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An Infinite System of Fractional Order with p -Laplacian Operator in a Tempered Sequence Space via Measure of Noncompactness Technique

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Abstract: In the current study, a new class of an infinite system of two distinct fractional orders with p -Laplacian operator is presented. Our mathematical model is introduced with the Caputo–Katugampola fractional derivative which is considered a generalization to the Caputo and Hadamard fractional derivatives. In a new sequence space associated with a tempered sequence and the sequence space c_0 (the space of convergent sequences to zero), a suitable new Hausdorff measure of noncompactness form is provided. This formula is applied to discuss the existence of a solution to our infinite system through applying Darbo’s theorem which extends both the classical Banach contraction principle and the Schauder fixed point theorem.

Keywords: p -Laplacian operator; infinite system; Darbo’s fixed point theorem; sequence space; measure of noncompactness

MSC: 34A12; 34A08; 47H10; 47H08



Citation: Salem, A.; Almaghamisi, L.; Alzahrani, F. An Infinite System of Fractional Order with p -Laplacian Operator in a Tempered Sequence Space via Measure of Noncompactness Technique. *Fractal Fract.* **2021**, *5*, 182. <https://doi.org/10.3390/fractalfract5040182>

Academic Editors: Giacomo Ascione, Alessandra Meoli and Enrica Pirozzi

Received: 22 September 2021

Accepted: 19 October 2021

Published: 25 October 2021

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1. Introduction

Differential and integral equations take place in the research work of p -Laplacian equation with n -dimensional space, a gas turbulent flow in porous media and non-newtonian fluid (see [1–3] and references cited therein). Infinite systems of fractional differential equations play a considerable role in several nonlinear analysis branches. These systems demonstrate some examples that cover the theorems for neural nets, dissociation of polymers and branching process. Therefore, the study of infinite systems drew the attention of a number of contributors (see [4–8] and references given therein).

It is well known that the concept of diffusion is associated with random motion of particles in space, usually denoted as Brownian motion [9]. Random fractional differential equations are useful mathematical tools to model problems involving memory effects and uncertainties. Since integer order differential equations cannot precisely describe the experimental and field measurement data, as an alternative approach, fractional order differential equation models are now being widely applied [10]. Mainradi and Pironi [9] have revisited the Brownian motion on the basis of the fractional Langevin equation. The two fluctuation-dissipation theorems and of the techniques of the Fractional Calculus have provided the analytical expressions of the correlation functions. The random force has been shown to be represented by a superposition of the usual white noise with a fractional noise. The fractional Langevin differential equation has been under consideration by several contributions [11–14]. Various types of this equation have been proposed and investigated using different approaches, see [15] and the references cited therein. Zhou et al. [15] considered the fractional Langevin differential equation subject to p -Laplacian operator in the Caputo sense of the form

$${}^c D^\mu \Phi_p[({}^c D^\nu + \lambda)u(t)] = f(t, u(t), {}^c D^\alpha u(t)), \quad t \in [0, 1].$$

In spite of the great significance of fractional Langevin equation with $\lambda = 0$ (the dissipative parameter), infinite systems and p -Laplacian operator in differential equations theory, there is no contributor, as far as we know, that has touched on bringing them together. This is what stimulated us in this paper to present the following system

$${}^{\rho_n} D^{\mu_n} \Phi_p({}^{\rho_n} D^{\nu_n} u_n(t)) = f_n(t, u(t), {}^{\rho_n} D^{\alpha_n} u(t)), \quad t \in [0, 1], n \in \mathbb{N} \quad (1)$$

where $\Phi_p(r) = |r|^{p-2}r$, $p > 1$ is the p -Laplacian operator, $u(t) = (u_n(t))$ is a convergent sequence, $f_n : [0, 1] \times c_0^\beta \times c_0^\beta \rightarrow c_0^\beta$ are continuously differentiable functions, $0 < \rho_n \leq 1$, $0 < \mu_n \leq 1$, $1 < \nu_n \leq 2$, $\alpha = (\alpha_n)$, $0 < \alpha_n < \nu_n$, ${}^{\rho_n} D^{\alpha_n}$ is the Caputo generalized fractional derivative due to Katugampola [16] and the space c_0^β is the tempered space with the tempering sequence $\beta = (\beta_n)$. This system subjects to the assumptions

$$u_n(0) = 0, \quad {}^{\rho_n} D^{\mu_n} u_n(0) = 0, \quad \lim_{t \rightarrow 0} t^{1-\rho_n} u'_n(t) = 0. \quad (2)$$

