



## Article

# A Variety of New Traveling Wave Packets and Conservation Laws to the Nonlinear Low-Pass Electrical Transmission Lines via Lie Analysis

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**Abstract:** This research is based on computing the new wave packets and conserved quantities to the nonlinear low-pass electrical transmission lines (NLETs) via the group-theoretic method. By using the group-theoretic technique, we analyse the NLETs and compute infinitesimal generators. The resulting equations concede two-dimensional Lie algebra. Then, we have to find the commutation relation of the entire vector field and observe that the obtained generators make an abelian algebra. The optimal system is computed by using the entire vector field and using the concept of abelian algebra. With the help of an optimal system, NLETs convert into nonlinear ODE. The modified Khater method (MKM) is used to find the wave packets by using the resulting ODEs for a supposed model. To represent the physical importance of the considered model, some 3D, 2D, and density diagrams of acquired results are plotted by using Mathematica under the suitable choice of involving parameter values. Furthermore, all derived results were verified by putting them back into the assumed equation with the aid of Maple software. Further, the conservation laws of NLETs are computed by the multiplier method.

**Keywords:** nonlinear low-pass electrical transmission lines; modified khater method; multiplier approach; lie analysis; travelling wave patterns; conservation laws



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## 1. Introduction

The nonlinear evolution equations (NLEEs) explain the physical problems in different branches of engineering and nonlinear science, for example, plasma physics, biology, fluid mechanics, optics, solid-state physics, etc. [1,2]. Numerical and analytical solutions of NLEEs play a significant role in understanding the nonlinear physical models that are appearing in nonlinear sciences as well as many other branches of science, including natural sciences. Furthermore, the effort in computing the exact travelling wave patterns of NLEEs through different methods has developed speedily in the last few years, which is one of the important and advanced subjects of nonlinear science, engineering, and theoretical physics. Subsequently, we know the hypothesis of the exceptional waves in a particular soliton. Solitons have an important role in several physical phenomena and they show up in different shapes, for example, kink, bright breather, pulse, dark, envelope, and many others.

Many numbers of significant techniques for the analytical and stable soliton and wave patterns of physical models have presently been created with the help of Matlab, Mathematica, etc., such as the differential transformation technique [3], the modified exponential-function [4] method, the  $(G'/G)$ -expansion [5] technique, the improved  $(G'/G)$ -expansion [6] technique,

the rational ( $G'/G$ ) [7] technique, the generalized Kudryashov [8] method, the homotopy analysis [9] scheme, the mean finite difference Monte-Carlo [10] technique, the Hirota's bilinear technique [11], the approach of modified simple [12] equation, the F-expansion [13] technique, the Exp-function [14] scheme, the sine-Gordon expansion [15] scheme, the modified Kudryashov [16] method, the extended trial Equation [17] scheme, the improved  $\tan(\Psi(\eta)/2)$ -expansion [18] scheme, the first integral [19] technique, the Adomian decomposition [20] method, the exponential rational function [21] technique, the rational function [22] technique, the unified scheme [23], the generalized projective Riccati [24] scheme, the multi-symplectic Runge-Kutta [25] technique, the modified extended tanh [26] scheme, the generalized unified technique [27], the modified auxiliary [28] technique, and other many methods and details are described in [29–45].

Many kinds of solutions are computed by using different analytical schemes such as a homogeneous balance technique, the auxiliary equation technique and the Jacobi elliptic expansion scheme, and many other techniques which are described and cited here. In this research, we are using the MKM, which has not been used previously for this model. Using this useful method on our supposed equation, we find some new kinds of wave patterns which are fruitful and very interesting results. We have used a MKM to get the required results and we obtain bright solutions, soliton-like solutions, singular bright solutions, periodic soliton solutions, and combined soliton solutions. These results are in the form of trigonometric and hyperbolic functions. There are different kinds of solutions for low pass electrical transmission lines that are computed in [46–50] and many other related data of this model can be seen in the literature which is cited in this article. As far as we know, the outcomes represented in this paper generally have not been described in the literature. The solutions we have developed here are new and very useful in the different branches of science.

Lie method [51–54] is employed to analyse the NLETs. We have to construct the classical symmetries of the assumed model. We see that the obtained vector field forms an abelian algebra. We find some new travelling wave results for the NLETs by utilizing an optimal system of Lie symmetry vectors by using the concept of abelian algebra. With the help of an optimal system, NLETs convert into nonlinear ODE. Using the theory of the Lie symmetry method [55–57], we get the various forms of significant results to the assumed PDE. Then, we explore the wave solutions with the help of the integration technique, namely the modified Khater method [58,59], to solve the NLPDEs depicting the wave proliferation in the NLETs. By using the translational vectors and their linear combinations, the NLETs are converted to an equation wave proliferation in ordinary differential equation NLETs. MKM is employed to find some new trigonometric and hyperbolic results which represent the consistency by Maple. To represent the physical importance of the considered model, some 3D, 2D, and density diagrams of the acquired results are plotted by using Mathematica under the suitable choice of involving parameters values. Furthermore, all derived results were verified by putting them back into the assumed equation with the aid of Maple software. Further, the conservation laws of NLPETs are computed by the multiplier method. In recent years, many scientists have developed the use of the Lie theory, including Ibragimov [60], Olver [61], A.F Cheviakov [62] and Bluman [63].

In this article, we will discuss the nonlinear low-pass electrical transmission line (NLPETL) Equation [64,65] of the following form:

$$U_{\tau\tau} - l_1(U^2)_{\tau\tau} + l_2(U^3)_{\tau\tau} - \delta^2 U_{\theta\theta} - \frac{\delta^4}{12} U_{\theta\theta\theta\theta} = 0, \quad (1)$$

where  $U = U(\theta, \tau)$  is the voltage and  $l_1, l_2, \delta$  are the constants. Spatial component  $\theta$  and temporal component  $\tau$  represent the proliferation distance and slow time, respectively. The actual parts of the derivation of Equation (1) applying Kirchhoff's laws are presented in [47], which are precluded here for brevity. The travelling wave and soliton solutions of Equation (1) are derived in [48]. The expansion method for describing given model and the

generalized projective Riccati equations method and its applications to non-linear PDEs describing the given equation are in [49,50].

The investigation of NLETs and their care arrangements are significant for different applied fields, for example, interfacing radio transmitters and collectors with their receiving wires, appropriating satellite TV signals, trunk lines directing calls between phone exchanging focuses, PC network associations, and rapid PC information transports. Furthermore, in interchanges and electronic designing, a transmission line is a specific medium or other construction intended to convey exchanging current of radio recurrence. NLETs are additionally given a valuable method to check how the nonlinear excitations act inside the medium and to show the colourful properties of new frameworks [66].

Conservation laws play a very important role in constructing the analytical results of different types of nonlinear physical models. Furthermore, conservation laws are used for the reduction of PDEs, to construct the numerical, analytical, and solitary wave results. Various methods are developed to find the conservation laws, including Naz et al. [67], Herman et al. [68] and A.F. Cheviakov [69], who constructed the package GeM on Maple to find the conservation laws for PDEs. Here, we are using the multiplier scheme [70] to find the conservation laws for the assumed model. To the best of our knowledge, the supposed equation is not described by the MKM and no one has computed the conservation laws of this physical model. The pattern of this paper is as follows, in Section 2, preliminaries are presented. Classical symmetries are computed in Section 3. The optimal system, wave patterns, and graphical diagrams are described in Section 4. Conservation laws of the NLETs are constructed in Section 5. At the end of this article, the conclusion is stated.

## 2. Preliminaries

### 2.1. Modified Khater Method

We describe the modified Khater method [71,72] to construct a new wave pattern for a supposed model. In this regard, some details of this method are given below;

(Step 1): Suppose a general  $n$ th order PDE:

$$H(U, U_\theta, U_{\theta\theta}, \dots, U_\tau, U_{\tau\tau}, \dots) = 0, \quad (2)$$

where  $U = U(\theta, \tau)$  is unknown and  $H$  is a polynomial function w.r.t specified variables.

(Step 2): Introducing the wave transformation

$$\varrho = \tau - a\theta, \quad U(\theta, \tau) = Q(\varrho), \quad (3)$$

using above transformation (3) to convert the partial differential Equation (2) into ordinary differential equation below:

$$F(Q, Q', Q'', \dots) = 0. \quad (4)$$

(Step 3): It is assumed that the general solution of nonlinear ODE (4) can be written as:

$$Q(\varrho) = \sum_{j=0}^m k_j \mathfrak{B}^j(\varrho), \quad (5)$$

where  $k_j$  ( $0 < j \leq n$ ) are the arbitrary constants and  $\mathfrak{B}(\varrho)$  is the solution of the equation:

$$\mathfrak{B}'(\varrho) = \ln(\vartheta)(\gamma_1 + \gamma_2 \mathfrak{B}(\varrho) + \gamma_3 \mathfrak{B}^2(\varrho)), \quad (6)$$

where  $\vartheta \neq 0, 1$  and  $\gamma_1, \gamma_2$ , and  $\gamma_3$  are the constants.

