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Article

Existence of the Class of Nonlinear Hybrid Fractional Langevin Quantum Differential Equation with Dirichlet Boundary Conditions

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Abstract: In this paper, we investigate the existence results for nonlinear fractional q-difference equations with two different fractional orders supplemented with the Dirichlet boundary conditions. Our main existence results are obtained by applying the contraction mapping principle and Krasnoselskii's fixed point theorem. An illustrative example is also discussed.

Keywords: fractional differential equations; hybrid differential equations; Krasnoselskii's fixed point theorems

1. Introduction

Fractional differential equations have been researched by a number of academics in recent years, with topics spanning from the theoretical concerns of existence and uniqueness to numerical techniques for finding solutions. Fractional differential equations have attracted a lot of attention as a result of its use in a variety of scientific and engineering applications arising from the study of precise descriptions of nonlinear processes. It has been discovered that fractional calculus-based models may accurately represent a variety of complex phenomena such as control, viscoelasticity, electrochemistry and porous media. The nonlinear oscillation of earthquakes can also be modeled with fractional derivatives and can eliminate the differences arising from the assumption of continuum traffic flow (see [1–10] and the related references cited therein).

Several scholars have looked at hybrid fractional differential equations. This class of equation involves the fractional order derivative of an unknown function hybrid with nonlinearity depending on it. The authors of [11] established the existence theorem for fractional hybrid differential equations and some fundamental differential inequalities. Dhage et al. [12–15] discussed the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations, initiating the study of the theory of such systems and proving, by utilizing the study of inequalities, the existence of extremal solutions and comparison results.

A series of publications provide some recent results on hybrid differential equations (see [11–18]).

The quantum calculus or q-difference calculus was initially developed by Jackson [19]. The quantum calculus has shown to have many applications in a number of fields, including quantum mechanics, hypergeometric series, particle physics and complex analysis [20,21].

Furthermore, Al-Salam [22] and Agarwal [23] contributed to the development of fractional q-difference calculus. Since fractional q-difference equations have become a hot subject, more academics are focusing on the field of quantum problems. Fractional difference equations have a wide range of applications in fields including economics, chemistry, physics and engineering. These q-fractional operators are expected to be important for the development of q-function theory, which is vital in combinatory analysis (see example [22,24–28]).

The Langevin equation is a helpful tool for analysing and evaluating physical phenomena that change over time. The ordinary Langevin equation, on the other hand, does not correctly represent complex media. Several generalisations of the Langevin equation with two distinct fractional orders have been developed to solve this issue, resulting in a more flexible model for fractal processes than the classic one specified by a single index. For more recent results on the Langevin equation, see [29–34] and the references therein.

Due to the tremendous scope and applications, several research studies have been devoted to the study of the existence of fractional differential equations, studied by many authors [35–43]. A q-variant of the nonlinear hybrid fractional Langevin equations with two distinct fractional orders complemented with Dirichlet boundary conditions has not been studied previously. This paper was motivated by some recent works [11,33,36,42]. Stimulated by the above discussion, we consider a q-variant of the nonlinear fractional difference integral equations of the following form:

$$
{}^cD_q^{\alpha_1}\left({}^cD_q^{\alpha_2}\frac{x(\varrho)}{f_1(\varrho,x(\varrho))}\right)+\lambda\,{}^cD_q^{\alpha_1}x(\varrho)=p_1f_2(\varrho,x(\varrho))+p_2I_q^{\xi}f_3(\varrho,x(\varrho)), \quad 0\le\varrho\le 1, 0<q<1 \tag{1}
$$

$$
x(0)=0, x(1)=1, \tag{2}
$$

where ${}^cD_q^{\alpha_1}$ and ${}^cD_q^{\alpha_2}$ are the Caputo type fractional q-derivative; $0<\alpha_1,\alpha_2\le 1$; $I_q^{\xi}(\cdot)$ denotes the Riemann–Liouville (R-L) integral with $0<\xi<1$; f_1, f_2 and f_3 are continuous functions; and λ, p_1 and p_2 are positive constants.