The value of the Caputo–Katugampola fractional derivative is found in the verity that it is a generalization of Hadamard and Caputo fractional approaches. It has drawn the attention of many authors who have construct their mathematical models based on it. For more details and properties, see [17]. Zeng et al. [18] provided a numerical method for solving generalized fractional differential equation of the Caputo–Katugampola derivative. Almeida et al. [17] proved an existence and uniqueness solution for a fractional Cauchy-type problem and then presented a simple numerical procedure to obtain a decomposition formula for the Caputo–Katugampola derivative. For more contributions, see [19,20] and references given therein.

Unquestionably, the fractional differential equation is a powerful mathematical case for presenting extra flexibility in treatment with numerous real-world implementations. More speciality, the fractional differential equations are extensively used in modeling different phenomena such as diffusion modeling [21], robot manipulators [22], economics [23], and many more. In view of its distinguished interest, several contributors have paid their attentions to deeply discuss the boundary value problems presented based on such equations (see [24–26]).

The measures of noncompactness theory make up an extremely remarkable branch of the nonlinear functional analysis. It permits us to choose an extremely important class of operators as generalizations of compact operators. Those operators satisfy the Darbo contractions with respect to a measure of noncompactness. The measures of noncompactness are widely used in fixed point theory. Darbo fixed point theorem is a good way to investigate the existence and uniqueness of solution to differential and integral equations via applying measures of noncompactness techniques [27,28].

The main goal of this article can be outlined as follows. In a tempered sequence space associated with the classical sequence space c_0 , a suitable Hausdorff measure on noncompactness is presented and used to investigate the infinite system (1) and (2) by aiding of Darbo fixed point theorem which extends both the classical Banach contraction principle and the Schauder fixed point theorem [29].

2. Preliminaries

This section is separated into two subsections. The first subsection inserts the main results and basic concepts needed in fractional calculus. The other subsection provides a short survey for the measure of noncompactness.

2.1. Fractional Calculus

Katugampola [30] defined a approach of fractional integral as

$${}^{\rho}I^{\mu}f(t) = \frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_0^t \frac{s^{\rho-1}f(s)}{(t^{\rho}-s^{\rho})^{1-\mu}} ds \quad (3)$$

where $0 < \rho \leq 1$ and $\mu > 0$, provided that the integral exists. He has shown that it satisfies the semigroup property

$${}^{\rho}I^{\nu} {}^{\rho}I^{\mu}f(t) = {}^{\rho}I^{\nu+\mu}f(t), \quad \nu > 0, \mu > 0. \quad (4)$$

It is clear that

$${}^{\rho}I^{\mu}t^{\rho\lambda} = \frac{\Gamma(\lambda+1)}{\rho^{\mu}\Gamma(\mu+\lambda+1)}t^{\rho(\lambda+\mu)}, \quad \mu > 0, \lambda > -1. \quad (5)$$

Jarad et al. [31] evolved a Katugampola fractional derivative in the Caputo sense [32] to be

$$\begin{aligned} {}^{\rho}_cD^{\mu}f(t) &= \left({}^{\rho}I^{n-\mu} \left(t^{1-\rho} \frac{d}{dt} \right)^n f \right)(t), \quad n-1 < \mu \leq n, n \in \mathbb{N} \\ &= \frac{\rho^{\mu-n+1}}{\Gamma(n-\mu)} \int_0^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{\mu-n+1}} \left(s^{1-\rho} \frac{d}{ds} \right)^n f(s) ds. \end{aligned} \quad (6)$$

It is remarkable to note that the approach above leads to a Caputo–Hadamard derivative when $\rho \rightarrow 0$ and Caputo derivative when $\rho \rightarrow 1$. Some of its properties are provided as [31]