After assuming  $\Delta = \gamma_2^2 - 4\gamma_3\gamma_1$ , the solutions of Equation (6) represented as:

1: If  $\Delta < 0$  and  $\gamma_3 \neq 0$ , then

$$\begin{aligned}\mathfrak{B}_1(\varrho) &= -\frac{\gamma_2}{2\gamma_3} + \frac{\sqrt{-\Delta}}{2\gamma_3} \tan_{\theta}\left(\frac{\sqrt{-\Delta}}{2}\varrho\right), \\ \mathfrak{B}_2(\varrho) &= -\frac{\gamma_2}{2\gamma_3} - \frac{\sqrt{-\Delta}}{2\gamma_3} \cot_{\theta}\left(\frac{\sqrt{-\Delta}}{2}\varrho\right), \\ \mathfrak{B}_3(\varrho) &= -\frac{\gamma_2}{2\gamma_3} + \frac{\sqrt{-\Delta}}{2\gamma_3} (\tan_{\theta}(\sqrt{-\Delta}\varrho) \pm \sqrt{rs} \sec_{\theta}(\sqrt{-\Delta}\varrho)), \\ \mathfrak{B}_4(\varrho) &= -\frac{\gamma_2}{2\gamma_3} - \frac{\sqrt{-\Delta}}{2\gamma_3} (\cot_{\theta}(\sqrt{-\Delta}\varrho) \pm \sqrt{rs} \csc_{\theta}(\sqrt{-\Delta}\varrho)), \\ \mathfrak{B}_5(\varrho) &= -\frac{\gamma_2}{2\gamma_3} + \frac{\sqrt{-\Delta}}{4\gamma_3} (\tan_{\theta}\left(\frac{\sqrt{-\Delta}}{4}\varrho\right) - \cot_{\theta}\left(\frac{\sqrt{-\Delta}}{4}\varrho\right)).\end{aligned}\quad (7)$$

2: If  $\Delta > 0$  and  $\gamma_3 \neq 0$ , then

$$\begin{aligned}\mathfrak{B}_6(\varrho) &= -\frac{\gamma_2}{2\gamma_3} - \frac{\sqrt{\Delta}}{2\gamma_3} \tanh_{\theta}\left(\frac{\sqrt{\Delta}}{2}\varrho\right), \\ \mathfrak{B}_7(\varrho) &= -\frac{\gamma_2}{2\gamma_3} - \frac{\sqrt{\Delta}}{2\gamma_3} \coth_{\theta}\left(\frac{\sqrt{\Delta}}{2}\varrho\right), \\ \mathfrak{B}_8(\varrho) &= -\frac{\gamma_2}{2\gamma_3} - \frac{\sqrt{\Delta}}{2\gamma_3} (\tanh_{\theta}(\sqrt{\Delta}\varrho) \pm \iota\sqrt{rs} \operatorname{sech}_{\theta}(\sqrt{\Delta}\varrho)), \\ \mathfrak{B}_9(\varrho) &= -\frac{\gamma_2}{2\gamma_3} - \frac{\sqrt{\Delta}}{2\gamma_3} (\coth_{\theta}(\sqrt{\Delta}\varrho) \pm \sqrt{rs} \operatorname{csch}_{\theta}(\sqrt{\Delta}\varrho)), \\ \mathfrak{B}_{10}(\varrho) &= -\frac{\gamma_2}{2\gamma_3} - \frac{\sqrt{\Delta}}{4\gamma_3} (\tanh_{\theta}\left(\frac{\sqrt{\Delta}}{4}\varrho\right) + \coth_{\theta}\left(\frac{\sqrt{\Delta}}{4}\varrho\right)).\end{aligned}\quad (8)$$

3: If  $\gamma_3\gamma_1 > 0$  and  $\gamma_2 = 0$ , then

$$\begin{aligned}\mathfrak{B}_{11}(\varrho) &= \sqrt{\frac{\gamma_1}{\gamma_3}} \tan_{\theta}(\sqrt{\gamma_3\gamma_1}\varrho), \\ \mathfrak{B}_{12}(\varrho) &= -\sqrt{\frac{\gamma_1}{\gamma_3}} \cot_{\theta}(\sqrt{\gamma_3\gamma_1}\varrho), \\ \mathfrak{B}_{13}(\varrho) &= \sqrt{\frac{\gamma_1}{\gamma_3}} (\tan_{\theta}(2\sqrt{\gamma_3\gamma_1}\varrho) \pm \sqrt{rs} \sec_{\theta}(2\sqrt{\gamma_3\gamma_1}\varrho)), \\ \mathfrak{B}_{14}(\varrho) &= \sqrt{\frac{\gamma_1}{\gamma_3}} (-\cot_{\theta}(2\sqrt{\gamma_3\gamma_1}\varrho) \pm \sqrt{rs} \csc_{\theta}(2\sqrt{\gamma_3\gamma_1}\varrho)), \\ \mathfrak{B}_{15}(\varrho) &= \frac{1}{2}\sqrt{\frac{\gamma_1}{\gamma_3}} (\tan_{\theta}\left(\frac{\sqrt{\gamma_3\gamma_1}}{2}\varrho\right) - \cot_{\theta}\left(\frac{\sqrt{\gamma_3\gamma_1}}{2}\varrho\right)).\end{aligned}\quad (9)$$

4: If  $\gamma_3\gamma_1 < 0$  and  $\gamma_2 = 0$ , then

$$\begin{aligned}\mathfrak{B}_{16}(\varrho) &= -\sqrt{-\frac{\gamma_1}{\gamma_3}} \tanh_{\theta}(\sqrt{-\gamma_3\gamma_1}\varrho), \\ \mathfrak{B}_{17}(\varrho) &= -\sqrt{-\frac{\gamma_1}{\gamma_3}} \coth_{\theta}(\sqrt{-\gamma_3\gamma_1}\varrho), \\ \mathfrak{B}_{18}(\varrho) &= -\sqrt{-\frac{\gamma_1}{\gamma_3}} (\tanh_{\theta}(2\sqrt{-\gamma_3\gamma_1}\varrho) \pm \iota\sqrt{rs} \operatorname{sech}_{\theta}(2\sqrt{-\gamma_3\gamma_1}\varrho)), \\ \mathfrak{B}_{19}(\varrho) &= -\sqrt{-\frac{\gamma_1}{\gamma_3}} (\coth_{\theta}(2\sqrt{-\gamma_3\gamma_1}\varrho) \pm \sqrt{rs} \operatorname{csch}_{\theta}(2\sqrt{-\gamma_3\gamma_1}\varrho)), \\ \mathfrak{B}_{20}(\varrho) &= -\frac{1}{2}\sqrt{-\frac{\gamma_1}{\gamma_3}} (\tanh_{\theta}\left(\frac{\sqrt{-\gamma_3\gamma_1}}{2}\varrho\right) + \coth_{\theta}\left(\frac{\sqrt{-\gamma_3\gamma_1}}{2}\varrho\right)).\end{aligned}\quad (10)$$

5: If  $\gamma_2 = 0$  and  $\gamma_3 = \gamma_1$ , then

$$\begin{aligned}\mathfrak{B}_{21}(\varrho) &= \tan_{\vartheta}(\gamma_1\varrho), \\ \mathfrak{B}_{22}(\varrho) &= -\cot_{\vartheta}(\gamma_1\varrho), \\ \mathfrak{B}_{23}(\varrho) &= \tan_{\vartheta}(2\gamma_1\varrho) \pm \sqrt{rs} \sec_{\vartheta}(2\gamma_1\varrho), \\ \mathfrak{B}_{24}(\varrho) &= -\cot_{\vartheta}(2\gamma_1\varrho) \pm \sqrt{rs} \csc_{\vartheta}(2\gamma_1\varrho), \\ \mathfrak{B}_{25}(\varrho) &= \frac{1}{2}(\tan_{\vartheta}(\frac{\gamma_1}{2}\varrho) - \cot_{\vartheta}(\frac{\gamma_1}{2}\varrho)).\end{aligned}\quad (11)$$

6: If  $\gamma_2 = 0$  and  $\gamma_3 = -\gamma_1$ , then

$$\begin{aligned}\mathfrak{B}_{26}(\varrho) &= -\tanh_{\vartheta}(\gamma_1\varrho), \\ \mathfrak{B}_{27}(\varrho) &= -\coth_{\vartheta}(\gamma_1\varrho), \\ \mathfrak{B}_{28}(\varrho) &= -\tanh_{\vartheta}(2\gamma_1\varrho) \pm \sqrt{rs} \operatorname{sech}_{\vartheta}(2\gamma_1\varrho), \\ \mathfrak{B}_{29}(\varrho) &= -\coth_{\vartheta}(2\gamma_1\varrho) \pm \sqrt{rs} \operatorname{csch}_{\vartheta}(2\gamma_1\varrho), \\ \mathfrak{B}_{30}(\varrho) &= -\frac{1}{2}(\tanh_{\vartheta}(\frac{\gamma_1}{2}\varrho) + \coth_{\vartheta}(\frac{\gamma_1}{2}\varrho)).\end{aligned}\quad (12)$$

7: If  $\gamma_2^2 = 4\gamma_3\gamma_1$ , then

$$\mathfrak{B}_{31}(\varrho) = \frac{-2\gamma_1(\gamma_2\varrho \ln(\vartheta) + 2)}{\gamma_2^2\varrho \ln(\vartheta)}.\quad (13)$$

8: If  $\gamma_2 = \lambda$ ,  $\gamma_1 = p\lambda$  ( $p \neq 0$ ) and  $\gamma_3 = 0$ , then

$$\mathfrak{B}_{32}(\varrho) = \vartheta^{\lambda\varrho} - p.\quad (14)$$

9: If  $\gamma_2 = \gamma_3 = 0$ , then

$$\mathfrak{B}_{33}(\varrho) = \gamma_1\varrho \ln(\vartheta).\quad (15)$$

10: If  $\gamma_2 = \gamma_1 = 0$ , then

$$\mathfrak{B}_{34}(\varrho) = \frac{-1}{\gamma_3\varrho \ln(\vartheta)}.\quad (16)$$

11: If  $\gamma_1 = 0$  and  $\gamma_2 \neq 0$ , then

$$\begin{aligned}\mathfrak{B}_{35}(\varrho) &= -\frac{r\gamma_2}{\gamma_3(\cosh_{\vartheta}(\gamma_2\varrho) - \sinh_{\vartheta}(\gamma_2\varrho) + r)}, \\ \mathfrak{B}_{36}(\varrho) &= -\frac{\gamma_2(\sinh_{\vartheta}(\gamma_2\varrho) + \cosh_{\vartheta}(\gamma_2\varrho))}{\gamma_3(\sinh_{\vartheta}(\gamma_2\varrho) + \cosh_{\vartheta}(\gamma_2\varrho) + s)}.\end{aligned}\quad (17)$$

12: If  $\gamma_2 = \lambda$ ,  $\gamma_3 = p\lambda$  ( $p \neq 0$ ) and  $\gamma_1 = 0$ , then

$$\mathfrak{B}_{37}(\varrho) = \frac{r\vartheta^{\lambda\varrho}}{s - pr\vartheta^{\lambda\varrho}}.\quad (18)$$

