r=0}^{m+j}\binom{m+j}{r}\frac{(-1)^r\,\Gamma\left(\frac{\alpha+r+j}{2}\right)}{\Gamma\left(1-\beta+\frac{\alpha+r-j}{2}\right)}.
\end{aligned} \tag{68}
$$

Theorem 13. *Let $\ell,\ \rho,\ m \in \mathbb{N}_0$, and $\alpha,\ \beta \in \mathbb{C}$. Furthermore, let $\alpha-\beta+m \in \mathbb{C} \setminus \mathbb{Z}^-$ and $c,\ d \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Further let $\Re(\beta) < \frac{2+m-\ell-\rho}{2}$. Then*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ c+\ell,\ d+\rho\ ; \\ 1+\alpha-\beta+m,\ c,\ d\ ;\end{matrix}\ -1\right] \\
&\quad= \frac{\Gamma(1+\alpha-\beta+m)}{2\,\Gamma(\alpha)\,(d)_\rho\,(c)_\ell\,(1-\beta)_m}\sum_{\nu=0}^{\rho}\sum_{j=0}^{\ell}\mathcal{A}_\nu(d,\rho)\,\mathcal{A}_j(c+\nu,\ell) \\
&\quad\times \sum_{r=0}^{m+\nu+j}\binom{m+\nu+j}{r}\frac{(-1)^r\,\Gamma\left(\frac{\alpha+r+\nu+j}{2}\right)}{\Gamma\left(1-\beta+\frac{\alpha+r-\nu-j}{2}\right)}.
\end{aligned} \tag{69}
$$

Theorem 14. *Let $t,\ \ell_1,\ \ldots,\ \ell_t,\ m \in \mathbb{N}_0$. Furthermore, let $\alpha,\ \beta \in \mathbb{C}$, and $\alpha-\beta+m \in \mathbb{C} \setminus \mathbb{Z}^-$, and $c_1,\ \ldots,\ c_t \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Further let $\Re(\beta) < \frac{2+m-l_t}{2}$. Then*

$$
\begin{aligned}
{}_{t+2}F_{t+1}\left[\begin{matrix}\alpha,\,\beta,\,c_1+\ell_1,\,\ldots,\,c_t+\ell_t\,;\\ 1+\alpha-\beta+m,\,c_1,\,\ldots,\,c_t\,;\end{matrix}\ -1\right] &= \frac{\Gamma(1+\alpha-\beta+m)}{2\,\Gamma(\alpha)\,(1-\beta)_m\,\prod_{j=1}^{t}(c_j)_{\ell_j}} \\
&\times \sum_{j_t=0}^{\ell_t}\cdots\sum_{j_1=0}^{\ell_1}\prod_{k=1}^{t}\mathcal{A}_{j_k}(c_k+\boldsymbol{j}_t-\boldsymbol{j}_k,\ell_k) \\
&\times \sum_{r=0}^{m+\boldsymbol{j}_t}\binom{m+\boldsymbol{j}_t}{r}\frac{\Gamma\left(\frac{\alpha+r+\boldsymbol{j}_t}{2}\right)}{\Gamma\left(1-\beta+\frac{\alpha+r-\boldsymbol{j}_t}{2}\right)},
\end{aligned} \tag{70}
$$

where $\boldsymbol{l}_t$ *and* $\boldsymbol{j}_k$ $(t,\,k\in\mathbb{N})$ *are the same as in* (61).

5.2. Generalized Summation Formulas Based on (17)

Theorem 15. *Let* $\ell,\,m\in\mathbb{N}_0$*, and* $\alpha,\,\beta\in\mathbb{C}$*. Furthermore, let* $c,\,\frac{1+\alpha+\beta+m}{2}\in\mathbb{C}\setminus\mathbb{Z}_0^-$*. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha,\,\beta,\,c+\ell\,;\\ \frac{1+\alpha+\beta+m}{2},\,c\,;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{\alpha-1}}{\Gamma(\alpha)\,(c)_\ell}\,\frac{\Gamma(\frac{1+\alpha+\beta+m}{2})\Gamma(\frac{1-\alpha+\beta-m}{2})}{\Gamma(\frac{1-\alpha+\beta+m}{2})} \\
&\times \sum_{j=0}^{\ell}(\beta)_j\,\mathcal{A}_j(c,\ell)\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\,\Gamma\left(\frac{\alpha+j+r}{2}\right)}{\Gamma(\frac{1+\beta+j+r-m}{2})}.
\end{aligned} \tag{71}
$$

