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Article

(k, ψ)-Proportional Fractional Integral Pólya–Szegö- and Grüss-Type Inequalities

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Abstract: The purpose of this research was to discover a novel method to recover k-fractional integral inequalities of the Pólya–Szegö-type. We employ these generalized inequalities to investigate some new fractional integral inequalities of the Grüss-type. More precisely, we generalize the proportional fractional operators with respect to another strictly increasing continuous function ψ. Then, we state and prove some of its properties and special cases. With the help of this generalized operator, we investigate some Pólya–Szegö- and Grüss-type fractional integral inequalities. The functions used in this work are bounded by two positive functions to obtain Pólya–Szegö- and Grüss-type k-fractional integral inequalities in a new sense. Moreover, we discuss some new special cases of the Pólya–Szegö- and Grüss-type inequalities through this work.

Keywords: Pólya–Szegö inequalities; Grüss inequalities; fractional inequalities; ψ-proportional fractional operators

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1. Introduction

The topic of fractional calculus, which is concerned with integral and differential operators of noninteger orders, is just about as old as the conventional calculus that transacts with integer orders. Because not all real-world events can be represented using standard calculus operators, researchers and scientists sought extensions of these operators. As of now, the field of fractional calculus is of significant interest for numerous researchers. There has been an abundance of literature and research, especially on conventional fractional calculus, such as the Riemann–Liouville (RL) and Caputo definitions. The RL derivative is the most often used definition and, in some ways, the most natural; nevertheless, it has significant limitations when used to simulate physical problems, due to the fact that the necessary initial conditions are fractional, which may be inappropriate for physical conditions. The Caputo derivative enjoys the benefit of being appropriate for physical conditions since it needs only initial conditions in the classical type [1].

This notwithstanding, these are far from the only ways to characterize and define fractional calculus. There are many definitions of fractional operators, including RL, Caputo, Riesz, Hilfer, Hadamard, Katugampola, and Erdélyi–Kober [2,3]. Each definition is subject to its own set of rules and conditions, which explains why a significant number of these definitions are not identical. Practically, the physical framework under consideration decides the determination of an appropriate fractional operator. As a consequence of this, researchers have introduced the various definitions of inequivalent fractional operators, every one of which is helpful in its own specific context. Therefore, it is sensible that we ought to explore and develop fractional operators that are the generalized categories of the current, specific cases. Certain very wide fractional operators may be called classes, since they include a large number of different fractional operators of acceptable values. This approach is more effective for developing theoretical analysis in this area than repeatedly eliciting the same proofs in many distinct, but similar fractional calculus models [4].

One such wide category is the class of fractional operators with analytic kernels, which was proposed in [5] as a convenient way of generalizing a variety of different types of fractional calculus that have been intensively studied in recent years. We refer to one model within this class, which will be a major topic of this work, called extended proportional fractional calculus [6,7].

Another general class of fractional operators has been referred to as ψ-fractional calculus at times. Erdélyi first proposed the idea in a 1964 paper [8], in which he discussed fractional integration with regard to a power function x^n. In 2011, Katugampola [9] presented a comparable operator to Erdélyi's operator, which has been known as Katugampola fractional calculus in recent years. Indeed, Osler inspired and developed this category in 1970 [10], and it was briefly mentioned by Oldham and Spanier [11], despite the fact that it is often cited instead in the book by Kilbas et al. [3], which delves further into the subject. For some works on ψ-fractional calculus, see [12,13].

In 2007, Daz and Pariguan [14] gave a k-gamma function $\Gamma_k(\cdot)$, which is the generalization of the Euler gamma function $\Gamma(\cdot)$, and for $k \to 1$, we obtain $\Gamma_k(\cdot) \to \Gamma(\cdot)$. Consequently, many definitions of fractional operators rely on $\Gamma(\cdot)$.

From the fact that $\Gamma_k(\cdot)$ is the natural generalization of $\Gamma(\cdot)$, it is natural to anticipate new concepts of fractional operators with the additional parameter k, and this has already been done by merging this parameter with many fractional operators and its generalizations; for more details, see the recent works [15–22].

On the other hand, in mathematical science, mathematical inequalities have great importance due to their useful applications, especially in classical differential equations and integrals; consequentially, only a few decades ago, many useful and noteworthy mathematical inequalities were investigated by numerous authors. One inequality, which has a well-known space in inequality theory, is the Chebyshev inequality; this inequality helps to build new inequalities of several different types and generates limit values for synchronous functions. The grounds of this inequality type lie in the following functional (Chebyshev (1882) [23]):

$$W(\eta,\varsigma) := \frac{1}{b-a}\int_a^b \eta(\omega)\varsigma(\omega)d\omega - \frac{1}{b-a}\left(\int_a^b \eta(\omega)d\omega\right)\left(\int_a^b \varsigma(\omega)d\omega\right), \tag{1}$$

where η and ς are two integrable functions on $[a,b]$. A number of investigators have given great attention to this functional, and several inequalities, generalizations, and extensions have appeared in the literature; for more details, see [24–26].

For η and ς, two differentiable functions, Dragomir (2000) [27] proved the following inequality for any $\mu,\sigma \in [a,b]$ (see also [28]):

$$2|W(\eta,\varsigma,\phi)| \le \|\eta'\|_{\varpi}\|\varsigma'\|_{\varepsilon}\left[\int_a^b |\mu-\sigma|\phi(\mu)\phi(\sigma)d\mu d\sigma\right], \tag{2}$$

where $\eta' \in L_{\varpi}(a,b)$, $\varsigma' \in L_{\varepsilon}(a,b)$, $\varpi > 1$, $\frac{1}{\varpi}+\frac{1}{\varepsilon}=1$.

Dahmani (2010) [29], for all $\gamma > 0$, $\omega > 0$, proved the following fractional version of Inequality (2) (see also [30,31]):

$$\begin{aligned} &2|I^\gamma\phi(\omega)I^\gamma(\phi\eta\varsigma)(\omega) - I^\gamma(\phi\eta)(\omega)I^\gamma(\phi\varsigma)(\omega)| \\ &\le \frac{\|\eta'\|_{\varpi}\|\varsigma'\|_{\varepsilon}}{\Gamma^2(\gamma)}\int_a^b\int_a^b(\omega-\mu)^{\gamma-1}(\omega-\sigma)^{\gamma-1}|\mu-\sigma|\phi(\mu)\phi(\sigma)d\mu d\sigma \\ &\le \|\eta'\|_{\varpi}\|\varsigma'\|_{\varepsilon}\omega(I^\gamma\phi(\omega))^2. \end{aligned} \tag{3}$$

One of the most important inequalities that is of great interest to the authors is the well-known Grüss inequality [32] (see also [33]):

$$\left|\frac{1}{b-a}\int_a^b \eta(\omega)\varsigma(\omega)d\omega - \frac{1}{(b-a)^2}\int_a^b \eta(\omega)d\omega\int_a^b \varsigma(\omega)d\omega\right| \le \frac{1}{4}(P-p)(Q-q), \quad (4)$$

where η, ς are two integrable functions on $[a,b]$, which satisfy the conditions:

$$p \le \eta(\omega) \le P, \quad q \le \varsigma(\omega) \le Q, \ \omega \in [a,b], \ p,P,q,Q \in \mathbb{R}.$$

The fractional version inequality of Inequality (4) was given by Dahmani et al. [34] in 2010 by employing the Riemann–Liouville fractional integral as follows:

$$\left|\frac{\omega^\gamma}{\Gamma(\gamma+1)}\mathcal{I}^\gamma(\eta\varsigma)(\omega) - \mathcal{I}^\gamma\eta(\omega)\mathcal{I}^\gamma\varsigma(\omega)\right| \le \left(\frac{\omega^\gamma}{\Gamma(\gamma+1)}\right)^2 (P-p)(Q-q), \quad (5)$$

for one parameter. The authors also gave the fractional integral version inequality on $[0,\infty)$ for two parameters (see [34]).

Tariboon et al. (2014) [35] replaced the bounds of the functions η and ς with four integrable functions on $[0,\infty)$, as:

$$G_1(\omega) \le \eta(\omega) \le G_2(\omega) \ \ and \ \ H_1(\omega) \le \varsigma(\omega) \le H_2(\omega),$$

and they gained the inequality:

$$\left|\frac{\omega^\gamma}{\Gamma(\gamma+1)}\mathcal{I}^\gamma(\eta\varsigma)(\omega) - \mathcal{I}^\gamma\eta(\omega)\mathcal{I}^\gamma\varsigma(\omega)\right| \le \sqrt{\xi(\eta,G_1,G_2)\xi(\eta,H_1,H_2)},$$

where $\xi(l,s,t)$ is defined as:

$$\xi(l,s,t) = (\mathcal{I}^\gamma t(x) - \mathcal{I}^\gamma l(\omega))(\mathcal{I}^\gamma l(\omega) - \mathcal{I}^\gamma s(\omega)) + \frac{\omega^\gamma}{\Gamma(\gamma+1)}\mathcal{I}^\gamma(ls)(\omega) - \mathcal{I}^\gamma l(\omega)\mathcal{I}^\gamma s(\omega) \quad (6)$$
$$+\frac{\omega^\gamma}{\Gamma(\gamma+1)}\mathcal{I}^\gamma(lt)(\omega) - \mathcal{I}^\gamma l(\omega)\mathcal{I}^\gamma t(\omega) - \frac{\omega^\gamma}{\Gamma(\gamma+1)}\mathcal{I}^\gamma(st)(\omega) + \mathcal{I}^\gamma s(\omega)\mathcal{I}^\gamma t(\omega).$$

for one parameter. The two-parameter fractional integral version of Inequality (4) for functional bounds was given by Aljaaidi and Pachpatte (2020) in [36] by using the Katugampola fractional integral; in 2020, they also presented the same inequality for functional bounds by using the ψ-Riemann–Liouville fractional integral (see [37]).