Here, we define the hyperbolic and trigonometric functions as follows:

$$\begin{aligned}
\sinh_{\theta}(\varrho) &= \frac{r\vartheta^{\varrho} - s\vartheta^{-\varrho}}{2}, & \cosh_{\theta}(\varrho) &= \frac{r\vartheta^{\varrho} + s\vartheta^{-\varrho}}{2}, \\
\tanh_{\theta}(\varrho) &= \frac{r\vartheta^{\varrho} - s\vartheta^{-\varrho}}{r\vartheta^{\varrho} + s\vartheta^{-\varrho}}, & \coth_{\theta}(\varrho) &= \frac{r\vartheta^{\varrho} + s\vartheta^{-\varrho}}{r\vartheta^{\varrho} - s\vartheta^{-\varrho}}, \\
\operatorname{csch}_{\theta}(\varrho) &= \frac{2}{r\vartheta^{\varrho} - s\vartheta^{-\varrho}}, & \operatorname{sech}_{\theta}(\varrho) &= \frac{2}{r\vartheta^{\varrho} + s\vartheta^{-\varrho}}, \\
\sin_{\theta}(\varrho) &= \frac{r\vartheta^{i\varrho} - s\vartheta^{-i\varrho}}{2i}, & \cos_{\theta}(\varrho) &= \frac{r\vartheta^{i\varrho} + s\vartheta^{-i\varrho}}{2}, \\
\tan_{\theta}(\varrho) &= -i \frac{r\vartheta^{i\varrho} - s\vartheta^{-i\varrho}}{r\vartheta^{i\varrho} + s\vartheta^{-i\varrho}}, & \cot_{\theta}(\varrho) &= i \frac{r\vartheta^{i\varrho} + s\vartheta^{-i\varrho}}{r\vartheta^{i\varrho} - s\vartheta^{-i\varrho}}, \\
\operatorname{csc}_{\theta}(\varrho) &= \frac{2i}{r\vartheta^{i\varrho} - s\vartheta^{-i\varrho}}, & \operatorname{sech}_{\theta}(\varrho) &= \frac{2}{r\vartheta^{i\varrho} + s\vartheta^{-i\varrho}},
\end{aligned} \tag{19}$$

where  $r$  and  $s$  are arbitrary constants.

(Step 4): By using the balancing method to find the value of  $m$  by comparing the highest order linear and nonlinear term in ODE (4), the value of  $j$  is always positive.

(Step 5): Plugging Equations (5) and (6) into Equation (4) and equating the coefficients of powers of  $\mathfrak{B}(\varrho)$  to zero, which gives us the system of algebraic equations which can be solved by Mathematica.

## 2.2. Multiplier Approach

In this portion, we will portray the technique in detail:

1. The total differential operator defined as:

$$D_i = \frac{\partial}{\partial \theta^i} + U_i \frac{\partial}{\partial U} + U_{ij} \frac{\partial}{\partial U_j} + \dots, \quad i = 1, 2, 3, \dots, m, \tag{20}$$

where  $U_i$  denotes the derivative w.r.t  $\theta^i$  and  $U_{ij}$  shows the derivative w.r.t  $\theta^i$  and  $\theta^j$ .

2. We define the Euler operator is of the form:

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_i \frac{\partial}{\partial U_i} + D_{ij} \frac{\partial}{\partial U_{ij}} - D_{ijk} \frac{\partial}{\partial U_{ijk}} + \dots, \tag{21}$$

3. An  $n$ -tuple  $\mathfrak{F} = (\mathfrak{F}^1, \mathfrak{F}^2, \mathfrak{F}^3, \dots, \mathfrak{F}^m)$ ,  $i = 1, 2, \dots, m$ , we have

$$D_i \mathfrak{F}^i = 0, \tag{22}$$

fulfils the all solutions of (2). Equation (22) is called the local conservation law.

4. The property of multiplier  $\Lambda(\theta, \tau, G)$  of the Equation (2):

$$D_i \mathfrak{F}^i = \Lambda H, \tag{23}$$

for some function  $U(\mu^1, \mu^2, \dots, \mu^m)$  [61].

5. We get the determining equations for multiplier  $\Lambda(\theta, \tau, U)$  when we take the derivative of Equation (23) (see [61]):

$$\frac{\delta}{\delta U} (\Lambda(\theta, \tau, U) H) = 0. \tag{24}$$

Equation (24) consists for some function  $U(\mu^1, \mu^2, \dots, \mu^m)$  not only for solutions of Equation (2).

When the multiplier  $\Lambda(\theta, \tau, U)$  are obtained with help of (24), the conservation laws can be determined by Equation (23) as the determining equation.

### 3. Lie Symmetry Analysis

Lie algebra of Equation (1) is generated by the vector field below:

$$\mathfrak{F} = \zeta^1(\theta, \tau, U) \frac{\partial}{\partial \theta} + \zeta^2(\theta, \tau, U) \frac{\partial}{\partial \tau} + \eta(\theta, \tau, U) \frac{\partial}{\partial U}, \quad (25)$$

we define the fourth prolongation of  $\mathfrak{F}$  as the form:

$$\mathfrak{F}^{[4]} = \mathfrak{F} + \eta^\tau \frac{\partial}{\partial U_\tau} + \eta^{\tau\tau} \frac{\partial}{\partial U_{\tau\tau}} + \eta^{\theta\theta} \frac{\partial}{\partial U_{\theta\theta}} + \eta^{\theta\theta\theta} \frac{\partial}{\partial U_{\theta\theta\theta}}, \quad (26)$$

applying  $\mathfrak{F}^{[4]}$  to Equation (1), which gives:

$$\mathfrak{F}^{[4]}(U_{\tau\tau} - l_1(U^2)_{\tau\tau} + l_2(U^3)_{\tau\tau} - \delta^2 U_{\theta\theta} - \frac{\delta^4}{12} U_{\theta\theta\theta\theta})|_{Equation (1)} = 0, \quad (27)$$

$$\begin{aligned} & -2l_1 U_{\tau\tau} + 6l_2 U_\tau^2 + 6l_2 U U_{\tau\tau} - 4l_1 \eta^\tau U_\tau + 12l_2 \eta^\tau U U_\tau + \eta^{\tau\tau} - 2l_1 \eta^{\tau\tau} U \\ & + 3l_2 \eta^{\tau\tau} U^2 - \delta^2 \eta^{\theta\theta} - \frac{\delta^4}{12} \eta^{\theta\theta\theta\theta} = 0, \end{aligned} \quad (28)$$

where  $U_{\tau\tau} = \frac{\partial^2 U}{\partial \tau^2}$ ,  $U_{\theta\theta} = \frac{\partial^2 U}{\partial \theta^2}$ , and  $\eta^\tau$ ,  $\eta^{\tau\tau}$ ,  $\eta^{\theta\theta}$ , and  $\eta^{\theta\theta\theta\theta}$ , etc., are the coefficients of  $\mathfrak{F}^{[4]}$ . Furthermore, we have

$$\begin{cases} \eta^\tau = D_\tau(\eta) - U_\theta D_\tau(\zeta^1) - U_\tau D_\tau(\zeta^2), \\ \eta^\theta = D_\theta(\eta) - U_\theta D_\theta(\zeta^1) - U_\tau D_\theta(\zeta^2), \\ \eta^{\theta\theta} = D_\theta(\eta^\theta) - U_{\theta\theta} D_\theta(\zeta^1) - U_{\tau\theta} D_\theta(\zeta^2), \\ \eta^{\tau\tau} = D_\tau(\eta^\tau) - U_{\tau\tau} D_\tau(\zeta^1) - U_{\theta\tau} D_\tau(\zeta^2), \\ \eta^{\theta\theta} = D_\theta(\eta^{\theta\theta}) - U_{\theta\theta\theta} D_\theta(\zeta^1) - U_{\tau\theta\theta} D_\theta(\zeta^2), \\ \eta^{\theta\theta\theta\theta} = D_\theta(\eta^{\theta\theta\theta\theta}) - U_{\theta\theta\theta\theta} D_\theta(\zeta^1) - U_{\tau\theta\theta\theta} D_\theta(\zeta^2). \end{cases} \quad (29)$$

Let  $(x^1, x^2) = (\theta, \tau)$ , we define the derivative operator  $D_i$  as the form:

$$D_i = \frac{\partial}{\partial x^i} + U_i \frac{\partial}{\partial U} + U_{ij} \frac{\partial}{\partial U_j} + \dots, \quad i = 1, 2.$$

Put (29) into (28), we obtain the following determining equations:

$$\begin{aligned} \zeta_\tau^1 &= 0, \quad \zeta_\theta^1 = 0, \quad \zeta_U^1 = 0, \quad \eta(\theta, \tau, U)_U = 0, \\ \zeta_\tau^2 &= 0, \quad \zeta_\theta^2 = 0, \quad \zeta_U^2 = 0, \end{aligned} \quad (30)$$

where  $\eta_U = \frac{\partial \eta}{\partial U}$ ,  $\zeta_\theta = \frac{\partial \zeta}{\partial \theta}$ ,  $\zeta_\tau = \frac{\partial \zeta}{\partial \tau}$ , etc. Solving the system (30), gives:

$$\zeta^1 = C_2, \quad \zeta^2 = C_1, \quad \eta(\theta, \tau, U) = 0, \quad (31)$$

where  $C_i, i = 1, 2$  are all constants.

Equation (31) gives the entire vector field of Equation (1):

$$Z_1 = \frac{\partial}{\partial \tau}, \quad Z_2 = \frac{\partial}{\partial \theta}. \quad (32)$$

We note that

$$[Z_r, Z_s] = 0, \quad \text{where } r, s = 1, 2.$$

#### 4. Optimal System

We observe that  $P = \{Z_1, Z_2\}$  make an abelian subalgebra. So the one-dimensional optimal system for the entire vector field (32) is:

$$\begin{aligned}\mathcal{L}_1 &= \langle Z_1 \rangle, \\ \mathcal{L}_2 &= \langle Z_1 + \kappa Z_2 \rangle.\end{aligned}\quad (33)$$

##### 4.1. Similarity Reduction of Equation (1)

In this portion, we aim to find the similarity variables for (33), obtained similarity variables are used to convert the considered PDE into nonlinear ODE and help us to compute the analytical solution for Equation (1).