Equation (1) reduces to a second order q-difference equation for the values $\alpha_1=1$ and $\alpha_2=1$ and places $f_1(\varrho,x(\varrho))=1$, which is the Langevin equation with two varying fractional orders in the limit $q\to 1^-$. The value is $\lambda=0$. Equation (1) is called a second order hybrid equation in the limit $q\to 1^-$. The integral type nonlinearity given in terms of q-difference of the Riemann–Liouville type of order $\xi\in(0,1)$ provides a flexible choice in terms of ξ.

In the sequel, we assume that the following conditions hold:

(A_1) $f_1:[0,1]\times\mathbb{R}\to\mathbb{R}/\{0\}$ and $f_2,f_3:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions such that

$$
|f_i(\varrho,x)-f_i(\varrho,y)|\le L_i|x-y|,
$$

$L_i>0$, $i=1,2,3$ and $x,y\in\mathbb{R}$ for all $\varrho\in[0,1]$;

(A_2) $|f_i(\varrho,x)|\le\mu_i(\varrho)$ for all $(\varrho,x)\in[0,1]\times\mathbb{R}$, $\mu_i\in C([0,1],\mathbb{R}^+)$ and $||\mu_i||=\sup_{\varrho\in[0,1]}|\mu_i(\varrho)|$, $i=1,2,3$.

The remainder of the paper is structured as follows: In Section 2, we review some basic concepts and results in fractional calculus and q-calculus. In Section 3, we establish a new conclusion for nonlinear q-fractional difference equations with Dirichlet boundary conditions. First, we used the contraction mapping principle to verify the existence and uniqueness of the problem (1). Following that, we applied Krasnoselskii's fixed point theorem to prove another new existence result for the problem (1). Finally, in Section 4, we looked at an example.

2. Preliminaries

We recall some important concepts and essential findings on quantum fractional calculus.

Definition 1 ([26])**.** *Let $\nu \geq 0, 0 < q < 1$, and the function $g \in [0,1]$. The R-L type fractional q-integral is $(I_q^0 g)(\varrho) = g(\varrho)$, and the following is the case:*

$$(I_q^\nu g)(\varrho) = \int_0^\varrho \frac{(\varrho - qs)^{(\nu-1)}}{\Gamma_q(\nu)} g(s) d_q s, \quad \nu > 0, \ \varrho \in [0,1],$$

where

$$\Gamma_q(\nu) = \frac{(1-q)^{(\nu-1)}}{(1-q)^{\nu-1}}, \quad 0 < q < 1$$

and $\Gamma_q(\nu+1) = [\nu]_q \Gamma_q(\nu)$. The following is the case:

$$[\nu]_q = \frac{q^\nu - 1}{q-1}, \ (1-q)^{(0)} = 1, \ (1-q)^{(n)} = \Pi_{k=0}^{n-1}(1-q^{k+1}),$$

$n \in \mathbb{N}$. Furthermore, if $\alpha \in \mathbb{R}$, then the following is the case.

$$(1-q)^{(\alpha)} = \Pi_{i=0}^{\infty} \frac{(1-q^{i+1})}{(1-q^{1+\alpha+i})}.$$

For $0 < q < 1$, then the function g is defined by the following.

$$D_q g(\varrho) = \frac{g(\varrho) - g(qt)}{(1-q)\varrho}, \quad \varrho \neq 0, \quad D_q g(0) = \lim_{n \to \infty} \frac{g(sq^n) - g(0)}{sq^n}, s \neq 0.$$

Definition 2 ([27])**.** *The R-L type fractional q-derivative of order $\nu \geq 0$ is defined by $(D_q^0 g)(\varrho) = g(\varrho)$ and the following:*

$$(D_q^\nu g)(\varrho) = (D_q^{[\nu]} I_q^{[\nu]-\nu} g)(\varrho), \quad \nu > 0,$$

where ν denotes the integer part and $[\nu] \geq \nu$.

Definition 3 ([27])**.** *The Caputo type fractional q-derivative of order $\nu \geq 0$ is defined by the following:*

$$({}^c D_q^\nu g)(\varrho) = (I_q^{[\nu]-\nu} D_q^{[\nu]} g)(\varrho), \quad \nu > 0,$$

where ν denotes the integer part and $[\nu] \geq \nu$.

Definition 4. *If $x, y > 0$, then the following is the case.*

$$B_q(x,y) = \int_0^1 \varrho^{(x-1)} (1-qt)^{(y-1)} d_q t$$

This is called a q-beta function, and we obtain the following.