Another inequality that is beneficial for this article is the Pólya–Szegö inequality, which was introduced by Pólya and Szegő [38] in 1925 as follows:

$$\frac{\int_a^b \{\eta^2\}(\omega)d\omega \int_a^b \{\varsigma^2\}(\omega)d\omega}{\left(\int_a^b \{\eta\}(\omega)\{\varsigma\}(\omega)d\omega\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{pq}{PQ}} + \sqrt{\frac{PQ}{pq}}\right)^2. \quad (7)$$

Dragomir and Diamond [39], in 2003, by employing the Pólya–Szegö inequality with the integrable functions η and ς on $[a,b]$, which satisfy the conditions:

$$0 < p \le \eta(\mu) \le P \le \infty, \quad 0 < q \le \varsigma(\sigma) \le Q \le \infty, \quad (8)$$

proved the following Grüss-type inequality:

$$|W(\eta,\varsigma)| \le \frac{(p-P)(q-Q)}{4\sqrt{pPqQ}}\frac{1}{(b-a)^2}\int_a^b \eta(\omega)d\omega\int_a^b \varsigma(\omega)d\omega. \quad (9)$$

The fractional integral version of the Pólya–Szegö inequality was given by Amber and Dahmani [40] in 2013 as follows:

$$\frac{\mathcal{I}^{\gamma}\{\eta^2\}(\omega)\mathcal{I}^{\gamma}\{\varsigma^2\}(\omega)}{(\mathcal{I}^{\gamma}\{\eta\}(\omega)\{\varsigma\}(\omega))^2} \leq \frac{1}{4}\left(\sqrt{\frac{pq}{PQ}}+\sqrt{\frac{PQ}{pq}}\right)^2, \tag{10}$$

where η and ς are two integrable functions on $[a,b]$ satisfying the condition (8).

Ntouyas et al. [41] in 2016 investigated some new Pólya–Szegö-type integral inequalities by employing the Riemann–Liouville fractional integral. They replaced the constants that restrict the functions it uses with four new integrable functions, and they used them to investigate some factional integral inequalities of the Grüss-type. The same authors [42] in the same year presented a study to prove new Pólya–Szegö integral inequalities via the generalized k-fractional integral operator and then employed it to establish some Chebyshev fractional integral inequalities. Set et al. [43] in 2018 established some new Pólya–Szegö inequalities using the generalized Katugampola fractional integral, and they used them to investigate some new fractional Chebyshev inequalities. Nikolova and Varosanes [44] in 2018 gave two variants of Cauchy inequalities and the Pólya–Szegö-type inequalities, and they obtained a new bound for the Chebyshev-type inequalities. Rahman et al. [45] in 2020 established some new weighted fractional Chebyshev–Pólya–Szegö-type integral inequalities by using the generalized weighted fractional integral containing another function in the kernel. Recently, in 2020 and 2021, several types of research on Pólya–Szegö-type inequalities were published; among them, we refer to [46–48].

Motivated by the above discussion and inspired by [7,41], in this paper, we establish an important connection between proportional and generalized RL-type fractional calculus with respect to another strictly increasing continuous function ψ. To the best of our knowledge, this has not yet been considered until now. Specifically, we propose a more generalized version of the proportional fractional operators, the so-called (k,ψ)-proportional fractional operators. We obtain several definitions of classical fractional operators as a special case of our generalized version. Some properties of (k,ψ)-proportional fractional operators are proven and applied to develop the (k,ψ)-proportional fractional calculus. In this regard, we introduce the Pólya–Szegö inequality and then employ it to prove some new Grüss-type inequalities. Moreover, we enrich this work by discussing some special cases related to the current research paper. In the Conclusion, we list future directions for extending and proving new properties of the proposed operator and its corresponding differential operator. We apply this operator in a qualitative analysis of fractional differential equations.

This paper is organized as follows: In the Section 2, we recollect some definitions, results, notations, and introductory information used throughout this work. The Section 3 is devoted to giving the proposed fractional integral in the frame of a generalized proportional operator. The major results of Pólya–Szegö- and Grüss-type inequalities are obtained in the Section 4. In the Conclusions, we list future directions relevant to the current work.

2. Essential Preliminaries

Here, we provide some definitions, notations, and clarifications related to our new generalized k-fractional integral used throughout this work. Let us start with the following well-known Gama function, which was first introduced by Leonhard Euler, and the k-Gamma function [14].

Definition 1. *The notation:*

$$\Gamma(\gamma) = \int_0^{\infty} e^{-\omega}\omega^{\gamma-1}d\omega,$$

is called the Gamma function, while the notation:

$$\Gamma_k(\gamma) = \int_0^{\infty} e^{-\frac{\omega^k}{k}}\omega^{\gamma-1}d\omega,$$

is called the k-Gama function, where $\omega, k > 0$. The k*-Gamma function has the following relationships:*

$$(i)\ \Gamma_k(k) = 1.$$
$$(ii)\ \Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right).$$
$$(iii)\ \Gamma_k(\gamma + k) = \gamma\Gamma_k(\gamma).$$
$$(iv)\ \Gamma_k(\gamma) = \Gamma(\gamma), k \longrightarrow 1.$$

Definition 2. *The notation:*

$$\beta(\gamma, \delta) = \int_0^1 \omega^{\gamma-1}(1-\omega)^{\delta-1}d\omega, (\gamma, \delta > 0),$$

is called the Beta function. In terms of the Gamma function, it has the following relationship:

$$\beta(\gamma, \delta) = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma + \delta)}.$$

The notation:

$$\beta_k(\gamma, \delta) = \frac{1}{k}\int_0^1 \omega^{\frac{\gamma}{k}-1}(1-\omega)^{\frac{\delta}{k}-1}d\omega, (\gamma, \delta > 0),$$

is called the k-Beta function. In terms of the k-Gamma function, it has the following relationship:

$$\beta_k(\gamma, \delta) = \frac{\Gamma_k(\gamma)\Gamma_k(\delta)}{\Gamma_k(\gamma + \delta)} = \frac{1}{k}\beta\left(\frac{\gamma}{k}, \frac{\delta}{k}\right).$$

Definition 3 ([3])**.** *For the integrable function η on $[a, b]$ and $a \geq 0$, we have for all $\gamma > 0$:*

$$\mathcal{I}_{a^+}^{\gamma}\eta(\omega) = \frac{1}{\Gamma(\gamma)}\int_a^{\omega}(\omega - \mu)^{\gamma-1}\eta(\mu)d\mu, \mu > a \tag{11}$$

and:

$$\mathcal{I}_{b^-}^{\gamma}\eta(\omega) = \frac{1}{\Gamma(\gamma)}\int_{\omega}^{b}(\mu - \omega)^{\gamma-1}\eta(\mu)d\mu, \mu < b, \tag{12}$$

The notations $\mathcal{I}_{a^+}^{\gamma}\eta(\omega)$ and $\mathcal{I}_{b^-}^{\gamma}\eta(\omega)$ are called the left- and right-sided Riemann–Liouville fractional integrals of a function η, respectively.

Definition 4 ([2,3])**.** *For the integrable function η on the interval $\digamma$ and for the increasing function ψ, where $\psi(\omega) \in C^1(\digamma, \mathbb{R})$, such that $\psi'(\omega) \neq 0$, $\omega \in \digamma$, we have for all $\gamma > 0$:*

$$\left({}^{\psi}\mathcal{I}_{a^+}^{\gamma}\eta\right)(\omega) = \frac{1}{\Gamma(\gamma)}\int_a^{\omega}\psi'(\mu)[\psi(\omega) - \psi(\mu)]^{\gamma-1}\eta(\mu)d\mu \tag{13}$$

and:

$$\left({}^{\psi}\mathcal{I}_{b^-}^{\gamma}\eta\right)(\omega) = \frac{1}{\Gamma(\gamma)}\int_{\omega}^{b}\psi'(\mu)[\psi(\mu) - \psi(\omega)]^{\gamma-1}\eta(\mu)d\mu. \tag{14}$$

The notations $\left({}^{\psi}\mathcal{I}_{a^+}^{\gamma}\eta\right)(\omega)$ and $\left({}^{\psi}\mathcal{I}_{b^-}^{\gamma}\eta\right)(\omega)$ are called the left- and right-sided ψ-Riemann–Liouville fractional integrals of the function η, respectively.