Theorem 16. *Let* $\rho,\,\ell,\,m\in\mathbb{N}_0$*, and* $\alpha,\,\beta\in\mathbb{C}$*. Furthermore, let* $c,\,d,\,\frac{1+\alpha+\beta+m}{2}\in\mathbb{C}\setminus\mathbb{Z}_0^-$*. Then*

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}\alpha,\,\beta,\,c+\ell,\,d+\rho\,;\\ \frac{1+\alpha+\beta+m}{2},\,c,\,d\,;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{\alpha-1}\Gamma(\frac{1+\alpha+\beta+m}{2})\Gamma(\frac{1-\alpha+\beta-m}{2})}{\Gamma(\alpha)\Gamma(\frac{1-\alpha+\beta+m}{2})\,(d)_\rho\,(c)_\ell} \\
&\times \sum_{\nu=0}^{\rho}\sum_{j=0}^{\ell}\mathcal{A}_j(c+\nu,\ell)\,\mathcal{A}_\nu(d,\rho)\,(\beta)_{\nu+j}\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\,\Gamma\left(\frac{\alpha+\nu+j+r}{2}\right)}{\Gamma(\frac{1+\beta+\nu+j+r-m}{2})}.
\end{aligned} \tag{72}
$$

Theorem 17. *Let* $t,\,\ell_1,\,\ldots,\,\ell_t,\,m\in\mathbb{N}_0$*, and* $\alpha,\,\beta\in\mathbb{C}$*. Furthermore, let* $c,\,d,\,\frac{1+\alpha+\beta+m}{2}\in\mathbb{C}\setminus\mathbb{Z}_0^-$*. Then*

$$
\begin{aligned}
{}_{t+2}F_{t+1}\left[\begin{matrix}\alpha,\,\beta,\,c_1+\ell_1,\,\ldots,\,c_t+\ell_t\,;\\ \frac{1+\alpha+\beta+m}{2},\,c_1,\,\ldots,\,c_t\,;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{\alpha-1}\Gamma(\frac{1+\alpha+\beta+m}{2})\Gamma(\frac{1-\alpha+\beta-m}{2})}{\Gamma(\alpha)\Gamma(\frac{1-\alpha+\beta+m}{2})\,\prod_{k=1}^{t}(c_k)_{\ell_k}} \\
&\times \sum_{j_t=0}^{\ell_t}\cdots\sum_{j_1=0}^{\ell_1}(\beta)_{\mathbf{j_t}}\,\mathcal{A}_{j_k}(c_k+\boldsymbol{j}_t-\boldsymbol{j}_k,\ell_k)\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\,\Gamma\left(\frac{\alpha+r+\mathbf{j_t}}{2}\right)}{\Gamma(\frac{1+\beta+r-m+\mathbf{j_t}}{2})},
\end{aligned} \tag{73}
$$

where $\boldsymbol{j}_k$ *is the same as in* (61).

5.3. Generalized Summation Formulas Based on (19)

Theorem 18. *Let* $\ell,\,m\in\mathbb{N}_0$*. Furthermore, let* $\alpha\in\mathbb{C}$*, and* $\beta,\,c\in\mathbb{C}\setminus\mathbb{Z}_0^-$*. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha,\,1-\alpha+m,\,c+\ell\,;\\ \beta,\,c\,;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{m-\alpha}\,\Gamma(\alpha-m)\,\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)\,(c)_\ell} \\
&\times \sum_{j=0}^{\ell}(-1)^j\,\mathcal{A}_j(c,\ell)\sum_{r=0}^{m+2j}(-1)^r\binom{m+2j}{r}\frac{\Gamma(\frac{\beta-\alpha+r}{2})}{\Gamma\left(\frac{\beta+\alpha+r}{2}-m-j\right)}.
\end{aligned} \tag{74}
$$

Theorem 19. *Let* $\rho,\,\ell,\,m\in\mathbb{N}_0$*. Furthermore, let* $\alpha\in\mathbb{C}$*, and* $\beta,\,c,\,d\in\mathbb{C}\setminus\mathbb{Z}_0^-$*. Then*