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.$$

Lemma 1 ([27])**.** *Let $\nu, \gamma \geq 0$ and the function $g \in [0,1]$. Then, we have the following*

(i) $(I_q^{\gamma} I_q^{\nu} g)(\varrho) = (I_q^{\nu+\gamma} g)(\varrho)$;
(ii) $(D_q^{\nu} I_q^{\nu} g)(\varrho) = g(\varrho)$.

Lemma 2 ([27])**.** *Let $\nu > 0$. Then, the following is the case.*

$$(I_q^{\nu c} D_q^{\nu} g)(\varrho) = g(\varrho) - \Sigma_{k=0}^{[\nu]-1} \frac{\varrho^k}{\Gamma_q(k+1)} (D_q^k g)(0).$$

Lemma 3 ([38])**.** *Let $\nu \geq 0$ and $n \in \mathbb{N}$. Then, the following is the case.*

$$(I_q^{\nu} D_q^{n} g)(\varrho) = D_q^{n} I_q^{\nu} g(\varrho) - \Sigma_{k=0}^{[\nu]-1} \frac{\varrho^{\nu-n+k}}{\Gamma_q(\nu-n+k)} (D_q^k g)(0).$$

Lemma 4 ([28])**.** *For $\nu \in \mathbb{R}^+$, $\rho \in (-1,\infty)$, then we obtain the following.*

$$I_q^{\nu}\Big((x-a)^{(\rho)}\Big) = \frac{\Gamma_q(\rho+1)}{\Gamma_q(\nu+\rho+1)}(x-a)^{(\nu+\rho)}, \quad 0 < a < x < b.$$

If $a = 0$, $\rho = 0$, we obtain the following.

$$(I_q^{\nu} 1)(x) = \frac{1}{\Gamma_q(\nu+1)} x^{(\nu)}.$$

3. Main Results

The following lemma is required to define the solution for the problems (1) and (2).

Lemma 5. *Let $h \in C([0,1],\mathbb{R})$, the function x is a unique solution for the fractional q-difference boundary value problem (BVP).*

$$\begin{aligned} &{}^cD_q^{\alpha_1}\left({}^cD_q^{\alpha_2} \frac{x(\varrho)}{f_1(\varrho, x(\varrho))}\right) + \lambda\, {}^cD_q^{\alpha_1} x(\varrho) = h(\varrho), \quad 0 \leq \varrho \leq 1, \quad 0 < q < 1, \\ &x(0) = 0, x(1) = 1, \end{aligned} \tag{3}$$

The above is given by the following:

$$\begin{aligned} x(\varrho) = &- \lambda f_1(\varrho, x(\varrho)) \int_0^{\varrho} \frac{(\varrho - qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \left(\int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\lambda \Gamma_q(\alpha_1)} h(\varsigma) d_q\varsigma - x(u)\right) d_q u \\ &- f_1(\varrho, x(\varrho))\varrho^{\alpha} \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \left(\int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} h(\varsigma) d_q\varsigma\right) d_q u \\ &+ \frac{\lambda f_1(\varrho, x(\varrho))\varrho^{2\alpha}}{\Gamma_q(1+\alpha_2)} + \frac{f_1(\varrho, x(\varrho))\varrho^{\alpha}}{\hat{f}_1}, \end{aligned} \tag{4}$$

where $f_1(0, x(0)) = \tilde{f}_1$ and $f_1(1, x(1)) = \hat{f}_1$.

Proof. A general solution x of the Equation (3) is given by the following.

$$\begin{aligned} x(\varrho) = &- \lambda f_1(\varrho, x(\varrho)) \int_0^{\varrho} \frac{(\varrho - qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} x(u) d_q u \\ &+ \lambda f_1(\varrho, x(\varrho)) \int_0^{\varrho} \frac{(\varrho - qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \left(\int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\lambda \Gamma_q(\alpha_1)} h(\varsigma) d_q\varsigma\right) d_q u - \frac{c_0 f_1(\varrho, x(\varrho))\varrho^{\alpha}}{\Gamma_q(1+\alpha_2)} \\ &- c_1 f_1(\varrho, x(\varrho)). \end{aligned} \tag{5}$$

By applying the boundary conditions for Equation (1), we obtain the following.