Definition 5 ([21])**.** *For the integrable function η on the interval $\digamma$ and for the increasing function ψ, where $\psi(\omega) \in C^1(\digamma, \mathbb{R})$, such that $\psi'(\omega) \neq 0$, $\omega \in \digamma$, we have for all $\gamma > 0$:*

$$\left({}^{(k,\psi)}\mathcal{I}_{a^+}^{\gamma}\eta\right)(\omega) = \frac{1}{\Gamma_k(\gamma)}\int_a^{\omega}\psi'(\mu)[\psi(\omega) - \psi(\mu)]^{\frac{\gamma}{k}-1}\eta(\mu)d\mu \tag{15}$$

and:

$$\left({}^{(k,\psi)}\mathcal{I}^{\gamma}_{b^-}\eta\right)(\omega) = \frac{1}{\Gamma_k(\gamma)}\int_{\omega}^{b}\psi'(\mu)[\psi(\mu)-\psi(\omega)]^{\frac{\gamma}{k}-1}\eta(\mu)d\mu. \tag{16}$$

The notations $\left({}^{(k,\psi)}\mathcal{I}^{\gamma}_{a^+}\eta\right)(\omega)$ *and* $\left({}^{(k,\psi)}\mathcal{I}^{\gamma}_{b^-}\eta\right)(\omega)$ *are called the left- and right-sided* (k,ψ)*-Riemann–Liouville fractional integrals of the function* η*, respectively.*

Definition 6 ([6])**.** *For the integrable function* η*, let* $v > 0$*; we have for all* $\gamma \in \mathbb{C}$*,* $Re(\gamma) \geq 0$*,*

$$\begin{aligned}({}_aD^{\gamma,v}\eta)(\omega) &= D^{m,v}\,{}_a\mathcal{I}^{m-\gamma,v}\,\eta(\omega) \\ &= \frac{D^{m,v}_{\omega}}{v^{m-\gamma}\Gamma(m-\gamma)}\int_a^{\omega}\exp\left[\frac{v-1}{v}(\omega-\mu)\right](\omega-\mu)^{m-\gamma-1}\eta(\mu)d\mu\end{aligned} \tag{17}$$

and:

$$\begin{aligned}(D^{\gamma,v}_{b}\eta)(\omega) &= {}_{\varrho}D^{m,v}\,\mathcal{I}^{m-\gamma,v}_{b}\,\eta(\omega) \\ &= \frac{{}_{\varrho}D^{m,v}_{\omega}}{v^{m-\gamma}\Gamma(m-\gamma)}\int_{\omega}^{b}\exp\left[\frac{v-1}{v}(\mu-\omega)\right](\mu-\omega)^{m-\gamma-1}\eta(\mu)d\mu,\end{aligned} \tag{18}$$

where:

$$D^{m,v} = \underbrace{D^{v}D^{v}\ldots D^{v}}_{m-times}, m = [Re(\gamma)]+1$$

and:

$$\left({}_{\varrho}D^{v}\eta\right)(\omega) = (1-v)\eta(\omega) - v\eta'(\omega),\ {}_{\varrho}D^{m,v} = \underbrace{{}_{\varrho}D^{v}_{\varrho}D^{v}\ldots{}_{\varrho}D^{v}}_{m-times}.$$

The notations $({}_aD^{\gamma,v}\eta)(\omega)$ *and* $(D^{\gamma,v}_{b}\eta)(\omega)$ *are called the left- and right-sided proportional fractional derivatives of a function* η*, respectively, for the order* γ*.*

Definition 7 ([6])**.** *For the integrable function* η*, let* $v > 0$*; we have for all* $\gamma \in \mathbb{C}$*,* $Re(\gamma) \geq 0$*,*

$$({}_a\mathcal{I}^{\gamma,v}\,\eta)(\omega) = \frac{1}{v^{\gamma}\Gamma(\gamma)}\int_a^{\omega}\exp\left[\frac{v-1}{v}(\omega-\mu)\right](\omega-\mu)^{\gamma-1}\eta(\mu)d\mu \tag{19}$$

and:

$$(\mathcal{I}^{\gamma,v}_{b}\,\eta)(\omega) = \frac{1}{v^{\gamma}\Gamma(\gamma)}\int_{\omega}^{b}\exp\left[\frac{v-1}{v}(\mu-\omega)\right](\mu-\omega)^{\gamma-1}\eta(\mu)d\mu. \tag{20}$$

The notations $({}_a\mathcal{I}^{\gamma,v}\,\eta)(\omega)$ *and* $(\mathcal{I}^{\gamma,v}_{b}\,\eta)(\omega)$ *are called the left- and right-sided proportional fractional integrals of a function* η*, respectively, for the order* γ*.*

Definition 8 ([7])**.** *For the integrable function* η *and for the strictly increasing continuous function* ψ *on* $[a,b]$*, let* $v \in (0,1]$*; we have for all* $\gamma \in \mathbb{C}$*,* $Re(\gamma) \geq 0$*,*

$$\begin{aligned}&\left({}^{\psi}_{a}D^{\gamma,v}\eta\right)(\omega) = {}^{\psi}D^{m,v}\,{}^{\psi}_{a}\mathcal{I}^{m-\gamma,v}\,\eta(\omega) \\ &= \frac{{}^{\psi}D^{m,v}_{\omega}}{v^{m-\gamma}\Gamma(m-\gamma)}\int_a^{\omega}\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right](\psi(\omega)-\psi(\mu))^{m-\gamma-1}\psi'(\mu)\eta(\mu)d\mu\end{aligned} \tag{21}$$

and:

$$\begin{aligned}&({}^{\psi}D^{\gamma,v}_{b}\eta)(\omega) = {}^{\psi}_{\varrho}D^{m,v}\,{}^{\psi}\mathcal{I}^{m-\gamma,v}_{b}\,\eta(\omega) \\ &= \frac{{}^{\psi}_{\varrho}D^{m,v}_{\omega}}{v^{m-\gamma}\Gamma(m-\gamma)}\int_{\omega}^{b}\exp\left[\frac{v-1}{v}(\psi(\mu)-\psi(\omega))\right](\psi(\mu)-\psi(\omega))^{m-\gamma-1}\psi'(\mu)\eta(\mu)d\mu,\end{aligned} \tag{22}$$

where:

$$ {}^{\psi}D^{m,v} = \underbrace{{}^{\psi}D^{v}\,{}^{\psi}D^{v}\ldots{}^{\psi}D^{v}}_{m-times}, m = [Re(\gamma)] + 1 $$

and:

$$ \left({}^{\psi}_{\varrho}D^{v}\eta\right)(\omega) = (1-v)\eta(\omega) - v\frac{\eta'(\omega)}{\psi'(\omega)}, {}^{\gamma}_{\varrho}D^{m,v} = \underbrace{{}^{\psi}_{\varrho}D^{v}\,{}^{\psi}_{\varrho}D^{v}\ldots{}^{\psi}_{\varrho}D^{v}}_{m-times}. $$

The notations $\left({}^{\psi}_{a}D^{\gamma,v}\eta\right)(\omega)$ *and* $({}^{\psi}D^{\gamma,v}_{b}\eta)(\omega)$ *are called, respectively, the left- and right-sided proportional fractional derivatives of a function* η *with respect to the function* ψ *for the order* γ*.*

Definition 9 ([7])**.** *For the integrable function* η *and for the strictly increasing continuous function* ψ *on* $[a,b]$*, let* $v \in (0,1]$*; we have for all* $\gamma \in \mathbb{C}$*,* $Re(\gamma) \geq 0$*,*

$$ \left({}^{\psi}_{a}\mathcal{I}^{\gamma,v}\eta\right)(\omega) = \frac{1}{v^{\gamma}\Gamma(\gamma)}\int_{s}^{\omega}\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right](\psi(\omega)-\psi(\mu))^{\gamma-1}\psi'(\mu)\eta(\mu)d\mu \tag{23} $$

and:

$$ ({}^{\psi}\mathcal{I}^{\gamma,v}_{b}\eta)(\omega) = \frac{1}{v^{\gamma}\Gamma(\gamma)}\int_{\omega}^{b}\exp\left[\frac{v-1}{v}(\psi(\mu)-\psi(\omega))\right](\psi(\mu)-\psi(\omega))^{\gamma-1}\psi'(\mu)\eta(\mu)d\mu, \tag{24} $$

where $\left({}^{\psi}_{a}\mathcal{I}^{\gamma,v}\eta\right)(\omega)$ *and* $({}^{\psi}\mathcal{I}^{\gamma,v}_{b}\eta)(\omega)$ *are called, respectively, the left- and right-sided proportional fractional integrals of a function* η *with respect to the function* ψ *for the order* γ*.*

Lemma 1 ([7])**.** *Let* ψ *be a continuous function on* $\omega \geq a$*. If* $v \in (0,1]$ *and* $Re(\gamma), Re(\delta) > 0$*, we have:*

$$ {}^{\psi}_{a}\mathcal{I}^{\gamma,v}\left({}^{\psi}_{a}\mathcal{I}^{\delta,v}\eta\right)(\omega) = {}^{\psi}_{a}\mathcal{I}^{\delta,v}\left({}^{\psi}_{a}\mathcal{I}^{\gamma,v}\eta\right)(\omega) = \left({}^{\psi}_{a}\mathcal{I}^{\gamma+\delta,v}\eta\right)(\omega), \tag{25} $$

$$ {}^{\psi}\mathcal{I}^{\gamma,v}_{b}\left({}^{\psi}\mathcal{I}^{\delta,v}_{b}\eta\right)(\omega) = {}^{\psi}\mathcal{I}^{\delta,v}_{b}\left({}^{\psi}\mathcal{I}^{\gamma,v}_{b}\eta\right)(\omega) = \left({}^{\psi}\mathcal{I}^{\gamma+\delta,v}_{b}\eta\right)(\omega). \tag{26} $$

Lemma 2 ([7])**.** *Let* ψ *be a integrable functions defined on* $[a,\omega]$ *or for* $\omega > a$*. If* $0 \leq m < [Re(\gamma)] + 1$*, then we have:*

$$ {}^{\psi}D^{m,v}\left({}^{\psi}_{a}\mathcal{I}^{\gamma,v}\eta\right)(\omega) = \left({}^{\psi}_{a}\mathcal{I}^{\gamma-m,v}\eta\right)(\omega), \tag{27} $$

$$ {}^{\psi}_{\varrho}D^{m,v}({}^{\psi}\mathcal{I}^{\gamma,v}_{b}\eta)(\omega) = \left({}^{\psi}\mathcal{I}^{\gamma-m,v}_{b}\eta\right)(\omega). \tag{28} $$

3. (k, ψ)-Proportional Fractional Integrals and Derivatives

In this section, we define the most generalized version of proportional fractional integrals and derivatives, namely (k,ψ)-proportional operators.