##### 4.2. $\mathcal{L}_1 = \langle Z_1 \rangle$

The characteristic equation for this vector field can be written as:

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dU}{0},$$

after solving the above characteristic equation, we obtain

$$U(\theta, \tau) = Q(\varrho), \quad \text{where } \varrho = \theta, \quad (34)$$

putting Equation (34) into Equation (1), which gives us the following ODE below:

$$\delta^2 Q''(\varrho) + \frac{\delta^4}{12} Q''''(\varrho) = 0, \quad (35)$$

integrating (35) once w.r.t  $\varrho$ , we get the following solution:

$$U(\theta, \tau) = C_1 + C_2\theta + C_3 \sin\left(\frac{2\sqrt{3}\theta}{\delta}\right) + C_4 \cos\left(\frac{2\sqrt{3}\theta}{\delta}\right), \quad (36)$$

where  $C_1, C_2, C_3$ , and  $C_4$  are all integration constants.

##### 4.3. $\mathcal{L}_2 = \langle Z_1 + \kappa Z_2 \rangle$

In this case, we have the transformation

$$U(\theta, \tau) = Q(\varrho), \quad \text{where } \varrho = \tau - \kappa\theta, \quad (37)$$

plugging Equation (37) into Equation (1), which gives us the following ODE:

$$Q'' - 2l_1(Q')^2 - 2l_1QQ'' + 6l_2Q(Q')^2 + 3l_2Q^2Q'' - \delta^2\kappa^2Q'' - \frac{\delta^4\kappa^4}{12}Q'''' = 0. \quad (38)$$

##### 4.4. Travelling Wave Solutions of Equation (1)

In this portion, wave patterns are computed by MKM for the NLETs from Equation (38).

Using the balancing scheme described in preliminaries to find the value of  $m$ , choosing the linear and nonlinear terms  $Q''''$  and  $Q^2Q''$  from Equation (38), we get  $m = 1$ , substituting in Equation (5), and we get:

$$Q(\varrho) = k_0 + k_1\mathfrak{B}(\varrho). \quad (39)$$

Assume that the  $\mathfrak{B}(\varrho)$  is the solution of the ODE:

$$\mathfrak{B}'(\varrho) = \ln(\vartheta)(\gamma_1 + \gamma_2\mathfrak{B}(\varrho) + \gamma_3\mathfrak{B}^2(\varrho)), \quad (40)$$

plugging Equations (39) and (40) into an (38), after some routine calculations we acquired a system of equations which gives us the following set of solutions:

$$\begin{aligned} \kappa &= \sqrt{\frac{2m_1 - 12\gamma_3^2}{m_2\delta}}, \\ k_0 &= \frac{\gamma_2}{\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}}, \\ k_1 &= \frac{2}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}}, \end{aligned} \quad (41)$$

where

$$m_1 = [24\gamma_3^5 \ln(\vartheta)^2 \gamma_1 \delta^2 - 6\gamma_3^4 \ln(\vartheta)^2 \gamma_2^2 \delta^2 + 36\gamma_3^4]^{\frac{1}{2}},$$

and

$$m_2 = [4\gamma_3^3 \ln(\vartheta)^2 \gamma_1 \delta^2 - \gamma_3^2 \ln(\vartheta)^2 \gamma_2^2 \delta^2],$$

we get the following set of solutions for Equation (1) with the use of Equation (41):

1: If  $\Delta < 0$  and  $\gamma_3 \neq 0$ , then

$$\begin{aligned} U_1(\theta, \tau) &= \frac{1}{\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{-\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \tan_{\vartheta}\left(\frac{\sqrt{-\Delta}}{2}\varrho\right), \\ U_2(\theta, \tau) &= \frac{-1}{\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{-\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \cot_{\vartheta}\left(\frac{\sqrt{-\Delta}}{2}\varrho\right), \\ U_3(\theta, \tau) &= \frac{1}{\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{-\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} (\tan_{\vartheta}(\sqrt{-\Delta}\varrho) \pm \sqrt{rs} \sec_{\vartheta}(\sqrt{-\Delta}\varrho)), \\ U_4(\theta, \tau) &= \frac{-1}{\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{-\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} (\cot_{\vartheta}(\sqrt{-\Delta}\varrho) \pm \sqrt{rs} \csc_{\vartheta}(\sqrt{-\Delta}\varrho)), \\ U_5(\theta, \tau) &= \frac{-1}{2\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{-\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left(\tan_{\vartheta}\left(\frac{\sqrt{-\Delta}}{4}\varrho\right) - \cot_{\vartheta}\left(\frac{\sqrt{-\Delta}}{4}\varrho\right)\right). \end{aligned}$$

2: If  $\Delta > 0$  and  $\gamma_3 \neq 0$ , then

$$\begin{aligned} U_6(\theta, \tau) &= \frac{-1}{\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \tanh_{\vartheta}\left(\frac{\sqrt{\Delta}}{2}\varrho\right), \\ U_7(\theta, \tau) &= \frac{-1}{\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \coth_{\vartheta}\left(\frac{\sqrt{\Delta}}{2}\varrho\right), \\ U_8(\theta, \tau) &= \frac{-1}{\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} (\tanh_{\vartheta}(\sqrt{\Delta}\varrho) \pm \sqrt{rs} \operatorname{sech}_{\vartheta}(\sqrt{\Delta}\varrho)), \\ U_9(\theta, \tau) &= \frac{-1}{\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} (\coth_{\vartheta}(\sqrt{\Delta}\varrho) \pm \sqrt{rs} \operatorname{csch}_{\vartheta}(\sqrt{\Delta}\varrho)), \\ U_{10}(\theta, \tau) &= \frac{-1}{2\delta\gamma_3 \ln(\vartheta)} \sqrt{\frac{\Delta(12\gamma_3^2 + 2m_1 + m_2)}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left(\tanh_{\vartheta}\left(\frac{\sqrt{\Delta}}{4}\varrho\right) + \coth_{\vartheta}\left(\frac{\sqrt{\Delta}}{4}\varrho\right)\right). \end{aligned}$$

3: If  $\gamma_3\gamma_1 > 0$  and  $\gamma_2 = 0$ , then

$$\begin{aligned}
 U_{11}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + 2\sqrt{\frac{\gamma_1}{\gamma_3}} \tan_{\vartheta}(\sqrt{\gamma_3\gamma_1}\varrho) \right), \\
 U_{12}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2\sqrt{\frac{\gamma_1}{\gamma_3}} \cot_{\vartheta}(\sqrt{\gamma_3\gamma_1}\varrho) \right), \\
 U_{13}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + 2\sqrt{\frac{\gamma_1}{\gamma_3}} (\tan_{\vartheta}(2\sqrt{\gamma_3\gamma_1}\varrho) \pm \sqrt{rs} \sec_{\vartheta}(2\sqrt{\gamma_3\gamma_1}\varrho)) \right), \\
 U_{14}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + 2\sqrt{\frac{\gamma_1}{\gamma_3}} (-\cot_{\vartheta}(2\sqrt{\gamma_3\gamma_1}\varrho) \pm \sqrt{rs} \csc_{\vartheta}(2\sqrt{\gamma_3\gamma_1}\varrho)) \right), \\
 U_{15}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + \sqrt{\frac{\gamma_1}{\gamma_3}} (\tan_{\vartheta}(\frac{\sqrt{\gamma_3\gamma_1}}{2}\varrho) - \cot_{\vartheta}(\frac{\sqrt{\gamma_3\gamma_1}}{2}\varrho)) \right).
 \end{aligned}$$

4: If  $\gamma_3\gamma_1 < 0$  and  $\gamma_2 = 0$ , then

$$\begin{aligned}
 U_{16}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2\sqrt{-\frac{\gamma_1}{\gamma_3}} \tanh_{\vartheta}(\sqrt{-\gamma_3\gamma_1}\varrho) \right), \\
 U_{17}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2\sqrt{-\frac{\gamma_1}{\gamma_3}} \coth_{\vartheta}(\sqrt{-\gamma_3\gamma_1}\varrho) \right), \\
 U_{18}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2\sqrt{-\frac{\gamma_1}{\gamma_3}} (\tanh_{\vartheta}(2\sqrt{-\gamma_3\gamma_1}\varrho) \pm \sqrt{rs} \operatorname{sech}_{\vartheta}(2\sqrt{-\gamma_3\gamma_1}\varrho)) \right), \\
 U_{19}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2\sqrt{-\frac{\gamma_1}{\gamma_3}} (\coth_{\vartheta}(2\sqrt{-\gamma_3\gamma_1}\varrho) \pm \sqrt{rs} \operatorname{csch}_{\vartheta}(2\sqrt{-\gamma_3\gamma_1}\varrho)) \right), \\
 U_{20}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - \sqrt{-\frac{\gamma_1}{\gamma_3}} (\tanh_{\vartheta}(\frac{\sqrt{-\gamma_3\gamma_1}}{2}\varrho) + \coth_{\vartheta}(\frac{\sqrt{-\gamma_3\gamma_1}}{2}\varrho)) \right).
 \end{aligned}$$

5: If  $\gamma_2 = 0$  and  $\gamma_3 = \gamma_1$ , then

$$\begin{aligned}
 U_{21}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + 2 \tan_{\vartheta}(\gamma_1\varrho) \right), \\
 U_{22}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2 \cot_{\vartheta}(\gamma_1\varrho) \right), \\
 U_{23}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + 2 \tan_{\vartheta}(2\gamma_1\varrho) \pm 2\sqrt{rs} \sec_{\vartheta}(2\gamma_1\varrho) \right), \\
 U_{24}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2 \cot_{\vartheta}(2\gamma_1\varrho) \pm 2\sqrt{rs} \csc_{\vartheta}(2\gamma_1\varrho) \right), \\
 U_{25}(\theta, \tau) &= \frac{1}{\delta \ln(\vartheta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + \tan_{\vartheta}(\frac{\gamma_1}{2}\varrho) - \cot_{\vartheta}(\frac{\gamma_1}{2}\varrho) \right).
 \end{aligned}$$