$$c_1 = 0,$$
$$c_0 = -\lambda \varrho^{\alpha} + \Gamma_q(1+\alpha_2) \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \left(\int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} h(\varsigma) d_q\varsigma \right) d_q u - \frac{\Gamma_q(1+\alpha_2)}{\hat{f}_1}. \tag{6}$$

By substituting Equation (6) in (5), we obtain the solution given by Equation (4). This completes the proof. □

Let us assume that $\mathcal{C} = C([0,1],\mathbb{R})$ is a Banach space enriched with the usual norm defined by $||x|| = \sup\{|x(\varrho)|, \varrho \in [0,1]\}$. As a result of Lemma 5, we define $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ as follows.

$$\begin{aligned}
(\mathcal{F}x)(\varrho) &= \frac{\lambda f_1(\varrho, x(\varrho))\varrho^{2\alpha}}{\Gamma_q(1+\alpha_2)} + \frac{f_1(\varrho, x(\varrho))\varrho^{\alpha}}{\hat{f}_1} - \lambda f_1(\varrho, x(\varrho)) \int_0^{\varrho} \frac{(\varrho-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
&\times \left(p_1 \int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\lambda\Gamma_q(\alpha_1)} f_2(\varsigma, x(\varsigma)) d_q\varsigma \right. \\
&\quad \left. + p_2 \int_0^u \frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\lambda\Gamma_q(\alpha_1+\xi)} f_3(\varsigma, x(\varsigma)) d_q\varsigma - x(u) \right) d_q u \\
&- f_1(\varrho, x(\varrho))\varrho^{\alpha} \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
&\times \left(p_1 \int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} f_2(\varsigma, x(\varsigma)) d_q\varsigma \right. \\
&\quad \left. + p_2 \int_0^u \frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)} f_3(\varsigma, x(\varsigma)) d_q\varsigma \right) d_q u.
\end{aligned} \tag{7}$$

Consider the fact that the problems (1) and (2) have solutions only if $x = \mathcal{F}x$ has fixed point, where $\mathcal{F}$ is defined in Equation (7). The following existence result is based on Banach's contraction principle.

Theorem 1. *Assume (A_1) holds, then the BVPs (1) and (2) have a unique solution $\Theta < 1$, where the following is the case.*

$$\Theta = \left(\frac{|\lambda|}{\Gamma_q(\alpha_2+1)} + \frac{2|p_1|L_2}{\Gamma_q(\alpha_1+\alpha_2+1)} + \frac{(|\lambda|+1)|p_2|L_3}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)} \right) \hat{f}_1. \tag{8}$$

Proof. Let us set $\sup_{\varrho\in[0,1]} |f_i(\varrho, 0)| = M_i,\ \ i = 1,2,3$ where M_i are finite numbers. Let us choose the following:

$$r \geq \frac{1}{1-\sigma} \left(1 + \left(\frac{|\lambda|}{\Gamma_q(\alpha_2+1)} + \frac{2|p_1|M_2}{\Gamma_q(\alpha_1+\alpha_2+1)} + \frac{(|\lambda|+1)|p_2|M_3}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)} \right) \hat{f}_1 \right), \tag{9}$$

where σ is such that $\Theta \leq \sigma < 1$. Now, we prove that $\mathcal{F}B_r \subset B_r$, where the following results.

$$B_r = \{x \in \mathcal{C} : ||x|| \leq r\}.$$

For $x \in B_r$, we obtain the following.