Definition 10. *For* $\varrho \in [0,1]$*, let the functions* $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ *be continuous such that for all* $\omega \in \mathbb{R}$*, we have:*

$$ \lim_{\varrho\to 0^+}\kappa_1(\varrho,\omega) = 1, \ \lim_{\varrho\to 0^+}\kappa_0(\varrho,\omega) = 0, \ \lim_{\varrho\to 1^-}\kappa_1(\varrho,\omega) = 0, \ \lim_{\varrho\to 1^-}\kappa_0(\varrho,\omega) = 1 $$

and $\kappa_1(\varrho,\omega) \neq 0, \varrho \in [0,1), \kappa_0(\varrho,\omega) \neq 0, \varrho \in (0,1]$*.*

Definition 11. *For the integrable function η and for the strictly increasing continuous function ψ on $[a,b]$, let $v \in (0,1]$, we have for all $\gamma \in \mathbb{C}$, $\Re(\gamma) \geq 0, k \in \mathbb{R}^+$,*

$$\begin{aligned}&\left({}^{(k,\psi)}{}_{a}\mathcal{I}^{\gamma,v}\eta\right)(\omega) \\ &= \frac{1}{v^{\frac{\gamma}{k}} k\Gamma_k(\gamma)} \int_a^\omega \exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right](\psi(\omega)-\psi(\mu))^{\frac{\gamma}{k}-1}\psi'(\mu)\eta(\mu)d\mu\end{aligned} \tag{29}$$

and:

$$\begin{aligned}&\left({}^{(k,\psi)}\mathcal{I}_b^{\gamma,v}\eta\right)(\omega) \\ &= \frac{1}{v^{\frac{\gamma}{k}} k\Gamma_k(\gamma)} \int_\omega^b \exp\left[\frac{v-1}{v}(\psi(\mu)-\psi(\omega))\right](\psi(\mu)-\psi(\omega))^{\frac{\gamma}{k}-1}\psi'(\mu)\eta(\mu)d\mu.\end{aligned}$$

The notations $\left({}^{(k,\psi)}{}_{a}\mathcal{I}^{\gamma,v}\eta\right)(\omega)$ and $\left({}^{(k,\psi)}\mathcal{I}_b^{\gamma,v}\eta\right)(\omega)$ are called, respectively, the left- and right-sided proportional k-fractional integrals of a function η with respect to the function ψ for the order γ.

Definition 12. *Let $\gamma \in \mathbb{C}, \Re(\gamma) \geq 0$, $k \in \mathbb{R}^+$, $v > 0$, and $n \in \mathbb{N}$ such that $n = [\Re(\frac{\gamma}{k})] + 1$. Then, for $\eta \in L^1[a,b]$ and $\psi \in C^1[a,b]$, where $\psi(t) > 0$, we have:*

$$\begin{aligned}&\left({}^{(k,\psi)}{}_{a}\mathcal{D}^{\gamma,v}\eta\right)(\omega) \\ &= \left(\frac{vk}{\psi'(\omega)}\frac{d}{d\omega}\right)^n \left({}^{(k,\psi)}{}_{a}\mathcal{I}^{n-\gamma,v}\eta\right)(\omega) \\ &= \frac{{}^{(k,\psi)}\mathcal{D}_\omega^{n,v}}{v^{\frac{n-\gamma}{k}} k\Gamma_k(n-\gamma)} \int_a^\omega \exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right](\psi(\omega)-\psi(\mu))^{\frac{n-\gamma}{k}-1}\psi'(\mu)\eta(\mu)d\mu\end{aligned}$$

and:

$$\begin{aligned}&\left({}^{(k,\psi)}\mathcal{D}_b^{\gamma,v}\eta\right)(\omega) \\ &= \left(-\frac{vk}{\psi'(\omega)}\frac{d}{dt}\right)^n \left({}^{(k,\psi)}\mathcal{I}_b^{n-\gamma,v}\eta\right)(\omega) \\ &= \frac{{}^{(k,\psi)}\mathcal{D}_\ominus^{n,v}}{v^{\frac{n-\gamma}{k}} k\Gamma_k(n-\gamma)} \int_\omega^b \exp\left[\frac{v-1}{v}(\psi(\mu)-\psi(\omega))\right](\psi(\mu)-\psi(\omega))^{\frac{n-\gamma}{k}-1}\psi'(\mu)\eta(\mu)d\mu,\end{aligned}$$

where $\left({}^{(k,\psi)}{}_{a}\mathcal{D}^{\gamma,v}\eta\right)(\omega)$ and $\left({}^{(k,\psi)}\mathcal{D}_b^{\gamma,v}\eta\right)(\omega)$ are, respectively, the left- and right-sided (k,ψ)-proportional fractional integrals of a function η with respect to ψ of order γ and type v. Moreover,

$${}^{(k,\psi)}\mathcal{D}_\omega^{n,v} = \underbrace{{}^{(k,\psi)}\mathcal{D}_\omega^{v}\,{}^{(k,\psi)}\mathcal{D}_\omega^{v}\ldots{}^{(k,\psi)}\mathcal{D}_\omega^{v}}_{n\ times}$$

$${}^{(k,\psi)}\mathcal{D}_\ominus^{n,v} = \underbrace{{}^{(k,\psi)}\mathcal{D}_\ominus^{v}\,{}^{(k,\psi)}\mathcal{D}_\ominus^{n}\ldots{}^{(k,\psi)}\mathcal{D}_\ominus^{v}}_{n\ times}$$

$$\left({}^{(k,\psi)}\mathcal{D}_\varkappa^{v}\eta\right)(\omega) = (1-\varrho)\eta(\omega) + \varrho\left(\frac{k}{\psi'(\omega)}\frac{d}{d\omega}\right)\eta(\omega)$$

and:

$$\left({}^{(k,\psi)}\mathcal{D}_\ominus^{n,v}\eta\right)(\omega) = (1-\varrho)\eta(\omega) - \varrho\left(\frac{k}{\psi'(\omega)}\frac{d}{d\omega}\right)\eta(\omega),$$

where $(1-\varrho) = \kappa_1(\varrho,\omega)$ and $\varrho = \kappa_0(\varrho,\omega)$.

Remark 1. *In particular, in Definitions 11 and 12, if $\psi(\omega) = \omega$, then we obtain the k-fractional proportional operators.*

Remark 2. *In Definitions 11 and 12, if we take:*

1. *$k = 1$, then we obtain the ψ-fractional proportional operators introduced by [7];*
2. *$k = 1$ and $\psi(\omega) = \omega$, then we obtain the fractional proportional operators introduced by [6];*
3. *$k = 1$ and $v = 1$, then we obtain the ψ-RL fractional operators introduced by Kilbas et al. [3];*
4. *$\psi(\omega) = \omega$ and $v = 1$, then we obtain the k-RL fractional operators introduced by [49];*
5. *$k = 1$, $v = 1$, and $\psi(\omega) = \omega$, then we obtain the standard RL fractional operators; see [3].*

Lemma 3. *Let $\gamma_1, \gamma_2 \in \mathbb{C}$ be such that $\Re(\gamma_1) > 0$ and $\Re(\gamma_2) > 0$ and $k \in \mathbb{R}^+$. Then, for any $v \in (0, 1]$, we have:*

$$^{(k,\psi)}{}_a\mathcal{I}^{\gamma_1,v}\ {}^{(k,\psi)}{}_a\mathcal{I}^{\gamma_2,v} = {}^{(k,\psi)}{}_a\mathcal{I}^{\gamma_1\gamma_2+,v}.$$

Proof. The proof of the lemma can easily be obtained by applying Definition 11, the Dirichlet formula, substituting $\psi(\mu) = \psi(a) + z[\psi(\omega) - \psi(a)]$, and the properties of the k-gamma function specified in Definition 1. Therefore, we omit the specifics. □

Proposition 1. *Let $\gamma, \delta \in \mathbb{C}$ be such that $\Re(\gamma) > 0$ and $\Re(\delta) > 0$ and $k \in \mathbb{R}^+$. Then, for any $v \in (0, 1]$, we have:*