6: If  $\gamma_2 = 0$  and  $\gamma_3 = -\gamma_1$ , then

$$\begin{aligned}
 U_{26}(\theta, \tau) &= \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2 \tanh_{\theta}(\gamma_1\varrho) \right), \\
 U_{27}(\theta, \tau) &= \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2 \coth_{\theta}(\gamma_1\varrho) \right), \\
 U_{28}(\theta, \tau) &= \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + 2(-\tanh_{\theta}(2\gamma_1\varrho) \pm \iota\sqrt{rs} \operatorname{sech}_{\theta}(2\gamma_1\varrho)) \right), \\
 U_{29}(\theta, \tau) &= \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + 2(-\coth_{\theta}(2\gamma_1\varrho) \pm \sqrt{rs} \operatorname{csch}_{\theta}(2\gamma_1\varrho)) \right), \\
 U_{30}(\theta, \tau) &= \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - 2\left(\tanh_{\theta}\left(\frac{\gamma_1}{2}\varrho\right) + \coth_{\theta}\left(\frac{\gamma_1}{2}\varrho\right)\right) \right).
 \end{aligned}$$

7: If  $\gamma_2^2 = 4\gamma_3\gamma_1$ , then

$$U_{31}(\theta, \tau) = \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - \frac{4\gamma_1(\gamma_2\varrho \ln(\theta) + 2)}{\gamma_2^2\varrho \ln(\theta)} \right).$$

8: If  $\gamma_2 = \lambda$ ,  $\gamma_1 = p\lambda$  ( $p \neq 0$ ) and  $\gamma_3 = 0$ , then

$$U_{32}(\theta, \tau) = \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + 2\theta^{\lambda\varrho} - 2p \right).$$

9: If  $\gamma_2 = \gamma_3 = 0$ , then

$$U_{33}(\theta, \tau) = \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + 2\gamma_1\varrho \ln(\theta) \right).$$

10: If  $\gamma_2 = \gamma_1 = 0$ , then

$$U_{34}(\theta, \tau) = \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - \frac{2}{\gamma_3\varrho \ln(\theta)} \right).$$

11: If  $\gamma_1 = 0$  and  $\gamma_2 \neq 0$ , then

$$U_{35}(\theta, \tau) = \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - \frac{2r\gamma_2}{\gamma_3(\cosh_{\theta}(\gamma_2\varrho) - \sinh_{\theta}(\gamma_2\varrho) + r)} \right),$$

$$U_{36}(\theta, \tau) = \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} - \frac{2\gamma_2(\sinh_{\theta}(\gamma_2\varrho) + 2\cosh_{\theta}(\gamma_2\varrho))}{\gamma_3(\sinh_{\theta}(\gamma_2\varrho) + \cosh_{\theta}(\gamma_2\varrho) + s)} \right).$$

12: If  $\gamma_2 = \lambda$ ,  $\gamma_3 = p\lambda$  ( $p \neq 0$ ) and  $\gamma_1 = 0$ , then

$$U_{37}(\theta, \tau) = \frac{1}{\delta \ln(\theta)} \sqrt{\frac{12\gamma_3^2 + 2m_1 + m_2}{l_2(16\gamma_1^2\gamma_3^2 - 8\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4)}} \left( \frac{\gamma_2}{\gamma_3} + \frac{2r\theta^{\lambda\varrho}}{s - pr\theta^{\lambda\varrho}} \right).$$

#### 4.5. Graphical Behaviour of Wave Patterns

Here, we represent the physical importance of the considered model, some 3D, 2D, and density diagrams of acquired results that are plotted by using Mathematica under the suitable choice of involving parameters values. Figure 1 represent the 3D and 2D graph  $U_1(\theta, \tau)$  for  $l_1 = 1.5$ ,  $l_2 = 0.75$ ,  $\delta = 0.5$ ,  $\gamma_1 = 0.50$ ,  $\gamma_2 = 0.50$ ,  $\gamma_3 = 0.75$ . We have shown

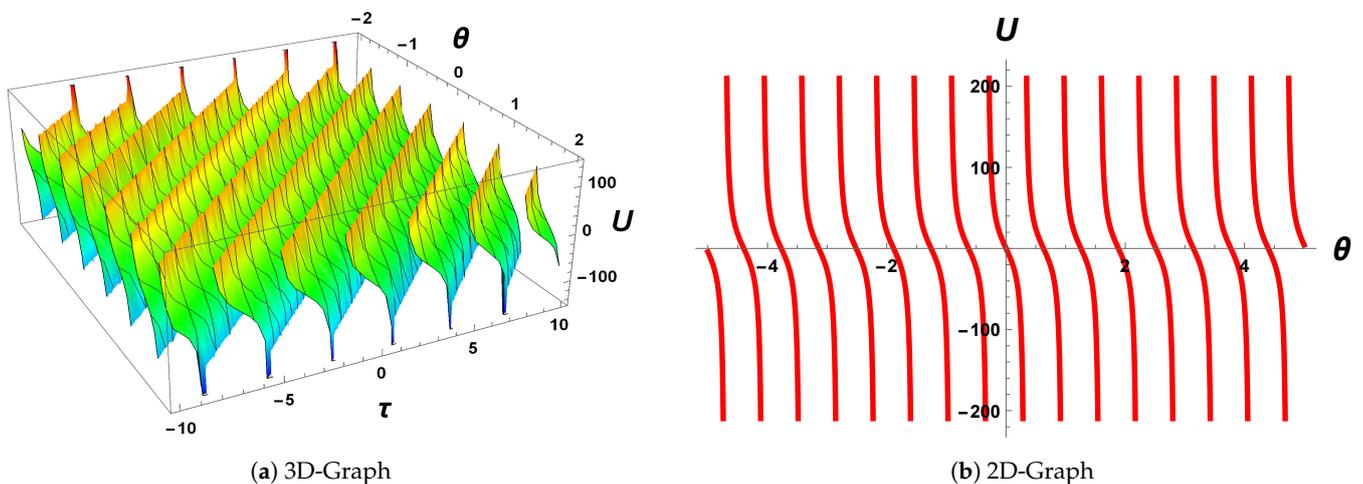
the graphical structure of  $U_2(\theta, \tau)$  for  $l_1 = 0.25, l_2 = 0.50, \delta = 0.25, \gamma_1 = 1.5, \gamma_2 = 0.10,$  and  $\gamma_3 = 1.5$  in Figure 2.

3D and 2D graphical behaviour of  $U_4(\theta, \tau)$  for  $l_1 = 1.5, l_2 = 0.75, \delta = 2, \gamma_1 = 2, \gamma_2 = 0.5,$  and  $\gamma_3 = 1.7$  is shown in Figure 3 Graphical representation of  $U_5(\theta, \tau)$  for  $l_1 = 1.5, l_2 = 0.75, \delta = 0.5, \gamma_1 = 0.50, \gamma_2 = 0.50,$  and  $\gamma_3 = 0.75$  is presented in Figure 4. Furthermore, 2D and 3D diagrams of  $U_6(\theta, \tau)$  for  $l_1 = 0.25, l_2 = 0.50, \delta = 0.25, \gamma_1 = 1.5, \gamma_2 = 0.10,$  and  $\gamma_3 = 1.5$  are shown in Figure 5.

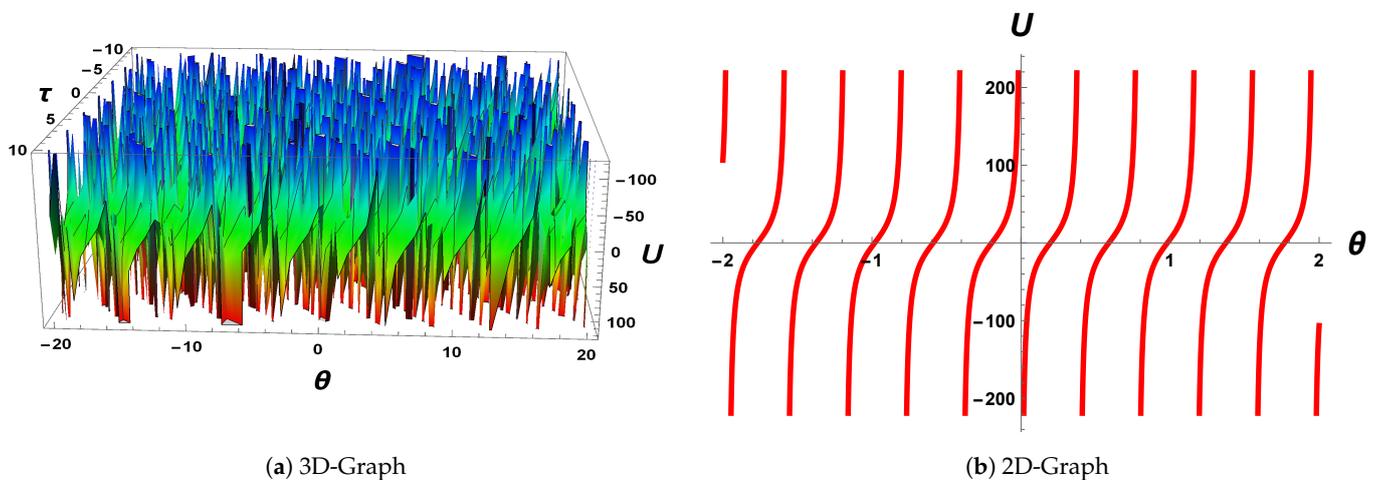
In Figure 6, we represent the 3D and 2D diagrams of  $U_7(\theta, \tau)$  for  $l_1 = 1.5, l_2 = 0.75, \delta = 2, \gamma_1 = 2, \gamma_2 = 0.5, \gamma_3 = 1.7.$  3D and 2D graphical representations of  $U_8(\theta, \tau)$  for  $l_1 = 0.2, l_2 = 0.1, \delta = 1.5, \gamma_1 = 0.5, \gamma_2 = 0.15, \gamma_3 = 1.7$  are shown in Figure 7.

Figure 8 show the 3D and 2D diagrams of  $U_9(\theta, \tau)$  for  $l_1 = 0.2, l_2 = 0.1, \delta = 1.5, \gamma_1 = 0.5, \gamma_2 = 0.15,$  and  $\gamma_3 = 1.7.$  We represent the 3D and 2D diagrams of  $U_{19}(\theta, \tau)$  for  $l_1 = 1.5, l_2 = 0.75, \delta = 0.1, \gamma_1 = 0.1, \beta = 0.15, \gamma_3 = 1.2$  in Figure 9.