Lemma 1. Suppose $n \in \mathbb{N}$, $n-1 < \mu \leq n$ and $0 < \rho \leq 1$. Then, we have

$$\begin{aligned} {}^{\rho}_cD^{\mu}t^{\rho\lambda} &= \frac{\rho^{\mu}\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}t^{\rho(\lambda-\mu)}, \quad \lambda > -1, \lambda \neq 0, 1, \dots, n-1 \\ {}^{\rho}_cD^{\mu}t^{\rho\lambda} &= 0, \quad \lambda = 0, 1, \dots, n-1 \\ {}^{\rho}_cD^{\mu}A &= 0, \quad A \text{ is a constant} \\ {}^{\rho}_cD^{\mu} {}^{\rho}I^{\nu}f(t) &= {}^{\rho}I^{\nu-\mu}f(t), \quad \nu \geq \mu \\ {}^{\rho}I^{\mu} {}^{\rho}_cD^{\mu}f(t) &= f(t) - \sum_{r=0}^{n-1} a_r t^{\rho r}, \quad a_r, r = 0, 1, \dots, n-1, \text{ are constants.} \end{aligned}$$

Lemma 2. The p -Laplacian operator Φ_p , $p > 1$ satisfies

- (1) Φ_p is increasing, continuous and invertible with inverse $\Phi_p^{-1} = \Phi_q$ where $q > 1$ and $1/p + 1/q = 1$.
- (2) $|\Phi_p(x)| = \Phi_p(|x|)$ and $\Phi_p(\lambda x) = \lambda^{p-1}\Phi_p(x)$, $\lambda > 0$
- (3) Ref. [3] For all $x, y > 0$

$$\begin{aligned} \Phi_p(x+y) &\leq \Phi_p(x) + \Phi_p(y), \quad \text{if } p < 2 \\ \Phi_p(x+y) &\leq 2^{p-2}(\Phi_p(x) + \Phi_p(y)), \quad \text{if } p \geq 2. \end{aligned}$$

- (4) Refs. [33,34] For all $x, y \in \mathbb{R}$, there exist $c_1 > 0$ and $c_2 > 0$ satisfy

$$\begin{aligned} |\Phi_p(x) - \Phi_p(y)| &\leq c_1|x-y|^{p-1} & \text{if } 1 < p \leq 2 \\ |\Phi_p(x) - \Phi_p(y)| &\leq c_2(|x| + |y|)^{p-2}|x-y| & \text{if } p > 2. \end{aligned}$$

Lemma 3. Suppose that the function $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function, $\rho \in (0, 1]$, $\mu \in (0, 1]$ and $\nu \in (1, 2]$. Then, the boundary value problem

$${}^{\rho}_cD^{\mu}\Phi_p({}^{\rho}_cD^{\nu}u(t)) = f(t), \quad t \in [0, 1], p \geq 1 \quad (7)$$

subject to the assumptions

$$u(0) = 0, \quad {}^{\rho}_c D^{\nu} u(0) = 0, \quad \lim_{t \rightarrow 0} t^{1-\rho} u'(t) = 0 \quad (8)$$

has the unique solution

$$u(t) = \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\nu}} \Phi_q \left(\frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_0^s \frac{r^{\rho-1} f(r)}{(s^{\rho} - r^{\rho})^{1-\mu}} dr \right) ds. \quad (9)$$

Proof. Operate by ${}^{\rho} I^{\mu}$ on both sides of (7) with using the last statement of Lemma 1 to obtain

$$\Phi_p({}^{\rho}_c D^{\nu} u(t)) = {}^{\rho} I^{\mu} f(t) + a_0.$$

Taking the p -Laplacian inverse and the second condition in (8) produces

$${}^{\rho}_c D^{\nu} u(t) = \Phi_q({}^{\rho} I^{\mu} f(t)).$$

Again, operating by ${}^{\rho} I^{\nu}$ on all sides of the former equation and inserting the last item of Lemma 1 obtains

$$u(t) = {}^{\rho} I^{\nu} \Phi_q({}^{\rho} I^{\mu} f(t)) + a_1 + a_2 t^{\rho}.$$

Inserting the first and last conditions produces $a_1 = a_2 = 0$. By the definition (3), one can obtain (3). Conversely, by substituting (9) into the left side in (7) and applying Lemma 1, we can obtain the right side of Equation (9). It is easy to view that the solution (9) verifies all assumptions of (8). \square