1. $\left(^{(k,\psi)}{}_a\mathcal{I}^{\gamma,v} e^{\frac{v-1}{v}[\psi(\mu)-\psi(a)]} (\psi(\mu) - \psi(a))^{\frac{\delta}{k}-1} \right)(\omega) = \frac{\Gamma_k(\delta)}{v^{\frac{\gamma}{k}}\Gamma_k(\gamma+\delta)} \frac{e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]}}{[\psi(\omega)-\psi(a)]^{1-\frac{\gamma+\delta}{k}}};$
2. $\left(^{(k,\psi)}\mathcal{I}_b^{\gamma,v} e^{\frac{1-v}{v}[\psi(b)-\psi(\mu)]} (\psi(b) - \psi(\mu))^{\frac{\delta}{k}-1} \right)(\omega) = \frac{\Gamma_k(\delta)}{v^{\frac{\gamma}{k}}\Gamma_k(\delta+\gamma)} \frac{e^{\frac{1-v}{v}[\psi(b)-\psi(\omega)]}}{(\psi(b)-\psi(\omega))^{1-\frac{\delta+\gamma}{k}}};$
3. $\left(^{(k,\psi)}{}_a\mathcal{D}^{\gamma,v} e^{\frac{v-1}{v}[\psi(\mu)-\psi(a)]} (\psi(\mu) - \psi(a))^{\frac{\delta}{k}-1} \right)(\omega) = \frac{\Gamma_k(\delta)}{v^{\frac{\gamma}{k}}\Gamma_k(\delta-\gamma)} \frac{e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]}}{(\psi(\omega)-\psi(a))^{1-\frac{\delta-\gamma}{k}}};$
4. $\left(^{(k,\psi)}\mathcal{D}_b^{\gamma,v} e^{\frac{1-v}{v}[\psi(b)-\psi(\mu)]} (\psi(b) - \psi(\mu))^{\frac{\delta}{k}-1} \right)(\omega) = \frac{\Gamma_k(\delta)}{v^{\frac{\gamma}{k}}\Gamma_k(\delta-\gamma)} \frac{e^{\frac{1-v}{v}[\psi(b)-\psi(\omega)]}}{(\psi(b)-\psi(\omega))^{1-\frac{\delta-\gamma}{k}}}.$

Proof. To prove Property (1) directly, by using Definition 11, we have:

$$\begin{aligned}
&\left(^{(k,\psi)}{}_a\mathcal{I}^{\gamma,v} e^{\frac{v-1}{v}[\psi(\mu)-\psi(a)]} (\psi(\mu) - \psi(a))^{\frac{\delta}{k}-1} \right)(\omega) \\
&= \frac{1}{v^{\frac{\gamma}{k}} k\Gamma_k(\gamma)} \int_a^\omega e^{\frac{v-1}{v}[\psi(\omega)-\psi(\mu)]} (\psi(\omega) - \psi(\mu))^{\frac{\gamma}{k}-1} \psi'(\mu) \frac{e^{\frac{v-1}{v}[\psi(\mu)-\psi(a)]}}{(\psi(\mu) - \psi(a))^{1-\frac{\delta}{k}}} d\mu \\
&= \frac{1}{v^{\frac{\gamma}{k}} k\Gamma_k(\gamma)} [\psi(\omega) - \psi(a)]^{\frac{\gamma}{k}-1} \int_a^\omega e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]\left[1-\frac{\psi(\mu)-\psi(a)}{\psi(\omega)-\psi(a)}\right]} \psi'(\mu) \\
&\times \left[1 - \frac{\psi(\mu) - \psi(a)}{\psi(\omega) - \psi(a)} \right]^{\frac{\gamma}{k}-1} e^{\frac{v-1}{v}[\psi(\mu)-\psi(a)]} [\psi(\mu) - \psi(a)]^{\frac{\delta}{k}-1} d\mu.
\end{aligned}$$

Put $z = \frac{\psi(\mu)-\psi(a)}{\psi(\omega)-\psi(a)}$; we obtain:

$$\begin{aligned}
&\left({}^{(k,\psi)}{}_{a}\mathcal{I}^{\gamma,v}\, e^{\frac{v-1}{v}[\psi(\mu)-\psi(a)]} (\psi(\mu)-\psi(a))^{\frac{\delta}{k}-1} \right)(\omega) \\
&= \frac{1}{v^{\frac{\gamma}{k}} k\Gamma_k(\gamma)} e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]} [\psi(\omega)-\psi(a)]^{\frac{\gamma+\delta}{k}-1} \int_0^1 (1-z)^{\frac{\gamma}{k}-1} z^{\frac{\delta}{k}-1} dz \\
&= \frac{1}{v^{\frac{\gamma}{k}} \Gamma_k(\gamma)} e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]} [\psi(\omega)-\psi(a)]^{\frac{\gamma+\delta}{k}-1} B_k(\gamma,\delta) \\
&= \frac{1}{v^{\frac{\gamma}{k}} \Gamma_k(\gamma)} e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]} [\psi(\omega)-\psi(a)]^{\frac{\gamma+\delta}{k}-1} \frac{\Gamma_k(\gamma)\Gamma_k(\delta)}{\Gamma_k(\gamma+\delta)} \\
&= \frac{\Gamma_k(\delta)}{v^{\frac{\gamma}{k}} \Gamma_k(\gamma+\delta)} e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]} [\psi(\omega)-\psi(a)]^{\frac{\gamma+\delta}{k}-1}.
\end{aligned}$$

The prove of Property (2) is analogous.

It is very easy to deal with the proof of Properties (3) and (4) by Properties (1) and (2) as follows: using the Definitions 12, we have:

$$\begin{aligned}
&\left({}^{(k,\psi)}{}_{a}\mathcal{D}^{\gamma,v}\, e^{\frac{v-1}{v}[\psi(\mu)-\psi(a)]} (\psi(\mu)-\psi(a))^{\frac{\delta}{k}-1} \right)(\omega) \\
&= \left(\frac{vk}{\psi'(\omega)} \frac{d}{d\omega} \right)^n \left({}^{(k,\psi)}{}_{a}\mathcal{I}^{n-\gamma,v}\, e^{\frac{v-1}{v}[\psi(\mu)-\psi(a)]} (\psi(\mu)-\psi(a))^{\frac{\delta}{k}-1} \right)(\omega) \\
&= \left(\frac{vk}{\psi'(\omega)} \frac{d}{d\omega} \right)^n \frac{\Gamma_k(\delta)}{v^{\frac{n-\gamma}{k}} \Gamma_k(n-\gamma+\delta)} e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]} [\psi(\omega)-\psi(a)]^{\frac{n-\gamma+\delta}{k}-1} \\
&= \frac{\Gamma_k(\delta)}{v^{\frac{n-\gamma}{k}} \Gamma_k(n-\gamma+\delta)} \left(\frac{vk}{\psi'(\omega)} \frac{d}{d\omega} \right)^{n-1} vk \left(\frac{n-\gamma+\delta}{k} - 1 \right) \left(\frac{e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]}}{[\psi(\omega)-\psi(a)]^{2-\frac{n-\gamma+\delta}{k}}} \right) \\
&= \frac{\Gamma_k(\delta)}{v^{\frac{n-\gamma}{k}} \Gamma_k(n-\gamma+\delta)} \left(\frac{vk}{\psi'(\omega)} \frac{d}{d\omega} \right)^{n-2} (vk)^2 \left(\frac{n-\gamma+\delta}{k} - 1 \right) \left(\frac{n-\gamma+\delta}{k} - 2 \right) \\
&\left(e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]} [\psi(\omega)-\psi(a)]^{\frac{n-\gamma+\delta}{k}-3} \right) \\
&\vdots \\
&= \frac{\Gamma_k(\delta)}{v^{\frac{n-\gamma}{k}} \Gamma_k(n-\gamma+\delta)} (vk)^n \left(\frac{n-\gamma+\delta}{k} - 1 \right) \left(\frac{n-\gamma+\delta}{k} - 2 \right) \ldots \left(\frac{\delta-\gamma}{k} \right) \\
&\times \left(e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]} [\psi(\omega)-\psi(a)]^{\frac{\delta-\gamma}{k}-1} \right) \\
&= \frac{\Gamma_k(\delta)}{v^{\frac{\gamma}{k}} \Gamma_k(\delta-\gamma)} e^{\frac{v-1}{v}[\psi(\omega)-\psi(a)]} (\psi(\omega)-\psi(a))^{\frac{\delta-\gamma}{k}-1},
\end{aligned}$$

which completes the proof. □

Remark 3. *Taking $k = 1$ and $\psi(\omega) = \omega$ in Lemma 5, we obtain the following results:*

1. $\left({}_{a}\mathcal{I}^{\gamma,v} e^{\frac{v-1}{v}[\mu-a]} (\mu-a)^{\delta-1} \right)(\omega) = \frac{\Gamma(\delta)}{v^{\gamma}\Gamma(\gamma+\delta)} e^{\frac{v-1}{v}(\omega-a)} (\omega-a)^{\gamma+\delta-1}$;
2. $\left(\mathcal{I}_b^{\gamma,v} e^{\frac{1-v}{v}(b-\mu)} (b-\mu)^{\delta-1} \right)(\omega) = \frac{\Gamma(\delta)}{v^{\gamma}\Gamma(\delta+\gamma)} e^{\frac{1-v}{v}(b-\omega)} (b-\omega)^{\delta+\gamma-1}$;
3. $\left({}_{a}\mathcal{D}^{\gamma,v} e^{\frac{v-1}{v}(\mu-a)} (\mu-a)^{\delta-1} \right)(\omega) = \frac{\Gamma(\delta)}{v^{\gamma}\Gamma(\delta-\gamma)} e^{\frac{v-1}{v}(\omega-a)} (\omega-a))^{\delta-\gamma-1}$;
4. $\left(\mathcal{D}_b^{\gamma,v} e^{\frac{1-v}{v}(b-\mu)} (b-\mu)^{\delta-1} \right)(\omega) = \frac{\Gamma(\delta)}{v^{\gamma}\Gamma(\delta-\gamma)} e^{\frac{1-v}{v}(b-\omega)} (b-\omega)^{\delta-\gamma-1}$.