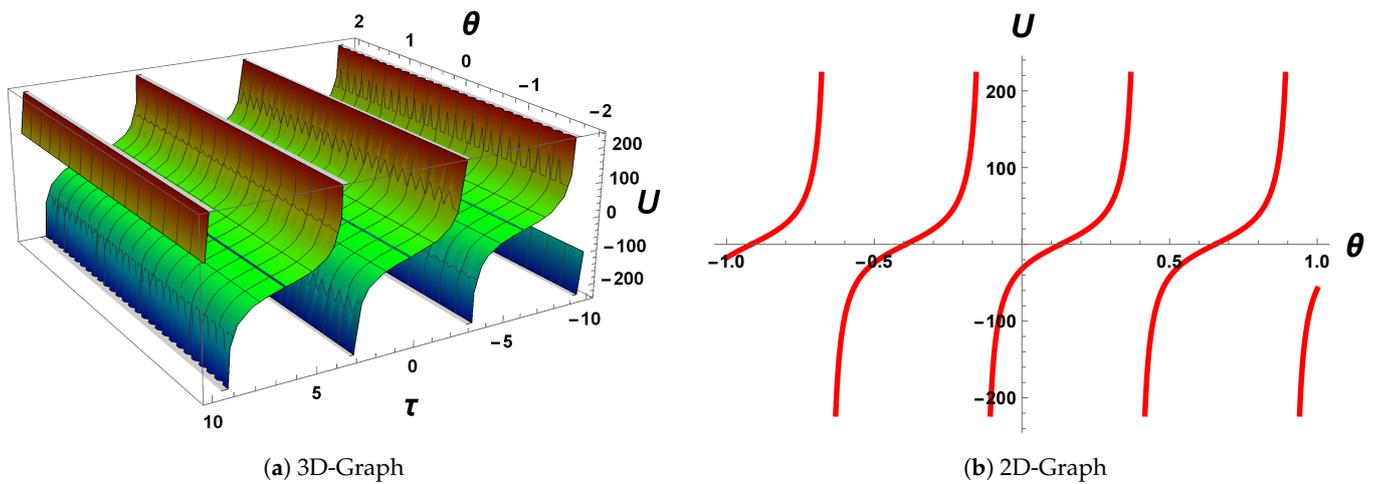
Moreover, In Figures 10 and 11, we have represented the 2D comparison graphical behaviour of choosing different values of  $U_8(\theta, \tau)$  for  $l_1 = 0.5, l_1 = 1.0,$  and  $l_1 = 1.5$  and  $l_2 = 0.5, l_2 = 1.0,$  and  $l_2 = 1.5.$



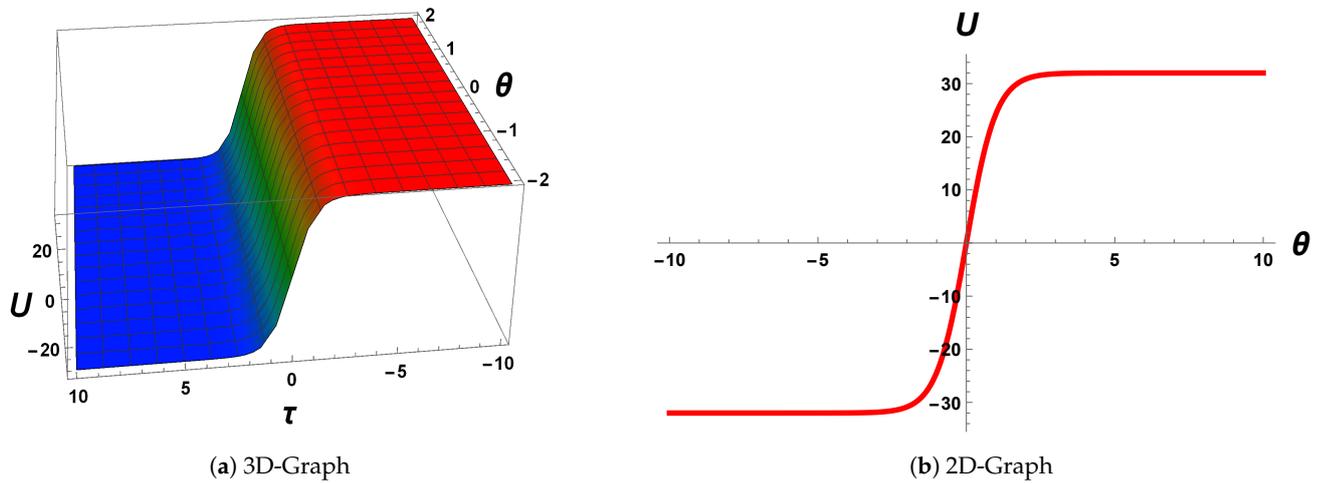
**Figure 1.** Diagrams a – 3D and b – 2D representation of  $U_1(\theta, \tau)$  for  $l_1 = 1.5, l_2 = 0.75, \delta = 0.50, \gamma_1 = 0.5, \gamma_2 = 0.50,$  and  $\gamma_3 = 0.75.$



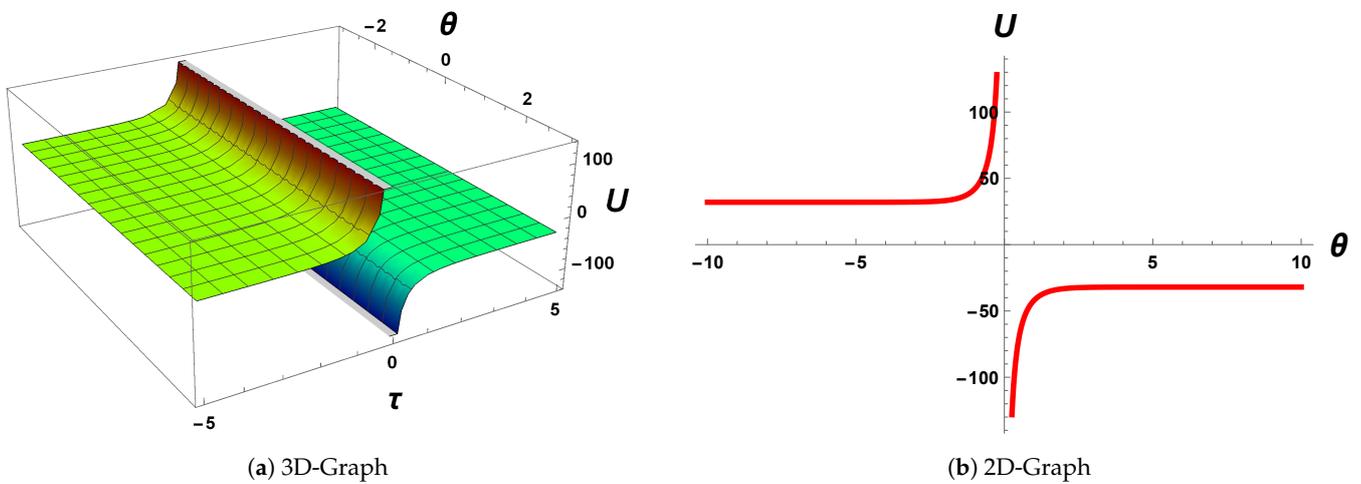
**Figure 2.** Diagrams a – 3D and b – 2D representation of  $U_2(\theta, \tau)$  for  $l_1 = 0.25, l_2 = 0.50, \delta = 0.25, \gamma_1 = 1.5, \gamma_2 = 0.10,$  and  $\gamma_3 = 1.5.$



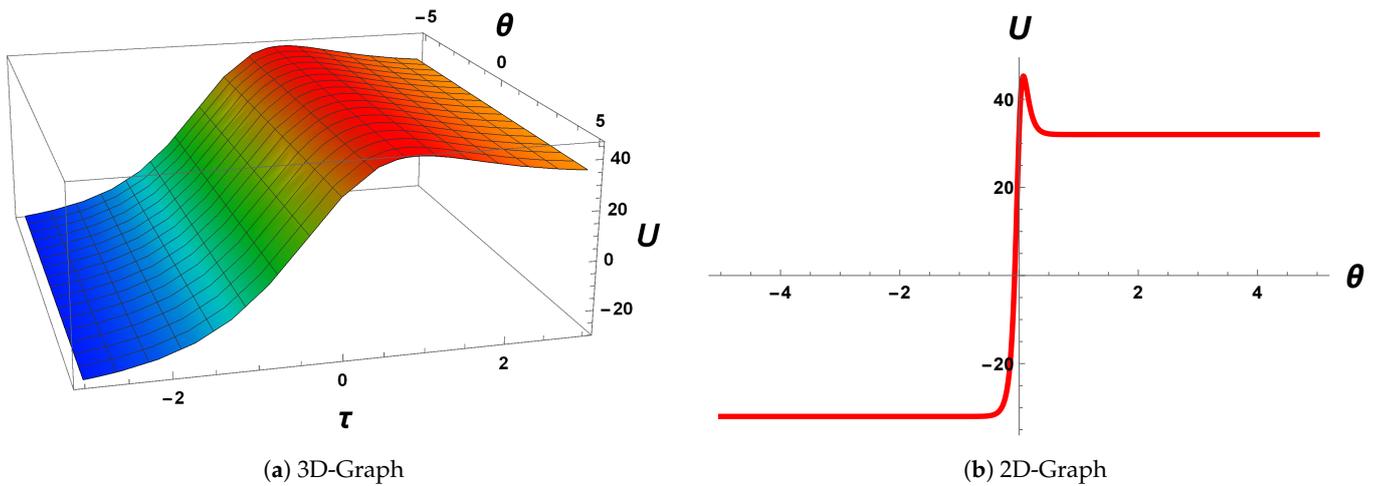
**Figure 3.** Diagrams *a* – 3D and *b* – 2D representation of  $U_4(\theta, \tau)$  for  $l_1 = 1.5, l_2 = 0.75, \delta = 2, \gamma_1 = 2, \gamma_2 = 0.5,$  and  $\gamma_3 = 1.7.$