2.2. Hausdorff Measure of Noncompactness and Tempered Sequence Space

Let c_0 be the sequence space of all sequences $u = (u_n)$ converging to zero and $\beta = (\beta_n)$ be a positive non-increasing real sequence. Such a sequence is called the tempering sequence. Banas and Krajewska [35] have presented the tempered sequence space

$$c_0^{\beta} = \{u \in c_0 | \beta_n u_n \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

and proved that the space c_0^{β} is a Banach space equipped the norm

$$\|u\|_{c_0^{\beta}} = \sup_{n \in \mathbb{N}} \{\beta_n |u_n|, u \in c_0\}.$$

It is worth pointing that if the tempering sequence β is a constant sequence or it is bounded from below, then the norm in the tempered sequence space c_0^{β} is equivalent to the classical supremum norm in c_0 . They have proved that the Hausdorff measure of noncompactness can be given by the formula

$$\chi_{c_0^{\beta}}(\mathcal{B}) = \lim_{m \rightarrow \infty} \sup_{n \geq m} \{\beta_n |u_n|, u \in \mathcal{B}\} \quad (10)$$

where \mathcal{B} is subset of a nonempty bounded set of the Banach space c_0^{β} and the Hausdorff measure of noncompactness is the mapping $\chi : \mathcal{B} \rightarrow [0, \infty)$ defined by

$$\chi(\mathcal{B}) = \inf\{\epsilon > 0 | \mathcal{B} \text{ has a finite } \epsilon - \text{net in Banach space}\}.$$

For extra details of Hausdorff measure of noncompactness, see [29,36].

Theorem 1 (Darbo Theorem [29]). Suppose \mathcal{B} is a closed, bounded, convex and nonempty subset of a Banach space X . Let $F : \mathcal{B} \rightarrow \mathcal{B}$ be a continuous map and there exists a positive constant $k \in [0, 1)$ satisfies the property $\chi(F\mathcal{B}) \leq k\chi(\mathcal{B})$. Then, F has a fixed point in \mathcal{B} .

3. Basic Constructions and Main Results

Consider $\mathcal{C}([0, 1], c_0^\beta)$ is the space contains continuous sequence functions defined on the interval $[0, 1]$ and belong to the space c_0^β : That is $u(t) \in \mathcal{C}([0, 1], c_0^\beta)$ leads to $\beta_n u_n(t) \rightarrow 0$ as $n \rightarrow \infty$ and $u_n : [0, 1] \rightarrow \mathbb{R}$ is continuous function for all $n \in \mathbb{N}$ where $\beta = (\beta_n)$ is a positive non-increasing sequence with $\beta_\infty \neq 0$. Consider the space

$$\mathbb{X} = \left\{ u(t) \mid u \in \mathcal{C}([0, 1], c_0^\beta) \text{ and } {}^R_c D^\alpha u(t) \in \mathcal{C}([0, 1], c_0^\beta) \right\}$$

where $0 < \alpha < \nu$ and $1 < \nu \leq 2$. It is easy to see that the space \mathbb{X} is Banach space under the norm

$$\|u\| = \|u\|_{c_0^\beta} + \|{}^R_c D^\alpha u\|_{c_0^\beta}. \quad (11)$$

Based on to the Formula (10) and Theorem 1.3 in [6], we can derive the Hausdorff measure of noncompactness in the form

$$\chi_{\mathbb{X}}(\mathcal{B}) = \lim_{m \rightarrow \infty} \sup_{n \geq m} \left\{ \max_{t \in [0, 1]} \{\beta_n |u_n(t)|\} + \max_{t \in [0, 1]} \{\beta_n |{}^R_c D^{\alpha_n} u_n(t)|\}, u \in \mathcal{B} \right\} \quad (12)$$

where \mathcal{B} is subset of a nonempty bounded set of the Banach space \mathbb{X} .

It is clear that the unique solution of the infinite system (1) and (2) comes immediately by replacing $u, \mu, \nu, \alpha, \rho$ and f by $u_n, \mu_n, \nu_n, \alpha_n, \rho_n$ and f_n , respectively, in Lemma 3. Our discussion of the existence results for the infinite system (1) and (2) will be investigated under the following assumptions:

(\mathcal{H}_1) The functions $f_n : [0, 1] \times c_0^\beta \times c_0^\beta \rightarrow c_0^\beta$ are continuous for all $n \in \mathbb{N}$ and satisfy the Lipschitz condition with Lipschitz constant L as

$$|f_n(t, u_1, v_1) - f_n(t, u_2, v_2)| \leq L(|u_1 - u_2| + |v_1 - v_2|), \quad u_i, v_i \in c_0^\beta, i = 1, 2.$$

(\mathcal{H}_2) There exist nonnegative sequence functions, $g_n(t)$ and $h_n(t)$, that satisfy the inequality

$$|f_n(t, u, v)| \leq g_n(t) + h_n(t)(|u(t)|^{p-1} + |v(t)|^{p-1})$$

for all $n \in \mathbb{N}, p > 1, t \in [0, 1]$ and $u, v \in \mathcal{C}([0, 1], c_0^\beta)$.