$$
\begin{aligned}
&||(\mathcal{F}x)|| = \\
&\sup_{\varrho\in[0,1]}\Bigg|\frac{\lambda f_1(\varrho,x(\varrho))\varrho^{2\alpha}}{\Gamma_q(1+\alpha_2)}+\frac{f_1(\varrho,x(\varrho))\varrho^{\alpha}}{\hat{f}_1}+\lambda f_1(\varrho,x(\varrho))\int_0^{\varrho}\frac{(\varrho-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
&\times\Bigg(p_1\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\lambda\Gamma_q(\alpha_1)}f_2(\varsigma,x(\varsigma))d_q\varsigma \\
&\quad+p_2\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}f_3(\varsigma,x(\varsigma))d_q\varsigma-x(u)\Bigg)d_qu \\
&+f_1(\varrho,x(\varrho))\varrho^{\alpha}\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
&\times\Bigg(p_1\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}f_2(\varsigma,x(\varsigma))d_q\varsigma \\
&\quad+p_2\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}f_3(\varsigma,x(\varsigma))d_q\varsigma\Bigg)d_qu\Bigg| \\
&\leq\sup_{\varrho\in[0,1]}|\lambda|\left|\frac{f_1(\varrho,x(\varrho))\varrho^{2\alpha}}{\Gamma_q(1+\alpha_2)}\right|+\left|\frac{f_1(\varrho,x(\varrho))\varrho^{\alpha}}{\hat{f}_1}\right|+|\lambda||f_1(\varrho,x(\varrho))|\int_0^{\varrho}\frac{(\varrho-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
&\times\Bigg(|p_1|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\lambda\Gamma_q(\alpha_1)}|f_2(\varsigma,x(\varsigma))|d_q\varsigma \\
&\quad+|p_2|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}|f_3(\varsigma,x(\varsigma))|d_q\varsigma-|x(u)|\Bigg)d_qu \\
&+|f_1(\varrho,x(\varrho))||\varrho^{\alpha}|\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
&\times\Bigg(|p_1|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}|f_2(\varsigma,x(\varsigma))|d_q\varsigma \\
&\quad+|p_2|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}|f_3(\varsigma,x(\varsigma))|d_q\varsigma\Bigg)d_qu
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{|\lambda|\hat{f}_1}{\Gamma_q(1+\alpha_2)}+1+|\lambda|\hat{f}_1\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
&\times\bigg(|p_1|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{|\lambda|\Gamma_q(\alpha_1)}|(f_2(\varsigma,x(\varsigma))-f_2(\varsigma,0)|+|f_2(\varsigma,0)|)d_q\varsigma \\
&+|p_2|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}|(f_3(\varsigma,x(\varsigma))-f_3(\varsigma,0)| \\
&\quad+|f_3(\varsigma,0)|)d_q\varsigma-|x(u)|\bigg)d_qu \\
&+\hat{f}_1\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
&\times\bigg(|p_1|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}|(f_2(\varsigma,x(\varsigma))-f_2(\varsigma,0)|+|f_2(\varsigma,0)|)d_q\varsigma \\
&+|p_2|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}(|f_3(\varsigma,x(\varsigma))-f_3(\varsigma,0)|+|f_3(\varsigma,0)|)d_q\varsigma\bigg)d_qu \\
&\leq \frac{|\lambda|\hat{f}_1}{\Gamma_q(1+\alpha_2)}+1+|\lambda|\hat{f}_1r\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}d_qu \\
&+\hat{f}_1|p_1|(L_2r+M_2)\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}d_q\varsigma d_qu \\
&+|\lambda|\hat{f}_1|p_2|(L_3r+M_3)\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}d_q\varsigma d_qu \\
&+\hat{f}_1|p_1|(L_2r+M_2)\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}d_q\varsigma d_qu \\
&+\hat{f}_1|p_2|(L_3r+M_3)\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}d_q\varsigma d_qu \\
&\leq\left(\frac{|\lambda|}{\Gamma_q(\alpha_2+1)}+\frac{2|p_1|L_2}{\Gamma_q(\alpha_1+\alpha_2+1)}+\frac{(|\lambda|+1)|p_2|L_3}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)}\right)\hat{f}_1r+ \\
&\left(\frac{|\lambda|}{\Gamma_q(\alpha_2+1)}+\frac{2|p_1|M_2}{\Gamma_q(\alpha_1+\alpha_2+1)}+\frac{(|\lambda|+1)|p_2|M_3}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)}\right)\hat{f}_1 \\
&\leq\Theta r+r(1-\sigma)=(\Theta+1-\sigma)r \\
&||(\mathcal{F}x)||\leq r.
\end{aligned}
$$