These were proven in [6].

Remark 4. *Taking $k = 1$, $v = 1$, and $\psi(\omega) = \omega$ in Lemma 5, we obtain the following results:*

1. $\left({}_a\mathcal{I}^{\gamma,1}(\mu - a)^{\delta-1}\right)(\omega) = \frac{\Gamma(\delta)}{\Gamma(\gamma+\delta)}(\omega - a)^{\gamma+\delta-1}$;
2. $\left(\mathcal{I}_b^{\gamma,1}(b - \mu)^{\delta-1}\right)(\omega) = \frac{\Gamma(\delta)}{\Gamma(\delta+\gamma)}(b - \omega)^{\delta+\gamma-1}$;
3. $\left({}_a\mathcal{D}^{\gamma,1}(\mu - a)^{\delta-1}\right)(\omega) = \frac{\Gamma(\delta)}{\Gamma(\delta-\gamma)}(\omega - a))^{\delta-\gamma-1}$;
4. $\left(\mathcal{D}_b^{\gamma,1}(b - \mu)^{\delta-1}\right)(\omega) = \frac{\Gamma(\delta)}{\Gamma(\delta-\gamma)}(b - \omega)^{\delta-\gamma-1}$.

These proven in Kilbas et al. [3].

4. Certain Pólya–Szegö- and Grüss-Type Inequalities Involving the Proportional *k*-Fractional Integral concerning Another Strictly Increasing Continuous Function

This section is concerned with the use of the left-sided (k, ψ)-proportional fractional integral (29) to obtain the main results. To reach our desired results, we first need to prove the following lemmas:

Lemma 4. *For the positive integrable functions η, ς on $[0, \infty)$ and for the positive integrable functions G_1, G_2, H_1, H_2, let $v \in (0, 1]$, $\gamma \in \mathbb{C}$, $\Re(\gamma) \geq 0$, and ψ be a strictly increasing continuous function. If the following conditions:*

$$(Z_1) \quad 0 < G_1(\mu) \leq \eta(\mu) \leq G_2(\mu), \quad 0 < H_1(\mu) \leq \varsigma(\mu) \leq H_2(\mu), \quad (\mu \in [0, \omega], \omega > 0), \tag{30}$$

hold, then the following inequality also holds:

$$\frac{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{H_1 H_2 \eta^2\right\}(\omega)\, {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{G_1 G_2 \varsigma^2\right\}(\omega)}{\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{(G_1 H_1 + G_2 H_2)\eta\varsigma\right\}(\omega)\right)^2} \leq \frac{1}{4}. \tag{31}$$

Proof. According to (Z_1), we can write for $(\mu \in [0, \omega], \omega > 0)$,

$$\left(\frac{G_2(\mu)}{H_1(\mu)} - \frac{\eta(\mu)}{\varsigma(\mu)}\right) \geq 0 \tag{32}$$

and:

$$\left(\frac{\eta(\mu)}{\varsigma(\mu)} - \frac{G_1(\mu)}{H_2(\mu)}\right) \geq 0. \tag{33}$$

By carrying out the multiplication between (32) and (33), we obtain:

$$\left(\frac{G_2(\mu)}{H_1(\mu)} - \frac{\eta(\mu)}{\varsigma(\mu)}\right)\left(\frac{\eta(\mu)}{\varsigma(\mu)} - \frac{G_1(\mu)}{H_2(\mu)}\right) \geq 0,$$

which leads to:

$$\{G_1(\mu)H_1(\mu) + G_2(\mu)H_2(\mu)\}\eta(\mu)\varsigma(\mu) \geq H_1(\mu)H_2(\mu)\eta^2(\mu) + G_1(\mu)G_2(\mu)\varsigma^2(\mu). \tag{34}$$

On both sides of (34), taking multiplication by the positive quantity $\frac{\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right]\psi'(\mu)}{v^{\frac{\gamma}{k}} k\Gamma_k(\gamma)(\psi(\omega)-\psi(\mu))^{1-\frac{\gamma}{k}}}$, then integrating the estimation with respect to μ over $(0, \omega)$, we obtain:

$${}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left(\{G_1 H_1 + G_2 H_2\}\eta\varsigma\right)(\omega) \geq {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left(H_1 H_2 \eta^2\right)(\omega) + {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left(G_1 G_2 \varsigma^2\right)(\omega). \tag{35}$$

Now, applying the arithmetic mean and geometric mean inequality *i.e.* ($s + t \geq 2\sqrt{st}$, $s, t \in \mathbb{R}^+$) on the right-hand side of Inequality (35), we conclude that:

$${}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left(\{G_1 H_1 + G_2 H_2\}\eta\varsigma\right)(\omega) \geq 2\sqrt{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left(H_1 H_2 \eta^2\right)(\omega)\, {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left(G_1 G_2 \varsigma^2\right)(\omega)},$$

which can be written as:

$${}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left(H_1 H_2 \eta^2\right)(\omega)\, {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left(G_1 G_2 \varsigma^2\right)(\omega) \leq \frac{1}{4}\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left(\{G_1 H_1 + G_2 H_2\}\eta\varsigma\right)(\omega)\right)^2.$$

The proof is complete. $\square$

The following corollary is a special case of Lemma 4.

Corollary 1. *For the positive integrable functions η, ς on $[0, \infty)$, let $v \in (0,1]$, $\gamma \in \mathbb{C}$, $\Re(\gamma) \geq 0$, and ψ be a strictly increasing continuous function. Then, for the real constants p, P, q, Q, if the following conditions:*

$$(Z_2) \quad 0 < p \leq \eta(\mu) \leq P, \quad 0 < q \leq \varsigma(\mu) \leq Q, \quad (\mu \in [0, \omega], \omega > 0), \tag{36}$$

hold, then the following inequality also holds:

$$\frac{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\varsigma^2\right\}(\omega)}{\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\varsigma\}(\omega)\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{pq}{PQ}} + \sqrt{\frac{PQ}{pq}}\right)^2. \tag{37}$$

Remark 5.

(i) *Putting $v = 1$ and $\psi(\omega) = \omega$, $(\forall \omega \in [a,b])$, $k = 1$ in Lemma 4, we obtain Lemma 3.1 proven by Ntouyas et al. [41];*

(ii) *Putting $v = 1$ and $\psi(\omega) = \omega$, $(\forall \omega \in [a,b])$, $k = 1$ in Corollary 1 , we obtain Lemma 3 proven by Amber and Dahmani [40];*

(iii) *Applying Corollary 1 for $\gamma = 1$, $v = 1$, $k = 1$, and $\psi(\omega) = \omega$, $(\forall \omega \in [a,b])$, we obtain Inequality (7), which was introduced by Pólya and Szegö [38].*

Lemma 5. *For the positive integrable functions η, ς on $[0, \infty)$ and for $v \in (0,1]$, $\gamma, \delta \in \mathbb{C}$, $\Re(\gamma), \Re(\delta) \geq 0$, let all assumptions of Lemma 4 hold, then the following inequality also holds:*

$$\frac{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_1H_2\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_2G_1\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\varsigma^2\right\}(\omega)}{\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_2\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_2\varsigma\}(\omega) + {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_1\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_1\varsigma\}(\omega)\right)^2} \leq \frac{1}{4}. \tag{38}$$

Proof. According to (Z_1), we can write for $(\mu, \sigma \in [0, \omega], \omega > 0)$,

$$\left(\frac{G_2(\mu)}{H_1(\sigma)} - \frac{\eta(\mu)}{\varsigma(\sigma)}\right) \geq 0 \tag{39}$$

and:

$$\left(\frac{\eta(\mu)}{\varsigma(\sigma)} - \frac{G_1(\mu)}{H_2(\sigma)}\right) \geq 0. \tag{40}$$

The multiplication between (39) and (40) leads to:

$$\left(\frac{G_2(\mu)}{H_1(\sigma)} + \frac{G_1(\mu)}{H_2(\sigma)}\right)\frac{\eta(\mu)}{\varsigma(\sigma)} \geq \frac{\eta^2(\mu)}{\varsigma^2(\sigma)} + \frac{G_2(\mu)G_1(\mu)}{H_1(\sigma)H_2(\sigma)}. \tag{41}$$

On both sides of (41), taking multiplication by $H_1(\sigma)H_2(\sigma)\varsigma^2(\sigma)$, we obtain:

$$G_2(\mu)\eta(\mu)H_2(\sigma)\varsigma(\sigma) + G_1(\mu)\eta(\mu)H_1(\sigma)\varsigma(\sigma) \geq \eta^2(\mu)H_1(\sigma)H_2(\sigma) + G_2(\mu)G_1(\mu)\varsigma^2(\sigma). \tag{42}$$