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}\alpha,\, 1-\alpha+m,\, c+\ell,\, d+\rho\,; \\ \beta,\, c,\, d\,;\end{matrix}\ \frac{1}{2}\right] &= \frac{2^{m-\alpha}\,\Gamma(\alpha-m)\,\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)\,(c)_\ell(d)_\rho} \\
&\times \sum_{\nu=0}^{\rho}\sum_{j=0}^{\ell}(-1)^{\nu+j}\,\mathcal{A}_j(c+\nu,\ell)\mathcal{A}_\nu(d,\rho) \\
&\times \sum_{r=0}^{m+2(\nu+j)}(-1)^r\binom{m+2(\nu+j)}{r}\frac{\Gamma(\frac{\beta-\alpha+r}{2})}{\Gamma\left(\frac{\beta+\alpha+r}{2}-m-\nu-j\right)}.
\end{aligned}
\tag{75}
$$

Theorem 20. *Let $t,\, \ell_1,\, \ldots,\, \ell_t,\, m \in \mathbb{N}_0$ Furthermore, let $\alpha \in \mathbb{C}$ and $\beta,\, c_1,\, \ldots,\, c_t \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
&{}_{t+2}F_{t+1}\left[\begin{matrix}\alpha,\, 1-\alpha+m,\, c_1+\ell_1,\, \ldots,\, c_t+\ell_t\,; \\ \beta,\, c_1,\, \ldots,\, c_t\,;\end{matrix}\ \frac{1}{2}\right] \\
&= \frac{2^{m-\alpha}\,\Gamma(\alpha-m)\,\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)\,\prod\limits_{k=1}^{t}(c_k)_{\ell_k}} \\
&\quad \times \sum_{j_t=0}^{\ell_t}\cdots\sum_{j_1=0}^{\ell_1}(-1)^{j_t}\prod_{k=1}^{t}\mathcal{A}_{j_k}(c_k+\boldsymbol{j}_t-\boldsymbol{j}_k,\ell_k) \\
&\quad \times \sum_{r=0}^{m+2\boldsymbol{j}_t}(-1)^r\binom{m+2\boldsymbol{j}_t}{r}\frac{\Gamma(\frac{\beta-\alpha+r}{2})}{\Gamma\left(\frac{\beta+\alpha+r}{2}-m-\boldsymbol{j}_t\right)},
\end{aligned}
\tag{76}
$$

where $\boldsymbol{j}_k$ is the same as in (61).

6. Formulas Involving Finite Sums of ${}_{t+1}F_t$

We provide formulae for finite sums of ${}_{t+1}F_t$ by using two identities in Theorems 3, 6, 9, 12, 15 and 18, which are stated in the following six theorems. This section contains just the proof of Theorem 21. The proofs of the other theorems are omitted since they would run concurrently with the proof of Theorem 21.

Theorem 21. *Let $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$ with $\ell \le m$, and $\alpha,\, \beta \in \mathbb{C}$. Furthermore, let $\alpha-\beta-m \in \mathbb{C}\setminus\mathbb{Z}^-$ and $c \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Further let $\Re(\beta) < \frac{2-m-\ell}{2}$. Then*

$$
\begin{aligned}
\sum_{j=1}^{\ell}\frac{1}{c+j-1}\,{}_4F_3\left[\begin{matrix}\alpha,\, \beta,\, c+\ell,\, c+j-1\,; \\ 1+\alpha-\beta-m,\, c,\, c+j\,;\end{matrix}\ -1\right] &= \frac{\Gamma(1+\alpha-\beta-m)}{2\,(c)_\ell\,\Gamma(\alpha)} \\
\times \sum_{j=0}^{\ell-1}(-1)^j(\beta)_j\,\mathcal{B}_j(c,\ell)\sum_{r=0}^{m-j}\binom{m-j}{r}&\frac{\Gamma\left(\frac{r+j+\alpha}{2}\right)}{\Gamma\left(\frac{r+j+\alpha}{2}+1-\beta-m\right)}.
\end{aligned}
\tag{77}
$$

Proof. Multiplying both sides of (56) by $(c)_\ell$, we get

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k\,(c)_{\ell+k}}{k!\,(1+\alpha-\beta-m)_k\,(c)_k}(-1)^k &= \frac{\Gamma(1+\alpha-\beta-m)}{2\,\Gamma(\alpha)} \\
\times \sum_{j=0}^{\ell}(-1)^j(\beta)_j\,\mathcal{A}_j(c,\ell)\sum_{r=0}^{m-j}\binom{m-j}{r}&\frac{\Gamma\left(\frac{r+j+\alpha}{2}\right)}{\Gamma\left(\frac{r+j+\alpha}{2}+1-\beta-m\right)}.
\end{aligned}
\tag{78}
$$