Then, for $x, y \in \mathcal{C}$ and for any $\varrho \in [0,1]$, we obtain the following:

$$
\begin{aligned}
&||(\mathcal{F}x)(\varrho) - (\mathcal{F}y)(\varrho)|| = \sup_{\varrho\in[0,1]} |(\mathcal{F}x)(\varrho) - (\mathcal{F}y)(\varrho)| \\
&\leq \hat{f}_1 \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \bigg(|p_1| \int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} |(f_2(\varsigma, x(\varsigma)) - (f_2(\varsigma, y(\varsigma))| d_q\varsigma \\
&+ |p_2| \int_0^u \frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)} |(f_3(\varsigma, x(\varsigma)) - (f_3(\varsigma, y(\varsigma))| d_q\varsigma \bigg) d_q u \\
&+ |\lambda| \hat{f}_1 \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} |x(u) - y(u)| d_q u + \hat{f}_1 \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
&\bigg(|p_1| \int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} |(f_2(\varsigma, x(\varsigma)) - (f_2(\varsigma, y(\varsigma))| d_q\varsigma \\
&+ |p_2| \int_0^u \frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)} |(f_3(\varsigma, x(\varsigma)) - (f_3(\varsigma, y(\varsigma))| d_q\varsigma \bigg) d_q u \\
&\leq \hat{f}_1 |p_1| L_2 ||x-y|| \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} d_q\varsigma d_q u \\
&+ |\lambda| \hat{f}_1 |p_2| L_3 ||x-y|| \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \int_0^u \frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)} d_q\varsigma d_q u \\
&+ |\lambda| \hat{f}_1 ||x-y|| \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} d_q u \\
&+ \hat{f}_1 |p_1| L_2 ||x-y|| \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \int_0^u \frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} d_q\varsigma d_q u \\
&+ \hat{f}_1 |p_2| L_3 ||x-y|| \int_0^1 \frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \int_0^u \frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)} d_q\varsigma d_q u \\
&\leq \hat{f}_1 |p_1| L_2 ||x-y|| \frac{1}{\Gamma_q(\alpha_1+\alpha_2+1)} + |\lambda| \hat{f}_1 |p_2| L_3 ||x-y|| \frac{1}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)} \\
&+ |\lambda| \hat{f}_1 ||x-y|| \frac{1}{\Gamma_q(\alpha_2+1)} + \hat{f}_1 |p_1| L_2 ||x-y|| \frac{1}{\Gamma_q(\alpha_1+\alpha_2+1)} \\
&+ \hat{f}_1 |p_2| L_3 ||x-y|| \frac{1}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)} \\
&\leq \left[\left(\frac{|\lambda|}{\Gamma_q(\alpha_2+1)} + \frac{2|p_1|L_2}{\Gamma_q(\alpha_1+\alpha_2+1)} + \frac{(|\lambda|+1)|p_2|L_3}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)} \right) \hat{f}_1 \right] ||x-y|| \\
&\leq \Theta ||x-y||
\end{aligned}
$$

which depends only on the parameters involved in the problem. As $\Theta < 1$, then $\mathcal{F}$ is a contraction. As a result, the contraction mapping principle results in the theorem's conclusion. This completes the proof. □

The second existence result is based on Krasnoselskii's fixed point theorem.

Lemma 6. *Let Y be a closed, convex, bounded and a nonempty subset of a Banach space X. Let Q_1, Q_2 be the operators such that we have the following:*

(i) $Q_1x + Q_2y \in Y$ *whenever* $x, y \in Y$;
(ii) Q_1 *is compact and continuous;*
(iii) Q_2 *is a contraction mapping.*

Then, there exists $z \in Y$ *such that* $z = Q_1z + Q_2z$.

Theorem 2. *Assume that (A_1)–(A_2) hold. If the following is the case:*

$$\frac{\hat{f}_1|p_1|L_2}{\Gamma_q(\alpha_1+\alpha_2+1)}+\frac{\hat{f}_1|p_2|L_3}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)}\leq 1, \tag{10}$$

then BVP (1) and (2) has at least one solution on [0, 1].

Proof. Let us define the following:

$$r\geq\left(1+\left(\frac{|\lambda|}{\Gamma_q(\alpha_2+1)}+\frac{2|p_1|\mu_2}{\Gamma_q(\alpha_1+\alpha_2+1)}+\frac{(|\lambda|+1)|p_2|\mu_3}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)}\right)\hat{f}_1\right) \tag{11}$$

and consider $B_r=\{x\in\mathcal{C}:||x||\leq r\}$. We define Q_1 and Q_2 on B_r as follows.