Now, on both sides of (42), taking multiplication by $\frac{\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right]\psi'(\mu)}{v^{\frac{\gamma}{k}}k\Gamma_k(\gamma)(\psi(\omega)-\psi(\mu))^{1-\frac{\gamma}{k}}}$, then integrating the estimation with respect to μ over $(0, \omega)$, we obtain:

$$\begin{aligned} &H_2(\sigma)\varsigma(\sigma)\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_2\eta\}(\omega) + H_1(\sigma)\varsigma(\sigma)\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_1\eta\}(\omega) \\ &\geq H_1(\sigma)H_2(\sigma)\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega) + \varsigma^2(\sigma)\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_2G_1\}(\omega). \end{aligned} \tag{43}$$

Again, on both sides of (43), taking multiplication by $\frac{\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\sigma))\right]\psi'(\sigma)}{v^{\frac{\delta}{k}}k\Gamma_k(\delta)(\psi(\omega)-\psi(\sigma))^{1-\frac{\delta}{k}}}$, then integrating the estimation with respect to σ over $(0,\omega)$, we obtain:

$$\begin{aligned}&{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_2\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_2\varsigma\}(\omega)+{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_1\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_1\varsigma\}(\omega)\\&\geq{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_1H_2\}(\omega)+{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_2G_1\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\varsigma^2\right\}(\omega).\end{aligned}\tag{44}$$

Applying the A.M.–G.M. inequality on the right-hand side of Inequality (44), we conclude that:

$$\begin{aligned}&{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_2\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_2\varsigma\}(\omega)+{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_1\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_1\varsigma\}(\omega)\\&\geq 2\sqrt{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta^2\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_1H_2\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_2G_1\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma^2\}(\omega)}.\end{aligned}\tag{45}$$

By rewriting Inequality (45), we obtain the desired Inequality (38), which completes the proof. □

The following result is a special case of Lemma 5.

Corollary 2. *Assume that the positive integrable functions η,ς on $[0,\infty)$ satisfy for the real constants p,P,q,Q the conditions (Z_2), then we have for a strictly increasing continuous function ψ and $v\in(0,1]$, $\gamma,\delta\in\mathbb{C}$, $\Re(\gamma),\Re(\delta)\geq 0$, the following inequality:*

$$\begin{aligned}&\frac{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta^2\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma^2\}(\omega)}{\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma\}(\omega)+{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma\}(\omega)\right)^2}\\&\times\frac{\frac{1}{\gamma\delta}[\psi(\omega)-\psi(0)]^{\frac{\gamma+\delta}{k}}}{v^{\frac{\gamma+\delta}{k}}\Gamma_k(\gamma)\Gamma_k(\delta)}\leq\frac{1}{4}\left(\sqrt{\frac{pq}{PQ}}+\sqrt{\frac{PQ}{pq}}\right)^2.\end{aligned}$$

Lemma 6. *Assume that all inputs of Lemma 4 are satisfied. Then, for $v\in(0,1]$, $\gamma,\delta\in\mathbb{C}$, $\Re(\gamma),\Re(\delta)\geq 0$, we have:*

$${}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\varsigma^2\right\}(\omega)\leq{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{(G_2\eta\varsigma)/H_1\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{(H_2\eta\varsigma)/G_1\}(\omega).\tag{46}$$

Proof. According to the conditions (Z_1), we can write:

$$\eta^2(\mu)\leq\frac{G_2(\mu)}{H_1(\mu)}\eta(\mu)\varsigma(\mu).\tag{47}$$

On both sides of (47), taking multiplication by $\frac{\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right]\psi'(\mu)}{v^{\frac{\gamma}{k}}k\Gamma_k(\gamma)(\psi(\omega)-\psi(\mu))^{1-\frac{\gamma}{k}}}$, then integrating the estimation with respect to μ over $(0,\omega)$, we obtain:

$${}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega)\leq{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{(G_2\eta\varsigma)/H_1\}(\omega).\tag{48}$$

Again, using the conditions (Z_1), we can write:

$$\varsigma^2(\mu)\leq\frac{H_2(\mu)}{G_1(\mu)}\eta(\mu)\varsigma(\mu),\tag{49}$$

which is analogously leads to:

$${}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\varsigma^2\right\}(\omega)\leq{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{(H_2\eta\varsigma)/G_1\}(\omega).\tag{50}$$

Clearly, the multiplication between Inequalities (48) and (50) ends the proof. □

Corollary 3. *Assume that the positive integrable functions η, ς on $[0, \infty)$ satisfy for the real constants p, P, q, Q the conditions (Z_2), then we have for a strictly increasing continuous function ψ and $v \in (0, 1]$, $\gamma, \delta \in \mathbb{C}$, $\Re(\gamma), \Re(\delta) \geq 0$, the following inequality:*

$$\frac{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\varsigma^2\right\}(\omega)}{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta\varsigma\right\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\eta\varsigma\right\}(\omega)} \leq \frac{PQ}{pq}. \tag{51}$$

In what follows, we employ the Pólya–Szegö fractional integral inequality to drive our main Chebyshev-fractional-integral-type inequalities with the help of the current generalized k-fractional integral.

Theorem 1. *For the positive integrable functions η, ς on $[0, \infty)$ and for the positive integrable functions G_1, G_2, H_1, H_2, let $v \in (0, 1]$, $\gamma, \delta \in \mathbb{C}$, $\Re(\gamma), \Re(\delta) \geq 0$, and ψ be a strictly increasing continuous function. If the conditions (Z_1) hold, then the following inequality also holds:*

$$\begin{aligned}
&\Bigg|\frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\varsigma\}(\omega) - {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma\}(\omega) \\
&+\frac{\frac{1}{\delta}[\psi(\omega)-\psi(0)]^{\frac{\delta}{k}}}{v^{\frac{\delta}{k}}\Gamma_k(\delta)}\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\varsigma\}(\omega) - {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\}(\omega)\Bigg| \\
&\leq |\xi_1(\eta, G_1, G_2)(\omega) + \xi_2(\eta, G_1, G_2)(\omega)|^{\frac{1}{2}} \times |\xi_1(\varsigma, H_1, H_2)(\omega) + \xi_2(\varsigma, H_1, H_2)(\omega)|^{\frac{1}{2}},
\end{aligned} \tag{52}$$

where:

$$\begin{aligned}
&\xi_1(l,s,t)(\omega) \\
&= \frac{\frac{1}{\delta}[\psi(\omega)-\psi(0)]^{\frac{\delta}{k}}}{4v^{\frac{\delta}{k}}\Gamma_k(\delta)}\frac{\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{(s+t)l\}(\omega)\right)^2}{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{st\}(\omega)} - {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{l\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{l\}(\omega),
\end{aligned}$$

$$\begin{aligned}
&\xi_2(l,s,t)(\omega) \\
&= \frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{4v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\frac{\left({}^{(k,\psi)}\mathcal{I}^{\delta,v}\{(s+t)l\}(\omega)\right)^2}{{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{st\}(\omega)} - {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{l\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{l\}(\omega).
\end{aligned}$$

Proof. We define the function $A(\mu, \sigma)$ as follows:

$$A(\mu,\sigma) = (\eta(\mu)-\eta(\sigma))(\varsigma(\mu)-\varsigma(\sigma)),$$

which can equivalently be rewritten as:

$$A(\mu,\sigma) = \eta(\mu)\varsigma(\mu) + \eta(\sigma)\varsigma(\sigma) - \eta(\mu)\varsigma(\sigma) - \eta(\sigma)\varsigma(\mu). \tag{53}$$

On both sides of (53), taking multiplication by the positive quantity:

$$\frac{\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right]\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\sigma))\right]\psi'(\mu)\psi'(\sigma)}{v^{\frac{\gamma+\delta}{k}}k^2\Gamma_k(\gamma)\Gamma_k(\delta)(\psi(\omega)-\psi(\mu))^{1-\frac{\gamma}{k}}(\psi(\omega)-\psi(\sigma))^{1-\frac{\delta}{k}}},$$

then double integrating the estimation with respect to μ and σ over $(0, \omega)$, we obtain:

$$\begin{aligned}
&\frac{1}{v^{\frac{\gamma+\delta}{k}}k^2\Gamma_k(\gamma)\Gamma_k(\delta)}\int_0^\omega\int_0^\omega \frac{\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right]\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\sigma))\right]}{(\psi(\omega)-\psi(\mu))^{1-\frac{\gamma}{k}}(\psi(\omega)-\psi(\sigma))^{1-\frac{\delta}{k}}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\psi'(\mu)\psi'(\sigma)A(\mu,\sigma)d\mu d\sigma \\
&=\frac{\frac{1}{\delta}[\psi(\omega)-\psi(0)]^{\frac{\delta}{k}}}{v^{\frac{\delta}{k}}\Gamma_k(\delta)}\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\varsigma\}(\omega)+\frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\varsigma\}(\omega) \\
&-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma\}(\omega)-{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega).
\end{aligned} \tag{54}$$