Differentiating $(c)_{\ell+k}/(c)_k = \Gamma(c+k+\ell)/\Gamma(c+k)$ with respect to c, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}c}\frac{(c)_{\ell+k}}{(c)_k} = \frac{(c)_{\ell+k}}{(c)_k}\{\psi(c+k+\ell)-\psi(c+k)\}.
$$

Using (22), we find

$$\frac{\mathrm{d}}{\mathrm{d}c}\frac{(c)_{\ell+k}}{(c)_k}=\frac{(c)_{\ell+k}}{(c)_k}\sum_{j=1}^{\ell}\frac{1}{c+j-1+k}.$$

Employing the fundamental identity $\Gamma(z+1)=z\,\Gamma(z)$, in view of (1), we have

$$\sum_{j=1}^{\ell}\frac{1}{c+j-1+k}=\sum_{j=1}^{\ell}\frac{\Gamma(c+j-1+k)}{\Gamma(c+j+k)}=\sum_{j=1}^{\ell}\frac{1}{c+j-1}\cdot\frac{(c+j-1)_k}{(c+j)_k}.$$

We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}c}\frac{(c)_{\ell+k}}{(c)_k}=\frac{(c)_\ell(c+\ell)_k}{(c)_k}\sum_{j=1}^{\ell}\frac{1}{c+j-1}\cdot\frac{(c+j-1)_k}{(c+j)_k}. \tag{79}$$

Differentiating both sides of (78) with respect to c and using (79), with the aid of (30) and (45), we can get the desired identity (77). □

Theorem 22. *Let $\ell\in\mathbb{N}$, $m\in\mathbb{N}_0$, and $\alpha,\beta\in\mathbb{C}$. Furthermore, let c, $\frac{1+\alpha+\beta-m}{2}\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then*

$$\begin{aligned}\sum_{j=1}^{\ell}\frac{1}{c+j-1}\,{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ c+\ell,\ c+j-1\,;\\ \frac{1+\alpha+\beta-m}{2},\ c,\ c+j\,;\end{matrix}\ \frac{1}{2}\right]\\ =\frac{2^{\beta-1}\,\Gamma\left(\frac{1+\alpha+\beta-m}{2}\right)}{\Gamma(\beta)\,(c)_\ell}\sum_{j=0}^{\ell-1}(\alpha)_j\,\mathcal{B}_j(c,\ell)\sum_{r=0}^{m}\binom{m}{r}\frac{\Gamma\left(\frac{\beta+j+r}{2}\right)}{\Gamma\left(\frac{1+\alpha-m+j+r}{2}\right)}.\end{aligned} \tag{80}$$

Theorem 23. *Let $\ell\in\mathbb{N}$, $m\in\mathbb{N}_0$ with $2\ell\le m$. Furthermore, let $\alpha\in\mathbb{C}$, and $\beta,\,c\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then*

$$\begin{aligned}\sum_{j=1}^{\ell}\frac{1}{c+j-1}\,{}_4F_3\left[\begin{matrix}\alpha,\ 1-\alpha-m,\ c+\ell,\ c+j-1\,;\\ \beta,\ c,\ c+j\,;\end{matrix}\ \frac{1}{2}\right]=\frac{2^{-\alpha-m}\Gamma(\beta)}{(c)_\ell\,\Gamma(\beta-\alpha)}\\ \times\sum_{j=0}^{\ell-1}(\alpha)_j\,(1-\alpha-m)_j\,\mathcal{B}_j(c,\ell)\sum_{r=0}^{m-2j}\binom{m-2j}{r}\frac{\Gamma\left(\frac{\beta-\alpha+r}{2}\right)}{\Gamma\left(\frac{\beta+\alpha+r}{2}+j\right)}.\end{aligned} \tag{81}$$