$$\begin{aligned}
(Q_1x)(\varrho)&=-\lambda f_1(\varrho,x(\varrho))\int_0^\varrho\frac{(\varrho-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\times\left(p_1\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\lambda\Gamma_q(\alpha_1)}f_2(\varsigma,x(\varsigma))d_q\varsigma\right.\\
&\left.+p_2\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\lambda\Gamma_q(\alpha_1+\xi)}f_3(\varsigma,x(\varsigma))d_q\varsigma-x(u)\right)d_qu\\
(Q_2x)(\varrho)&=-f_1(\varrho,x(\varrho))\varrho^\alpha\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\\
&\times\left(p_1\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}f_2(\varsigma,x(\varsigma))d_q\varsigma+p_2\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}f_3(\varsigma,x(\varsigma))d_q\varsigma\right)d_qu\\
&+\frac{\lambda f_1(\varrho,x(\varrho))\varrho^{2\alpha}}{\Gamma_q(1+\alpha_2)}+\frac{f_1(\varrho,x(\varrho))\varrho^\alpha}{\hat{f}_1}.
\end{aligned}$$

For $x,y\in B_r$, we obtain the following.

$$||Q_1x+Q_2y||\leq 1+\frac{|\lambda|\hat{f}_1}{\Gamma_q(\alpha_2+1)}+\frac{|\lambda|\hat{f}_1r}{\Gamma_q(\alpha_2+1)}+\frac{2|p_1|\mu_2\hat{f}_1}{\Gamma_q(\alpha_1+\alpha_2+1)}+\frac{(|\lambda|+1)\hat{f}_1|p_2|\mu_3}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)}\leq r.$$

Thus, $Q_1x+Q_2y\in B_r$. Continuity of f_2 and f_3 implies that Q_1 is continuous. Furthermore, Q_1 is uniformly bounded on B_r as follows.

$$||Q_1x||\leq\frac{|\lambda|\hat{f}_1r}{\Gamma_q(\alpha_2+1)}+\frac{|\lambda|\hat{f}_1|p_1|\mu_2}{\Gamma_q(\alpha_1+\alpha_2+1)}+\frac{|\lambda|\hat{f}_1|p_2|\mu_3}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)}.$$

Now, we prove the compactness of the operator Q_1. In view of (A_1), we define the following.

$$\sup_{(\varrho,x)\in[0,1]\times B_r}|f_i(\varrho,x)|=\bar{f}_i.$$

Consequently, the following is obtained.

$$\begin{aligned}
||(Q_1x)(\varrho_2)-(Q_1x)(\varrho_1)|| \leq & \Bigg(|\lambda|\bar{f}_1\int_0^{\varrho_2}\frac{(\varrho_2-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
& \times\Bigg(|p_1|\bar{f}_2\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\lambda\Gamma_q(\alpha_1)}d_q\varsigma \\
& +|p_2|\bar{f}_3\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\lambda\Gamma_q(\alpha_1+\xi)}d_q\varsigma+r\Bigg)d_qu\Bigg) \\
& -\Bigg(|\lambda|\bar{f}_1\int_0^{\varrho_1}\frac{(\varrho_1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} \\
& \times\Bigg(|p_1|\bar{f}_2\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\lambda\Gamma_q(\alpha_1)}d_q\varsigma \\
& +|p_2|\bar{f}_3\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\lambda\Gamma_q(\alpha_1+\xi)}d_q\varsigma+r\Bigg)d_qu\Bigg) \\
& \leq\bar{f}_1\bar{f}_2|p_1|\left(\frac{\varrho_2^{(\alpha_1+\alpha_2)}}{\Gamma_q(\alpha_1+\alpha_2+1)}-\frac{\varrho_1^{(\alpha_1+\alpha_2)}}{\Gamma_q(\alpha_1+\alpha_2+1)}\right) \\
& +|\lambda|\bar{f}_1\bar{f}_3|p_2|\left(\frac{\varrho_2^{(\alpha_1+\alpha_2+\xi)}}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)}-\frac{\varrho_1^{(\alpha_1+\alpha_2+\xi)}}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)}\right) \\
& +|\lambda|\bar{f}_1r\left(\frac{\varrho_2^{(\alpha_2)}}{\Gamma_q(\alpha_2+1)}-\frac{\varrho_1^{(\alpha_2)}}{\Gamma_q(\alpha_2+1)}\right)
\end{aligned}$$

Observe that the above inequality is independent of x and tends to zero as $\varrho_2 \to \varrho_1$. Thus, Q_1 is relatively compact on B_r, and the Arzelá-Ascoli Theorem implies that Q_1 is compact on B_r. Now, we shall show that Q_2 is a contraction.