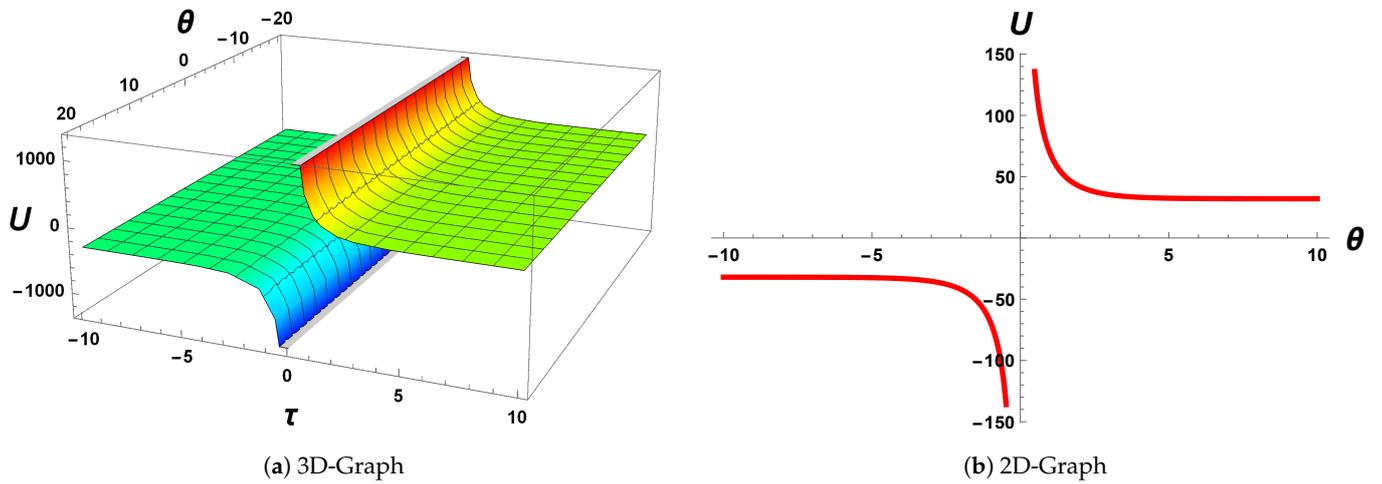
**Figure 4.** Diagrams *a* – 3D and *b* – 2D representation of  $U_5(\theta, \tau)$  for  $l_1 = 1.5, l_2 = 0.75, \delta = 0.5, \gamma_1 = 0.50, \gamma_2 = 0.50,$  and  $\gamma_3 = 0.75.$



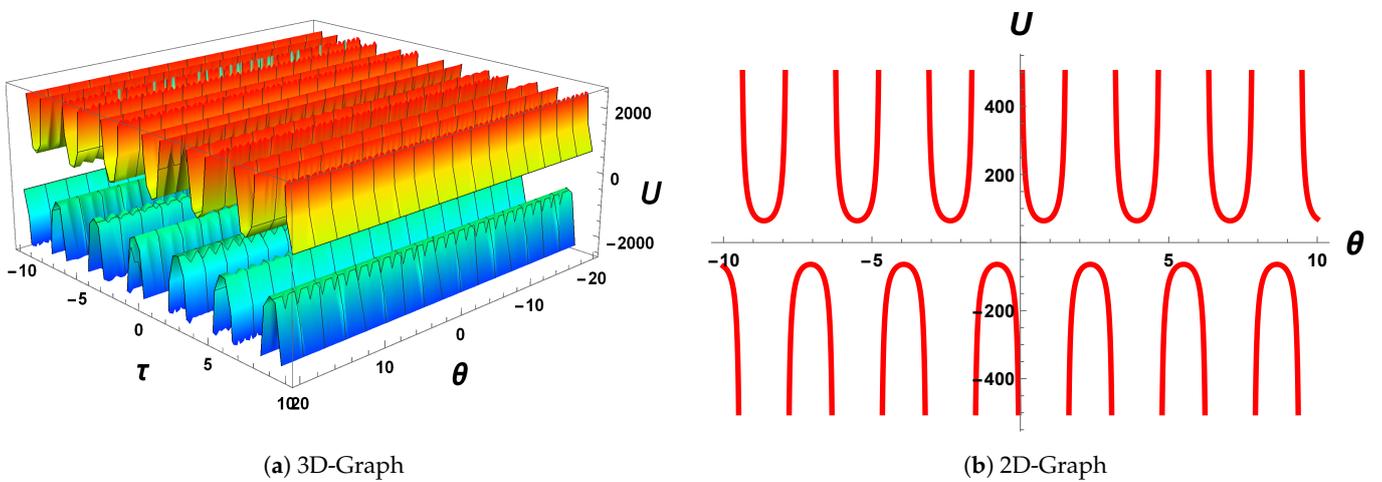
**Figure 5.** Diagrams *a* – 3D and *b* – 2D representation of  $U_6(\theta, \tau)$  for  $l_1 = 0.25, l_2 = 0.50, \delta = 0.25, \gamma_1 = 1.5, \gamma_2 = 0.10,$  and  $\gamma_3 = 1.5.$



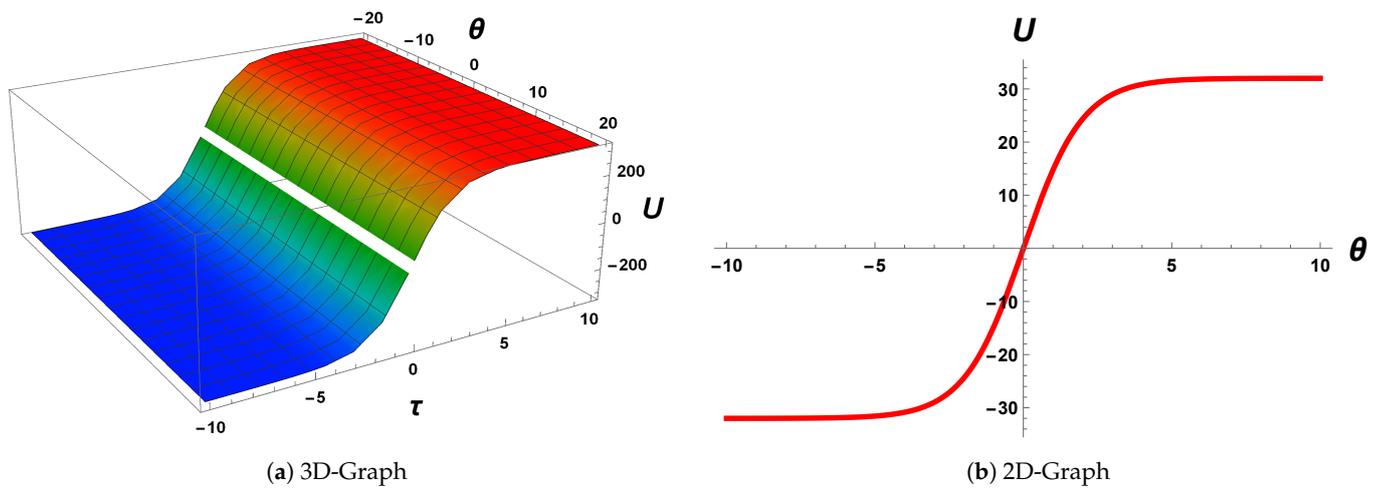
**Figure 6.** Diagrams *a* – 3D and *b* – 2D representation of  $U_7(\theta, \tau)$  for  $l_1 = 1.5, l_2 = 0.75, \delta = 2, \gamma_1 = 2, \gamma_2 = 0.5,$  and  $\gamma_3 = 1.7$ .



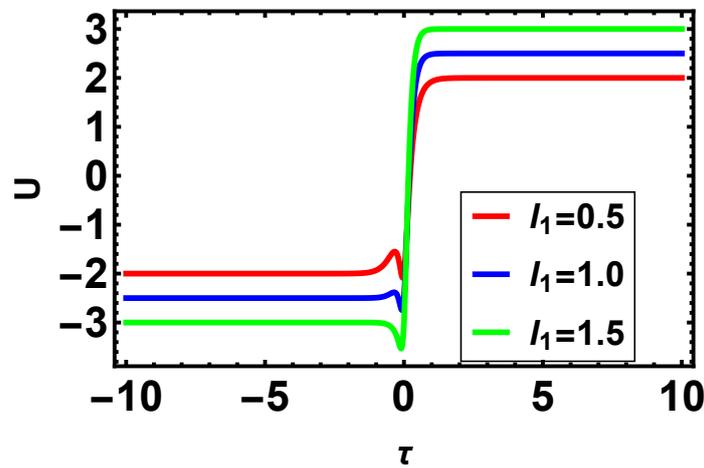
**Figure 7.** Diagrams *a* – 3D and *b* – 2D representation of  $U_8(\theta, \tau)$  for  $l_1 = 0.2, l_2 = 0.1, \delta = 1.5, \gamma_1 = 0.5, \gamma_2 = 0.15,$  and  $\gamma_3 = 1.7$ .



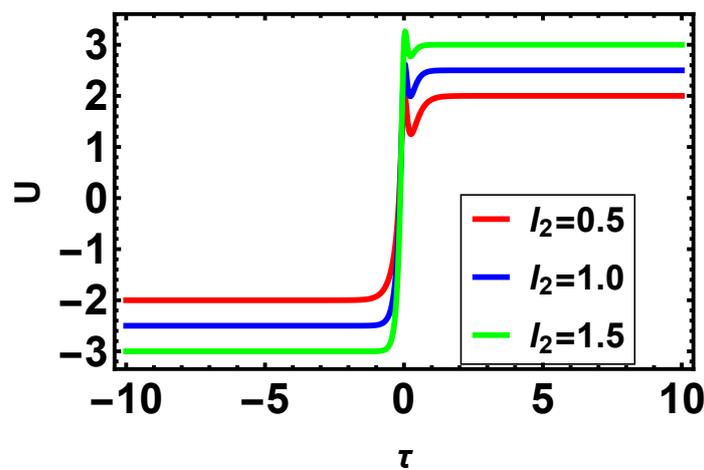
**Figure 8.** Diagrams *a* – 3D and *b* – 2D show graphically  $U_9(\theta, \tau)$  for  $l_1 = 0.2, l_2 = 0.1, \delta = 1.5, \gamma_1 = 0.5, \gamma_2 = 0.15,$  and  $\gamma_3 = 1.7$ .