(\mathcal{H}_3) There are positive constants A and C such that

$$A = \sup_{n \in \mathbb{N}} \max_{t \in [0, 1]} \beta_n |g_n(t)|^{q-1} \quad \text{and} \quad C = \sup_{n \in \mathbb{N}} \max_{t \in [0, 1]} |h_n(t)|^{q-1}, \quad q > 1.$$

For the convenience of computations, let

$$\Delta_\nu = \sup_{n \in \mathbb{N}} \frac{\rho_n^{\mu_n(1-q)-\nu_n} \Gamma(\mu_n(q-1) + 1)}{\Gamma^{q-1}(\mu_n + 1) \Gamma(\mu_n(q-1) + \nu_n + 1)}. \quad (13)$$

Theorem 2. Under the suppositions above, the infinite system (1) and (2) has at least one solution in \mathbb{X} for all $p > 1$ provided that $\Delta < 1$ where

$$\Delta = 4^{(q-2)H(q-2)} C(\Delta_\nu + \Delta_{\nu-\alpha}), \quad \frac{1}{p} + \frac{1}{q} = 1$$

and $H(\cdot)$ is the Heaviside step function.

Proof. Define the operator $\mathbf{F}_p : \mathbb{X} \rightarrow \mathbb{X}$ such that

$$(\mathbf{F}_p u_n)(t) = {}^{\rho_n} I^{\nu_n} \Phi_q \left({}^{\rho_n} I^{\mu_n} f_n(t, u(t), {}^{\rho} D^{\alpha} u(t)) \right).$$

By the continuity of $f_n, n \in \mathbb{N}$ according to (\mathcal{H}_1) , the operator \mathbf{F}_p is continuous for all $p > 1$. Furthermore, it is not difficult with using Lemmas 1 and 2 to show

$$|(\mathbf{F}_p u_n)(t)| = \left| {}^{\rho_n} I^{\nu_n} \Phi_q \left({}^{\rho_n} I^{\mu_n} f_n(t, u(t), {}^{\rho} D^{\alpha} u(t)) \right) \right| \leq {}^{\rho_n} I^{\nu_n} \Phi_q \left({}^{\rho_n} I^{\mu_n} |f_n(t, u(t), {}^{\rho} D^{\alpha} u(t))| \right)$$

In view of (\mathcal{H}_2) and (\mathcal{H}_3) with using the relation (5) and the facts $\Phi_q(xy) = \Phi_q(x)\Phi_q(y)$ and $\Phi_q(x) = x^{q-1}, x > 0$, we find that

$$\begin{aligned} & \beta_n \Phi_q \left({}^{\rho_n} I^{\mu_n} |f_n(t, u(t), {}^{\rho} D^{\alpha} u(t))| \right) \\ & \leq \beta_n \Phi_q \left({}^{\rho_n} I^{\mu_n} \left[g_n(t) + h_n(t) (|u_n(t)|^{p-1} + |{}^{\rho} D^{\alpha} u(t)|^{p-1}) \right] \right) \\ & = \Phi_q \left({}^{\rho_n} I^{\mu_n} \left[\beta_n^{p-1} g_n(t) + h_n(t) ((\beta_n |u_n(t)|)^{p-1} + (\beta_n |{}^{\rho} D^{\alpha} u(t)|)^{p-1}) \right] \right) \\ & = \Phi_q (A^{p-1} + C^{p-1} (\|u\|_{c_0^{\beta}}^{p-1} + \|{}^{\rho} D^{\alpha} u\|_{c_0^{\beta}}^{p-1})) \Phi_q ({}^{\rho_n} I^{\mu_n} 1) \\ & = \Phi_q \left(A^{p-1} + C^{p-1} (\|u\|_{c_0^{\beta}}^{p-1} + \|{}^{\rho} D^{\alpha} u\|_{c_0^{\beta}}^{p-1}) \right) \Phi_q \left(\frac{s^{\rho_n \mu_n}}{\rho_n^{\mu_n} \Gamma(\mu_n + 1)} \right) \\ & = \frac{s^{\rho_n \mu_n (q-1)}}{\rho_n^{\mu_n (q-1)} \Gamma^{q-1}(\mu_n + 1)} \Phi_q \left(A^{p-1} + C^{p-1} (\|u\|_{c_0^{\beta}}^{p-1} + \|{}^{\rho} D^{\alpha} u\|_{c_0^{\beta}}^{p-1}) \right) \end{aligned}$$