Now, applying the inequality of Cauchy–Schwartz for double integrals, we obtain:

$$\begin{aligned}
&\left|\frac{1}{v^{\frac{\gamma+\delta}{k}}k^2\Gamma_k(\gamma)\Gamma_k(\delta)}\int_0^\omega\int_0^\omega \frac{\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right]\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\sigma))\right]}{(\psi(\omega)-\psi(\mu))^{1-\frac{\gamma}{k}}}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.\times\frac{\psi'(\mu)\psi'(\sigma)A(\mu,\sigma)d\mu}{(\psi(\omega)-\psi(\sigma))^{1-\frac{\delta}{k}}}\right|d\sigma \\
&\leq\left[\frac{\frac{1}{\delta}[\psi(\omega)-\psi(0)]^{\frac{\delta}{k}}}{v^{\frac{\delta}{k}}\Gamma_k(\delta)}\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega)\right. \\
&\quad\left.+\frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\eta^2\right\}(\omega)-2\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\}(\omega)\right]^{\frac{1}{2}} \\
&\times\left[\frac{\frac{1}{\delta}[\psi(\omega)-\psi(0)]^{\frac{\delta}{k}}}{v^{\frac{\delta}{k}}\Gamma_k(\delta)}\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\varsigma^2\right\}(\omega)\right. \\
&\left.+\frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\varsigma^2\right\}(\omega)-2\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma\}(\omega)\right]^{\frac{1}{2}}.
\end{aligned} \tag{55}$$

Applying Lemma 4 for $H_1(\omega)=H_2(\omega)=\varsigma(\omega)=1$, we obtain:

$$ {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega)\leq\frac{\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{(G_1+G_2)\eta\}(\omega)\right)^2}{4\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_1G_2\}(\omega)}, $$

Therefore, we can write:

$$\begin{aligned}
&\frac{\frac{1}{\delta}[\psi(\omega)-\psi(0)]^{\frac{\delta}{k}}}{v^{\frac{\delta}{k}}\Gamma_k(\delta)}\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\eta^2\right\}(\omega)-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\}(\omega) \\
&\leq\frac{\frac{1}{\delta}[\psi(\omega)-\psi(0)]^{\frac{\delta}{k}}}{4v^{\frac{\delta}{k}}\Gamma_k(\delta)}\frac{\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{(G_1+G_2)\eta\}(\omega)\right)^2}{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{G_1G_2\}(\omega)}-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\}(\omega) \\
&=\xi_1(\eta,G_1,G_2)(\omega),
\end{aligned} \tag{56}$$

and:

$$
\begin{aligned}
&\frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\eta^2\right\}(\omega)-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\}(\omega)\\
&\le\frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{4v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\frac{\left({}^{(k,\psi)}\mathcal{I}^{\delta,v}\{(G_1+G_2)\eta\}(\omega)\right)^2}{{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{G_1G_2\}(\omega)}-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\}(\omega)\\
&=\xi_2(\eta,G_1,G_2)(\omega),
\end{aligned}
\tag{57}
$$

Again, applying Lemma 4 for $G_1(\omega)=G_2(\omega)=\eta(\omega)=1$, we can conclude that:

$$
\begin{aligned}
&\frac{\frac{1}{\delta}[\psi(\omega)-\psi(0)]^{\frac{\delta}{k}}}{v^{\frac{\delta}{k}}\Gamma_k(\delta)}\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\left\{\varsigma^2\right\}(\omega)-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma\}(\omega)\\
&\le\frac{\frac{1}{\delta}[\psi(\omega)-\psi(0)]^{\frac{\delta}{k}}}{4v^{\frac{\delta}{k}}\Gamma_k(\delta)}\frac{\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{(H_1+H_2)\varsigma\}(\omega)\right)^2}{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{H_1H_2\}(\omega)}-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma\}(\omega)\\
&=\xi_1(\varsigma,H_1,H_2)(\omega),
\end{aligned}
\tag{58}
$$

and:

$$
\begin{aligned}
&\frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\left\{\varsigma^2\right\}(\omega)-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma\}(\omega)\\
&\le\frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{4v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\frac{\left({}^{(k,\psi)}\mathcal{I}^{\delta,v}\{(H_1+H_2)\varsigma\}(\omega)\right)^2}{{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{H_1H_2\}(\omega)}-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\varsigma\}(\omega)\\
&=\xi_2(\varsigma,H_1,H_2)(\omega).
\end{aligned}
\tag{59}
$$

By adding (57) to (58) and (59) to (60) and replacing the estimations in (56), we obtain:

$$
\begin{aligned}
&\left|\frac{1}{v^{\frac{\gamma+\delta}{k}}k^2\Gamma_k(\gamma)\Gamma_k(\delta)}\int_0^{\omega}\int_0^{\omega}\frac{\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\mu))\right]\exp\left[\frac{v-1}{v}(\psi(\omega)-\psi(\sigma))\right]}{(\psi(\omega)-\psi(\mu))^{1-\frac{\gamma}{k}}}\right.\\
&\qquad\left.\times\frac{\psi'(\mu)\psi'(\sigma)A(\mu,\sigma)d\mu}{(\psi(\omega)-\psi(\sigma))^{1-\frac{\delta}{k}}}d\sigma\right|\\
&\le[\xi_1(\eta,G_1,G_2)(\omega)+\xi_2(\eta,G_1,G_2)(\omega)]^{\frac{1}{2}}[\xi_1(\varsigma,H_1,H_2)(\omega)+\xi_2(\varsigma,H_1,H_2)(\omega)]^{\frac{1}{2}}.
\end{aligned}
\tag{60}
$$

By comparing (55) with (61), we obtain the required Inequality (53). The proof is complete. □

The next result is a special case of Theorem 1.

Theorem 2. *For the positive integrable functions η,ς on $[0,\infty)$ and for the positive integrable functions G_1,G_2,H_1,H_2, let $v\in(0,1]$, $\gamma\in\mathbb{C}$, $\Re(\gamma)\ge 0$, and ψ be a strictly increasing continuous function. If the conditions (Z_1) hold, then the following inequality also holds:*

$$
\begin{aligned}
&\left|\frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{v^{\frac{\gamma}{k}}\Gamma_k(\gamma)}\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\varsigma\}(\omega)-{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\,{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega)\right|\\
&\le\sqrt{\xi(\eta,G_1,G_2)(\omega)\xi(\varsigma,H_1,H_2)(\omega)},
\end{aligned}
\tag{61}
$$

where:

$$\xi_2(l,s,t)(\omega) = \frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{4v^{\frac{\gamma}{k}}\Gamma_k(\gamma)} \frac{\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{(s+t)l\}(\omega)\right)^2}{{}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{st\}(\omega)} - \left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{l\}(\omega)\right)^2.$$

Proof. Setting $\delta = \gamma$, in (53), we obtain the required result (61). □

We introduce the next corollary as a special case of Theorem 1.

Corollary 4. *Assume that the positive integrable functions η, ς on $[0,\infty)$ satisfy for the real constants p, P, q, Q the conditions (Z_2), then we have for a strictly increasing continuous function ψ and $v \in (0,1]$, $\gamma \in \mathbb{C}$, $\Re(\gamma) \geq 0$, the following inequality:*

$$\begin{aligned} &\left| \frac{\frac{1}{\gamma}[\psi(\omega)-\psi(0)]^{\frac{\gamma}{k}}}{v^{\frac{\gamma}{k}}\Gamma_k(\gamma)} {}^{(k,\psi)}\mathcal{I}^{\delta,v}\{\eta\varsigma\}(\omega) - {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\, {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega) \right| \\ &\leq \frac{(p-P)(q-Q)}{4\sqrt{pPqQ}} {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\, {}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega). \end{aligned} \tag{62}$$

Remark 6.

(i) *When $G_1 = p$, $G_2 = P$, $H_1 = q$, $H_2 = Q$, we have:*

$$\xi(\eta,p,P)(\omega) = \frac{(p-P)^2}{4pP}\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\eta\}(\omega)\right)^2,$$

$$\xi(\varsigma,q,Q)(\omega) = \frac{(q-Q)^2}{4qQ}\left({}^{(k,\psi)}\mathcal{I}^{\gamma,v}\{\varsigma\}(\omega)\right)^2;$$

(ii) *Putting $v = 1$ and $\psi(\omega) = \omega$, $(\forall\omega \in [a,b])$, $k = 1$ in Theorem 1, we obtain Theorem 3.6 proven by Ntouyas et al. [41];*

(iii) *Putting $v = 1$ and $\psi(\omega) = \omega$, $(\forall\omega \in [a,b])$, $k = 1$ in Theorem 2, we obtain Theorem 9 proven by Tariboon et al. [40];*

(iv) *Applying Theorem 2 for $\gamma = 1$, $v = 1$, $k = 1$ and $\psi(\omega) = \omega$, $(\forall\omega \in [a,b])$, we obtain Inequality (4), which was introduced by Grüss [32] (see also [33]).*

5. Conclusions

We developed some new way k-fractional integral inequalities of the Pólya–Szegö-type. Then, we employed these generalized inequalities to investigate some new fractional integral inequalities of the Grüss-type. We firstly generalized the proportional fractional operators with respect to another strictly increasing continuous function ψ. Some important properties and special cases of these generalized fractional operators were proven. By means of the generalized proportional fractional integral operator, we investigated some Pólya–Szegö- and Grüss-type fractional integral inequalities. The functions used in this work were bounded by two positive functions to obtain Pólya–Szegö- and Grüss-type k-fractional integral inequalities in a new sense. Furthermore, some new special cases of the Pólya–Szegö- and Grüss-type inequalities were discussed. The obtained results in this work are recent and open the door for researchers to study more fractional inequalities and problems in different fields. Therefore, we recommend that researchers study it deeply, as well as give it more attention.

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