Theorem 24. *Let $\ell\in\mathbb{N}$, $m\in\mathbb{N}_0$, and $\alpha,\beta\in\mathbb{C}$. Furthermore, let $\alpha-\beta+m\in\mathbb{C}\setminus\mathbb{Z}^-$ and $c\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Further let $\Re(\beta)<\frac{2+m-\ell}{2}$. Then*

$$\begin{aligned}\sum_{j=1}^{\ell}\frac{1}{c+j-1}\,{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ c+\ell,\ c+j-1\,;\\ 1+\alpha-\beta+m,\ c,\ c+j\,;\end{matrix}\ -1\right]\\ =\frac{\Gamma(1+\alpha-\beta+m)}{2\,(c)_\ell\,\Gamma(\alpha)\,(1-\beta)_m}\\ \times\sum_{j=0}^{\ell-1}\mathcal{B}_j(c,\ell)\sum_{r=0}^{m+j}\binom{m+j}{r}\frac{(-1)^r\,\Gamma\left(\frac{r+\alpha+j}{2}\right)}{\Gamma\left(\frac{r+\alpha-j}{2}+1-\beta\right)}.\end{aligned} \tag{82}$$

Theorem 25. *Let $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$, and α, $\beta \in \mathbb{C}$. Furthermore, let c, $\frac{1+\alpha+\beta+m}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$\begin{aligned}\sum_{j=1}^{\ell} \frac{1}{c+j-1} {}_4F_3\left[\begin{matrix} \alpha,\, \beta,\, c+\ell,\, c+j-1\,; \\ \frac{1+\alpha+\beta+m}{2},\, c,\, c+j\,; \end{matrix} \; \frac{1}{2}\right] &= \frac{2^{\alpha-1}}{\Gamma(\alpha)\,(c)_\ell} \frac{\Gamma(\frac{1+\alpha+\beta+m}{2})\Gamma(\frac{1-\alpha+\beta-m}{2})}{\Gamma(\frac{1-\alpha+\beta+m}{2})} \\ &\times \sum_{j=0}^{\ell-1} (\beta)_j\, \mathcal{B}_j(c,\ell) \sum_{r=0}^{m} \binom{m}{r} \frac{(-1)^r\, \Gamma\left(\frac{\alpha+j+r}{2}\right)}{\Gamma(\frac{1+\beta+j+r-m}{2})}.\end{aligned} \tag{83}$$

Theorem 26. *Let $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$. Furthermore, let $\alpha \in \mathbb{C}$, and β, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$\begin{aligned}\sum_{j=1}^{\ell} \frac{1}{c+j-1} {}_4F_3\left[\begin{matrix} \alpha,\, 1-\alpha+m,\, c+\ell,\, c+j-1\,; \\ \beta,\, c,\, c+j\,; \end{matrix} \; \frac{1}{2}\right] &= \frac{2^{m-\alpha}\,\Gamma(\alpha-m)\,\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)\,(c)_\ell} \sum_{j=0}^{\ell-1} (-1)^j\, \mathcal{B}_j(c,\ell) \\ &\times \sum_{r=0}^{m+2j} (-1)^r \binom{m+2j}{r} \frac{\Gamma(\frac{\beta-\alpha+r}{2})}{\Gamma\left(\frac{\beta+\alpha+r}{2}-m-j\right)}.\end{aligned} \tag{84}$$

7. Particular Cases

We address the straightforward special instances of Theorems 3, 6, 9, 12, 15 and 18 when $\ell = 1$, which are specified in the following corollaries. The following are the identities from Section 2: $\mathcal{A}_0(c,1) = c$ and $\mathcal{A}_1(c,1) = 1$.

Corollary 1. *Let $m \in \mathbb{N}_0$, and α, $\beta \in \mathbb{C}$. Furthermore, let $\alpha - \beta - m \in \mathbb{C} \setminus \mathbb{Z}^-$ and $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Further let $\Re(\beta) < \frac{1-m}{2}$. Then*

$$\begin{aligned}{}_3F_2\left[\begin{matrix} \alpha,\, \beta,\, c+1\,; \\ 1+\alpha-\beta-m,\, c\,; \end{matrix} \; -1\right] &= \frac{\Gamma(1+\alpha-\beta-m)}{2\,\Gamma(\alpha)} \left\{ \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma(\frac{r+\alpha}{2})}{\Gamma(\frac{r+\alpha}{2}+1-\beta-m)} \right. \\ &\left. - \frac{\beta}{c} \sum_{r=0}^{m-1} \binom{m-1}{r} \frac{\Gamma\left(\frac{r+\alpha+1}{2}\right)}{\Gamma\left(\frac{r+\alpha+1}{2}+1-\beta-m\right)} \right\}.\end{aligned} \tag{85}$$