From the third statement of Lemma 2, we have

$$\Phi_q \left(A^{p-1} + C^{p-1} (\|u\|_{c_0^{\beta}}^{p-1} + \|{}^{\rho} D^{\alpha} u\|_{c_0^{\beta}}^{p-1}) \right) \leq 2^{(q-2)H(q-2)} A + 4^{(q-2)H(q-2)} C \|u\|$$

Hence, we obtain

$$\begin{aligned} \|\mathbf{F}_p u\|_{c_0^{\beta}} &= \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} \beta_n |(\mathbf{F}_p u_n)(t)| \\ &= \left(2^{(q-2)H(q-2)} A + 4^{(q-2)H(q-2)} C \|u\| \right) \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} \frac{\rho_n^{\mu_n(1-q)}}{\Gamma^{q-1}(\mu_n + 1)} {}^{\rho_n} I^{\nu_n} t^{\rho_n \mu_n (q-1)} \\ &= \left(2^{(q-2)H(q-2)} A + 4^{(q-2)H(q-2)} C \|u\| \right) \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} \frac{\rho_n^{\mu_n(1-q)-\nu_n} \Gamma(\mu_n(q-1) + 1) t^{\rho_n(\mu_n(q-1)+\nu_n)}}{\Gamma^{q-1}(\mu_n + 1) \Gamma(\mu_n(q-1) + \nu_n + 1)} \\ &= \left(2^{(q-2)H(q-2)} A + 4^{(q-2)H(q-2)} C \|u\| \right) \sup_{n \in \mathbb{N}} \frac{\rho_n^{\mu_n(1-q)-\nu_n} \Gamma(\mu_n(q-1) + 1)}{\Gamma^{q-1}(\mu_n + 1) \Gamma(\mu_n(q-1) + \nu_n + 1)} \\ &= \left(2^{(q-2)H(q-2)} A + 4^{(q-2)H(q-2)} C \|u\| \right) \Delta_{\nu} \end{aligned}$$

where Δ_{ν} is defined in (13). From the 4th statement of Lemma 1, we have

$$({}^{\rho} D^{\alpha} \mathbf{F}_p u_n)(t) = {}^{\rho_n} I^{\nu_n - \alpha_n} \Phi_q \left({}^{\rho_n} I^{\mu_n} |f_n(t, u(t), {}^{\rho} D^{\alpha} u(t))| \right)$$

Similarly, we have

$$\|{}^{\rho} D^{\alpha} \mathbf{F}_p u\|_{c_0^{\beta}} \leq \left(2^{(q-2)H(q-2)} A + 4^{(q-2)H(q-2)} C \|u\| \right) \Delta_{\nu - \alpha}$$

These conclude that

$$\|\mathbf{F}_p u\| = \|\mathbf{F}_p u\|_{c_0^\beta} + \| {}^\rho_c D^\alpha \mathbf{F}_p u \|_{c_0^\beta} \leq \left(2^{(q-2)H(q-2)} A + 4^{(q-2)H(q-2)} C \|u\| \right) (\Delta_v + \Delta_{v-\alpha})$$

which leads to the boundedness of the operator $\mathbf{F}_p u$ for all $p > 1$. Now, we present the set $\mathcal{B} \subset \mathbb{X}$ such that

$$\mathcal{B} = \{u \in \mathbb{X} \mid \|u\| \leq r, u \text{ holds the conditions (2)}\}$$

The subspace \mathcal{B} is bounded, closed and convex, with fix radius r satisfies

$$\left(2^{(q-2)H(q-2)} A + 4^{(q-2)H(q-2)} C r \right) (\Delta_v + \Delta_{v-\alpha}) \leq r.$$