**Figure 9.** Diagrams *a* – 3D and *b* – 2D representation of  $U_{19}(\theta, \tau)$  for  $l_1 = 1, l_1 = 1.5, l_2 = 0.75, \delta = 0.1, \gamma_1 = 0.1, \gamma_2 = 0.15,$  and  $\gamma_3 = 1.2.$



**Figure 10.** Comparison of different values if  $l_1.$



**Figure 11.** Comparison of different values if  $l_2.$

### 5. Conservation Laws

Here the multiplier technique is used to construct the conservation laws for (1). From Equation (24), we obtain the determining equation for lambda  $\Lambda(\theta, \tau, U)$ :

$$\frac{\delta}{\delta U} \left[ \Lambda(\theta, \tau, U)(U_{\tau\tau} - l_1(U^2)_{\tau\tau} + l_2(U^3)_{\tau\tau} - \delta^2 U_{\theta\theta} - \frac{\delta^4}{12} U_{\theta\theta\theta\theta} = 0) \right] = 0. \tag{42}$$

From Equation (21), the Euler operator is defined as

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_\tau \frac{\partial}{\partial U_\tau} - D_\theta \frac{\partial}{\partial U_\theta} + D_\tau^2 \frac{\partial}{\partial U_{\tau\tau}} + D_\theta^2 \frac{\partial}{\partial U_{\theta\theta}} + D_\theta D_\tau \frac{\partial}{\partial U_{\tau\theta}} + \dots, \tag{43}$$

total derivative operator  $D_\tau$  and  $D_\theta$  can be written as with help of Equation (20):

$$\begin{aligned} D_\theta &= \frac{\partial}{\partial \theta} + U_\theta \frac{\partial}{\partial U} + U_{\theta\theta} \frac{\partial}{\partial U_\theta} + U_{\tau\theta} \frac{\partial}{\partial U_\tau} \dots, \\ D_\tau &= \frac{\partial}{\partial \tau} + U_\tau \frac{\partial}{\partial U} + U_{\tau\tau} \frac{\partial}{\partial U_\tau} + U_{\tau\theta} \frac{\partial}{\partial U_\theta} \dots, \end{aligned} \tag{44}$$

solving Equation (42), and we obtain the eight multipliers:

$$\begin{aligned} \Lambda^{(1)}(\theta, \tau, U) &= \tau, & \Lambda^{(2)}(\theta, \tau, U) &= \tau\theta, \\ \Lambda^{(3)}(\theta, \tau, U) &= \tau \sin\left(\frac{2\sqrt{3}\theta}{\delta}\right), & \Lambda^{(4)}(\theta, \tau, U) &= \tau \cos\left(\frac{2\sqrt{3}\theta}{\delta}\right), \\ \Lambda^{(5)}(\theta, \tau, U) &= 1, & \Lambda^{(6)}(\theta, \tau, U) &= \theta, \\ \Lambda^{(7)}(\theta, \tau, U) &= \sin\left(\frac{2\sqrt{3}\theta}{\delta}\right), & \Lambda^{(8)}(\theta, \tau, U) &= \cos\left(\frac{2\sqrt{3}\theta}{\delta}\right) \end{aligned} \tag{45}$$

Equations (23) and (45) give us conservation laws. Two components,  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$ , of the conservation laws for Equation (1).

**Case 1.** Conserved vectors relating to

$$\Lambda^{(1)}(\theta, \tau, U) = \tau,$$

can be written as:

$$\begin{aligned} \mathfrak{F}_\tau^{(1)} &= 3l_2\tau U^2 U_\tau - 2l_1\tau U U_\tau - l_2 U^3 + l_1 U^2 + \tau U_\tau - U, \\ \mathfrak{F}_\theta^{(1)} &= -\tau\delta^2 U_\theta - \frac{1}{12}\tau\delta^4 U_{\theta\theta\theta}. \end{aligned} \tag{46}$$

**Case 2.** The conservation laws relating to

$$\Lambda^{(2)}(\theta, \tau, U) = \tau\theta,$$

can be written as:

$$\begin{aligned} \mathfrak{F}_\tau^{(2)} &= 3l_2\tau\theta U^2 U_\tau - 2l_1\tau\theta U U_\tau - l_2\theta U^3 + l_1\theta U^2 + \tau\theta U_\tau - \theta U, \\ \mathfrak{F}_\theta^{(2)} &= \tau\delta^2 U - \tau\theta\delta^2 U_\theta + \frac{1}{12}\tau\delta^4 U_{\theta\theta} - \frac{1}{12}\tau\theta\delta^4 U_{\theta\theta\theta}. \end{aligned} \tag{47}$$

**Case 3.** The conservation laws relating to

$$\Lambda^{(3)}(\theta, \tau, U) = \tau \sin\left(\frac{2\sqrt{3}\theta}{\delta}\right),$$

can be written as:

$$\begin{aligned}\mathfrak{F}_\tau^{(3)} &= -\sin\left(\frac{2\sqrt{3}\theta}{\delta}\right)(2l_1\tau UU_\tau - 3l_2\tau U^2U_\tau + l_2U^3 - l_1U^2 - \tau U_\tau + U), \\ \mathfrak{F}_\theta^{(3)} &= \frac{1}{12}\tau\delta^3\left[2\sqrt{3}U_{\theta\theta}\cos\left(\frac{2\sqrt{3}\theta}{\delta}\right) - 2\delta U_{\theta\theta\theta}\sin\left(\frac{2\sqrt{3}\theta}{\delta}\right)\right].\end{aligned}\quad (48)$$

**Case 4.** The conservation laws relating to

$$\Lambda^{(4)}(\theta, \tau, U) = \tau \cos\left(\frac{2\sqrt{3}\theta}{\delta}\right),$$

can be written as:

$$\begin{aligned}\mathfrak{F}_\tau^{(4)} &= -\cos\left(\frac{2\sqrt{3}\theta}{\delta}\right)(2l_1\tau UU_\tau - 3l_2\tau U^2U_\tau + l_2U^3 - l_1U^2 - \tau U_\tau + U), \\ \mathfrak{F}_\theta^{(4)} &= \frac{-1}{12}\tau\delta^3\left[2\sqrt{3}U_{\theta\theta}\sin\left(\frac{2\sqrt{3}\theta}{\delta}\right) + \delta U_{\theta\theta\theta}\cos\left(\frac{2\sqrt{3}\theta}{\delta}\right)\right].\end{aligned}\quad (49)$$

**Case 5.** The conservation laws relating to

$$\Lambda^{(5)}(\theta, \tau, U) = 1,$$

can be written as:

$$\begin{aligned}\mathfrak{F}_\tau^{(5)} &= 3l_2U^2U_\tau - 2l_1UU_\tau + U_\tau, \\ \mathfrak{F}_\theta^{(5)} &= -\delta^2U_\theta - \frac{1}{12}\delta^4U_{\theta\theta\theta}.\end{aligned}\quad (50)$$

**Case 6.** The conservation laws relating to

$$\Lambda^{(6)}(\theta, \tau, U) = \theta,$$

can be written as:

$$\begin{aligned}\mathfrak{F}_\tau^{(6)} &= 3l_2\theta U^2U_\tau - 2l_1\theta UU_\tau + \theta U_\tau, \\ \mathfrak{F}_\theta^{(6)} &= \delta^2U - \theta\delta^2U_\theta + \frac{1}{12}\delta^4U_{\theta\theta} - \frac{1}{12}\delta^4\theta U_{\theta\theta\theta}.\end{aligned}\quad (51)$$

**Case 7.** The conservation laws relating to

$$\Lambda^{(7)}(\theta, \tau, U) = \sin\left(\frac{2\sqrt{3}\theta}{\delta}\right),$$

can be written as:

$$\begin{aligned}\mathfrak{F}_\tau^{(7)} &= -U_\tau \sin\left(\frac{2\sqrt{3}\theta}{\delta}\right)(-3l_2U^2 + 2l_1U - 1), \\ \mathfrak{F}_\theta^{(7)} &= \frac{1}{12}\delta^3\left[2\sqrt{3}U_{\theta\theta}\cos\left(\frac{2\sqrt{3}\theta}{\delta}\right) - \delta U_{\theta\theta\theta}\sin\left(\frac{2\sqrt{3}\theta}{\delta}\right)\right].\end{aligned}\quad (52)$$

**Case 8.** The conservation laws relating to

$$\Lambda^{(8)}(\theta, \tau, U) = \cos\left(\frac{2\sqrt{3}\theta}{\delta}\right),$$

can be written as:

$$\begin{aligned}\mathfrak{F}_{\tau}^{(8)} &= -U_{\tau} \cos\left(\frac{2\sqrt{3}\theta}{\delta}\right) (-3l_2U^2 + 2l_1U - 1), \\ \mathfrak{F}_{\theta}^{(8)} &= \frac{-1}{12}\delta^3 \left[ \delta U_{\theta\theta\theta} \cos\left(\frac{2\sqrt{3}\theta}{\delta}\right) + 2\sqrt{3}U_{\theta\theta} \sin\left(\frac{2\sqrt{3}\theta}{\delta}\right) \right].\end{aligned}\quad (53)$$

## 6. Conclusions

In this work, we have studied the nonlinear model depicting the wave proliferation in NLETs employing integration scheme modified Khater method. NLETs have been discussed by the Lie analysis approach. This method is employed to analyse the NLETs and construct the infinitesimal generators. We found the entire vector field and discuss the different solutions of the NLETs. We can see that generated vector field forms an abelian Lie algebra. The assumed NLETs were converted into nonlinear ODE by using an optimal system. Then, we explored the wave solutions with the help of the integration technique, namely the MKM, to solve the nonlinear partial differential depicting the wave proliferation in the NLETs. We have used a MKM to get the new wave solutions. These results are in the form of trigonometric and hyperbolic functions. The benefits of the proposed technique are that they are simple, direct, reliable, and it addresses its wide-range appropriateness. It is hoped that our proposed technique is much better and can be used for the many types of NLPDE's. To represent the physical importance of the considered model, some 3D, 2D, and density diagrams of acquired results are plotted by using Mathematica under the suitable choice of involving parameters values. We have shown some comparison graphical behaviour of the solution for taking different values of  $l_1$  and  $l_2$ . Further, the conservation laws of NLETs are computed by the multiplier method.

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