From (A_1) and for $x, y \in B_r$, we have the following.

$$\begin{aligned}
||Q_2x-Q_2y|| \leq & \hat{f}_1\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\times\Bigg(|p_1|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}|f_2(\varsigma,x(\varsigma))-f_2(\varsigma,y(\varsigma))|d_q\varsigma \\
& +|p_2|\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}|f_3(\varsigma,x(\varsigma))-f_3(\varsigma,y(\varsigma))|d_q\varsigma\Bigg)d_qu+\frac{|\lambda|\hat{f}_1}{\Gamma_q(1+\alpha_2)}+1 \\
\leq & \hat{f}_1|p_1|L_2||x-y||\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\int_0^u\frac{(u-q\varsigma)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}d_q\varsigma d_qu \\
& +\hat{f}_1|p_2|L_3||x-y||\int_0^1\frac{(1-qu)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\int_0^u\frac{(u-q\varsigma)^{(\alpha_1+\xi-1)}}{\Gamma_q(\alpha_1+\xi)}d_q\varsigma d_qu+\frac{|\lambda|\hat{f}_1}{\Gamma_q(1+\alpha_2)}+1 \\
\leq & \left(\hat{f}_1|p_1|L_2\frac{1}{\Gamma_q(\alpha_1+\alpha_2+1)}+\hat{f}_1|p_2|L_3\frac{1}{\Gamma_q(\alpha_1+\alpha_2+\xi+1)}\right)||x-y||+\frac{|\lambda|\hat{f}_1}{\Gamma_q(1+\alpha_2)}+1,
\end{aligned}$$

By Equation (10), Q_2 is a contraction mapping. Hence, we deduce by the conclusion of Lemma 6 that problems (1) and (2) have at least one solution on [0, 1]. □

4. Example

Example 1. *Consider a Dirichlet boundary BVP for quantum fractional nonlinear differential equations given by the following:*

$$\begin{aligned} {}^{c}D_q^{\frac{1}{4}}\left({}^{c}D_q^{\frac{1}{2}}\frac{x(\varrho)}{f_1(\varrho,x(\varrho))}\right)+\frac{\Gamma_q(\frac{3}{2})}{12}\,{}^{c}D_q^{\frac{1}{4}}x(\varrho) &= \frac{\Gamma_q(\frac{7}{4})}{9}f_2(\varrho,x(\varrho)) \\ &+\Gamma_q(\frac{9}{4})\left(\frac{12}{12+\Gamma_q(\frac{3}{2})}\right)I_q^{\xi}f_3(\varrho,x(\varrho)), 0\le\varrho\le 1, 0<q<1 \\ & x(0)=0, x(1)=1. \end{aligned} \tag{12}$$

where $\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{2}, f_1(\varrho,x(\varrho)) = \varrho^2(1+x)$, $q = \frac{1}{2}, \lambda = \frac{\Gamma_q(\frac{3}{2})}{12}, p_1 = \frac{\Gamma_q(\frac{7}{4})}{9}$, $p_2 = \Gamma_q(\frac{9}{4})\left(\frac{12}{12+\Gamma_q(\frac{3}{2})}\right)$, $\hat{f}_1 = 2$, $f_2(\varrho,x(\varrho)) = \frac{1}{4}(x+\tan^{-1}x+\sin\varrho)$ *and* $f_3(\varrho,x(\varrho)) = \frac{1}{4}(\varrho^2+\cos\varrho+\tan^{-1}x)$. *With the given data, we obtain* $L_2 = \frac{1}{2}, L_3 = \frac{1}{4}$ *as* $|f_2(\varrho,x(\varrho)) - f_2(\varrho,y(\varrho))| \le \frac{1}{2}|x-y|, |f_3(\varrho,x(\varrho)) - f_3(\varrho,y(\varrho))| \le \frac{1}{4}|x-y|$. *Thus,* $\Theta \approx 0.\bar{8} < 1$. *Clearly, all the conditions of Theorem 3.1 are satisfied; therefore, BVP (12) has a unique solution.*

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