It is obvious that the operator $\mathbf{F}_p : \mathcal{B} \rightarrow \mathcal{B}$ is bounded. In order to prove the continuity of \mathbf{F}_p on the set \mathcal{B} , let $u, v \in \mathcal{B}$ and for all $\epsilon > 0$ there exists

$$0 < \delta < \max \left\{ \frac{\epsilon^{p-1} \beta_1^{2-p}}{L c_1^{p-1} \Delta_v^{p-1}}, \frac{\epsilon^{p-1} \beta_1^{2-p}}{L c_1^{p-1} \Delta_{v-\alpha}^{p-1}} \right\}, \quad 1 < q \leq 2, p > 2$$

$$0 < \delta < \max \left\{ \frac{\epsilon \beta_\infty^{2-p} (A + 4^{q-2} C r)^{p-2}}{2^{q-1} L c_2 \Delta_v}, \frac{\epsilon \beta_\infty^{2-p} (A + 4^{q-2} C r)^{p-2}}{2^{q-1} L c_2 \Delta_{v-\alpha}} \right\}, \quad 1 < p \leq 2, q > 2$$

where L is Lipschitz constant (\mathcal{H}_1) such that $\|u - v\| < \delta$. In fact, we obtain

$$|(\mathbf{F}_p u_n)(t) - (\mathbf{F}_p v_n)(t)| \leq {}^{\rho_n} I^{\nu_n} \left| \Phi_q \left({}^{\rho_n} I^{\mu_n} f_n(t, u(t), {}^\rho_c D^\alpha u(t)) \right) - \Phi_q \left({}^{\rho_n} I^{\mu_n} f_n(t, v(t), {}^\rho_c D^\alpha v(t)) \right) \right|$$

When $1 < q \leq 2$, using the 4th statement in Lemma 2 and (\mathcal{H}_1) obtains

$$\begin{aligned} \|\mathbf{F}_p u - \mathbf{F}_p v\|_{c_0^\beta} &= \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} \beta_n |(\mathbf{F}_p u_n)(t) - (\mathbf{F}_p v_n)(t)| \\ &\leq c_1 \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} \beta_n^{2-q} {}^{\rho_n} I^{\nu_n} \left({}^{\rho_n} I^{\mu_n} \beta_n |f_n(t, u(t), {}^\rho_c D^\alpha u(t)) - f_n(t, v(t), {}^\rho_c D^\alpha v(t))| \right)^{q-1} \\ &\leq c_1 L^{q-1} \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} \beta_n^{2-q} {}^{\rho_n} I^{\nu_n} \left(\beta_n |u(t) - v(t)| + \beta_n |{}^\rho_c D^\alpha u(t) - {}^\rho_c D^\alpha v(t)| \right)^{q-1} \\ &\leq c_1 L^{q-1} \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} \beta_n^{2-q} {}^{\rho_n} I^{\nu_n} ({}^{\rho_n} I^{\mu_n} 1)^{q-1} \|u - v\|^{q-1} \\ &\leq c_1 L^{q-1} \beta_1^{2-q} \Delta_n u \|u - v\|^{q-1} \leq c_1 L^{q-1} \beta_1^{2-q} \Delta_v \delta^{q-1} < \frac{\epsilon}{2} \end{aligned}$$

and

$$\| {}^\rho_c D^\alpha \mathbf{F}_p u - {}^\rho_c D^\alpha \mathbf{F}_p v \|_{c_0^\beta} \leq c_1 L^{q-1} \beta_1^{2-q} \Delta_n u \|u - v\|^{q-1} \leq c_1 L^{q-1} \beta_1^{2-q} \Delta_{v-\alpha} \delta^{q-1} < \frac{\epsilon}{2}$$

which mean that $\|\mathbf{F}_p u - \mathbf{F}_p v\| \leq \epsilon$ and so the operator \mathbf{F}_p is continuous on the set \mathcal{B} when $1 < q \leq 2$. Similarly, when $q > 2$, we find that

$$\begin{aligned} \|\mathbf{F}_p u - \mathbf{F}_p v\|_{c_0^\beta} &\leq c_2 \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} {}^{\rho_n} I^{\nu_n} \left[{}^{\rho_n} I^{\mu_n} \beta_n |f_n(t, u(t), {}^\rho_c D^\alpha u(t)) - f_n(t, v(t), {}^\rho_c D^\alpha v(t))| \right. \\ &\quad \left. \times \left({}^{\rho_n} I^{\mu_n} [|f_n(t, u(t), {}^\rho_c D^\alpha u(t))| + |f_n(t, v(t), {}^\rho_c D^\alpha v(t))|] \right)^{q-2} \right]. \end{aligned}$$