Corollary 2. *Let $m \in \mathbb{N}_0$, and α, $\beta \in \mathbb{C}$. Furthermore, let c, $\frac{1+\alpha+\beta-m}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$\begin{aligned}{}_3F_2\left[\begin{matrix} \alpha,\, \beta,\, c+1\,; \\ \frac{1+\alpha+\beta-m}{2},\, c\,; \end{matrix} \; \frac{1}{2}\right] &= \frac{2^{\beta-1}\,\Gamma\left(\frac{1+\alpha+\beta-m}{2}\right)}{\Gamma(\beta)} \\ &\times \sum_{r=0}^{m} \binom{m}{r} \left\{ \frac{\Gamma\left(\frac{\beta+r}{2}\right)}{\Gamma\left(\frac{1+\alpha-m+r}{2}\right)} + \frac{\alpha\,\Gamma\left(\frac{\beta+1+r}{2}\right)}{c\,\Gamma(\frac{2+\alpha-m+r}{2})} \right\}.\end{aligned} \tag{86}$$

Corollary 3. *Let ℓ, $m \in \mathbb{N}_0$. Furthermore, let $\alpha \in \mathbb{C}$, and β, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha,\, 1-\alpha-m,\, c+1\,; \\ \beta,\, c\,;\end{matrix}\ \frac{1}{2}\right]
&= \frac{2^{-\alpha-m}\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{\Gamma\left(\frac{\beta-\alpha+r}{2}\right)}{\Gamma\left(\frac{\beta+\alpha+r}{2}\right)}\right. \\
&\quad \left. + \frac{\alpha\,(1-\alpha-m)}{c}\sum_{r=0}^{m-2}\binom{m-2}{r}\frac{\Gamma\left(\frac{\beta-\alpha+r}{2}\right)}{\Gamma\left(\frac{\beta+\alpha+r}{2}+1\right)}\right\}.
\end{aligned} \tag{87}
$$

Corollary 4. *Let ℓ, $m \in \mathbb{N}_0$, and α, $\beta \in \mathbb{C}$. Furthermore, let $\alpha - \beta + m \in \mathbb{C} \setminus \mathbb{Z}^-$ and $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Further let $\Re(\beta) < \frac{2+m-\ell}{2}$. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha,\, \beta,\, c+1\,; \\ 1+\alpha-\beta+m,\, c\,;\end{matrix}\ -1\right]
&= \frac{\Gamma(1+\alpha-\beta+m)}{2\,\Gamma(\alpha)\,(1-\beta)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\,\Gamma\left(\frac{r+\alpha}{2}\right)}{\Gamma\left(\frac{r+\alpha}{2}+1-\beta\right)}\right. \\
&\quad \left. + \frac{1}{c}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\,\Gamma\left(\frac{r+\alpha+1}{2}\right)}{\Gamma\left(\frac{r+\alpha+1}{2}-\beta\right)}\right\}.
\end{aligned} \tag{88}
$$

Corollary 5. *Let $m \in \mathbb{N}_0$, and α, $\beta \in \mathbb{C}$. Furthermore, let c, $\frac{1+\alpha+\beta+m}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha,\, \beta,\, c+1\,; \\ \frac{1+\alpha+\beta+m}{2},\, c\,;\end{matrix}\ \frac{1}{2}\right]
&= \frac{2^{\alpha-1}}{\Gamma(\alpha)}\,\frac{\Gamma\left(\frac{1+\alpha+\beta+m}{2}\right)\Gamma\left(\frac{1-\alpha+\beta-m}{2}\right)}{\Gamma\left(\frac{1-\alpha+\beta+m}{2}\right)} \\
&\quad \times \sum_{r=0}^{m}\binom{m}{r}(-1)^r\left\{\frac{\Gamma\left(\frac{\alpha+r}{2}\right)}{\Gamma\left(\frac{1+\beta+r-m}{2}\right)} + \frac{\beta\,\Gamma\left(\frac{1+\alpha+r}{2}\right)}{c\,\Gamma\left(\frac{2+\beta+r-m}{2}\right)}\right\}.
\end{aligned} \tag{89}
$$

Corollary 6. *Let $m \in \mathbb{N}_0$. Furthermore, let $\alpha \in \mathbb{C}$, and β, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha,\, 1-\alpha+m,\, c+1\,; \\ \beta,\, c\,;\end{matrix}\ \frac{1}{2}\right]
&= \frac{2^{m-\alpha}\,\Gamma(\beta)\,\Gamma(\alpha-m)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)} \\
&\quad \times \left\{\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{\Gamma\left(\frac{\beta-\alpha+r}{2}\right)}{\Gamma\left(\frac{\beta+\alpha+r}{2}-m\right)}\right. \\
&\quad \left. - \frac{1}{c}\sum_{r=0}^{m+2}(-1)^r\binom{m+2}{r}\frac{\Gamma\left(\frac{\beta-\alpha+r}{2}\right)}{\Gamma\left(\frac{\beta+\alpha+r}{2}-m-1\right)}\right\}.
\end{aligned} \tag{90}
$$

Additionally, the following corollary demonstrates the special case of Theorem 4 where $\ell = 1 = \rho$.

Corollary 7. *Let $m \in \mathbb{N}_0$, and $\alpha, \beta \in \mathbb{C}$. Furthermore, let $\alpha-\beta-m \in \mathbb{C}\setminus\mathbb{Z}^-$ and $c, d \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Further let $\Re(\beta) < -\frac{m}{2}$. Then*

$$\begin{aligned}
{}_4F_3\left[\begin{matrix}\alpha,\,\beta,\,c+1,\,d+1\,;\\ 1+\alpha-\beta-m,\,c,\,d\,;\end{matrix}\; -1\right] &= \frac{\Gamma(1+\alpha-\beta-m)}{2\,c\,d\,\Gamma(\alpha)} \\
&\times\left\{cd\sum_{r=0}^{m}\binom{m}{r}\frac{\Gamma\left(\frac{r+\alpha}{2}\right)}{\Gamma\left(\frac{r+\alpha}{2}+1-\beta-m\right)}\right. \\
&\quad -\beta\,d\sum_{r=0}^{m-1}\binom{m-1}{r}\frac{\Gamma\left(\frac{r+1+\alpha}{2}\right)}{\Gamma\left(\frac{r+1+\alpha}{2}+1-\beta-m\right)} \\
&\quad -\beta\,(c+1)\sum_{r=0}^{m-1}\binom{m-1}{r}\frac{\Gamma\left(\frac{r+1+\alpha}{2}\right)}{\Gamma\left(\frac{r+1+\alpha}{2}+1-\beta-m\right)} \\
&\quad \left.+\beta(\beta+1)\sum_{r=0}^{m-2}\binom{m-2}{r}\frac{\Gamma\left(\frac{r+2+\alpha}{2}\right)}{\Gamma\left(\frac{r+2+\alpha}{2}+1-\beta-m\right)}\right\}.
\end{aligned} \tag{91}$$

8. Concluding Remarks and Posing Problems

Beginning with Gauss's celebrated summation formula for ${}_2F_1(1)$ (5), an astoundingly huge number of summation formulae for ${}_pF_q$ $(p,\, q \in \mathbb{N}_0)$, with a variety of arguments, have been given (see, e.g., [9]). Following this trend, we established families of generalized summation formulas for ${}_{t+2}F_{t+1}$ $(t \in \mathbb{N})$ with its arguments -1 and $1/2$ in Sections 4 and 5. We did so by introducing two sequences of new numbers in Definition 1 and their derivatives in Definition 2, as well as by selecting the six generalized summation formulas (15)–(20) above. Furthermore, in Section 6, we demonstrated two identities related to finite sums of ${}_4F_3$ by differentiating both sides of two formulae given here with respect to a particular parameter, among many others. Further, in Section 7, we provided simple specific identities for a few selected formulae in Sections 4 and 5.

In this study, the sequences of new numbers

$$\left\{\mathcal{A}_j(\alpha,\ell)\right\}_{j=0}^{\ell} \quad \text{and} \quad \left\{\mathcal{B}_j(\alpha,\ell)\right\}_{j=0}^{\ell}$$

in Section 2 were helpful in establishing certain generalized summation formulas for ${}_pF_q$ with specific arguments. Further, it is expected that the newly introduced numbers would be used substantially in other fields of study.

We conclude this paper by posing some problems:

(i) Give more detailed accounts of omitted proofs of Theorems in Sections 4 and 5.
(ii) Try to give more general formulas than those in Theorems 5, 8 and 11 as in the shape of the left-handed member of (10).
(iii) Try to establish generalized summation formulas for ${}_pF_q$ based on certain known ones in the literature, by using the similar technique in this paper, with a particular aid of the sequences of newly introduced numbers in Section 2.
(iv) Try to directly prove Equation (33) and Equation (35) from Definitions 1 and 2.

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