It is easy to see that

$${}^{\rho_n} I^{\mu_n} \beta_n |f_n(t, u(t), {}^\rho_c D^\alpha u(t)) - f_n(t, v(t), {}^\rho_c D^\alpha v(t))| \leq \frac{L t^{\rho_n \mu_n}}{\rho_n^{\mu_n} \Gamma(\mu_n + 1)} \|u - v\|$$

and, with noting $q - 2 = (2 - p)(q - 1)$,

$$\begin{aligned} & \left({}^{\rho_n} I^{\mu_n} [|f_n(t, u(t), {}^{\rho} D^{\alpha} u(t))| + |f_n(t, v(t), {}^{\rho} D^{\alpha} v(t))|] \right)^{q-2} \\ & \leq \left({}^{\rho_n} I^{\mu_n} [2g_n(t) + h_n(t)(|u|^{p-1} + |{}^{\rho} D^{\alpha} u(t)|^{p-1} + |v|^{p-1} + |{}^{\rho} D^{\alpha} v(t)|^{p-1})] \right)^{(2-p)(q-1)} \\ & \leq \beta_n^{p-2} (2^{q-1} A + 2^{3q-6} C (\|u\| + \|v\|))^{2-p} ({}^{\rho_n} I^{\mu_n} 1)^{q-2} \\ & \leq \beta_n^{p-2} (2^{q-1} A + 2^{3q-5} Cr)^{2-p} \left(\frac{t^{\rho_n \mu_n}}{\rho_n^{\mu_n} \Gamma(\mu_n + 1)} \right)^{q-2} \\ & = 2^{q-2} \beta_n^{p-2} (A + 4^{q-1} Cr)^{2-p} \left(\frac{t^{\rho_n \mu_n}}{\rho_n^{\mu_n} \Gamma(\mu_n + 1)} \right)^{q-2} \end{aligned}$$

Hence, for $q > 2$ and $1 < p < 2$, we have

$$\begin{aligned} \|\mathbf{F}_p u - \mathbf{F}_p v\|_{c_0^{\beta}} & \leq 2^{q-2} (A + 4^{q-1} Cr)^{2-p} Lc_2 \\ & \quad \times \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} \beta_n^{p-2} {}^{\rho_n} I^{\nu_n} \left(\frac{t^{\rho_n \mu_n (q-1)}}{\rho_n^{\mu_n (q-1)} \Gamma^{q-1}(\mu_n + 1)} \right) \|u - v\| \\ & \leq 2^{q-2} \beta_{\infty}^{p-2} (A + 4^{q-1} Cr)^{2-p} Lc_2 \Delta_{\nu} \delta \end{aligned}$$

and

$$\|{}^{\rho} D^{\alpha} \mathbf{F}_p u - {}^{\rho} D^{\alpha} \mathbf{F}_p v\|_{c_0^{\beta}} \leq 2^{q-2} \beta_{\infty}^{p-2} (A + 4^{q-1} Cr)^{2-p} Lc_2 \Delta_{\nu-\alpha} \delta$$

which mean that $\|\mathbf{F}_p u - \mathbf{F}_p v\| \leq \epsilon$ and so the operator \mathbf{F}_p is continuous on the set \mathcal{B} when $q > 2$. To prove that the operator $\mathbf{F}_p u(t)$ is continuous uniformly on the interval $[0, 1]$, let $0 \leq t_1 < t_2 \leq 1$. Then, we can obtain that

$$\begin{aligned} |\mathbf{F}_p u(t_2) - \mathbf{F}_p u(t_1)| & = \left| \frac{\rho_n^{1-\nu_n}}{\Gamma(\nu_n + 1)} \int_0^{t_2} \frac{s^{\rho_n-1}}{(t_2^{\rho_n} - s^{\rho_n})^{1-\nu_n}} \Phi_q \left({}^{\rho_n} I^{\mu_n} f_n(s, u(s), {}^{\rho} D^{\alpha} u(s)) \right) ds \right. \\ & \quad \left. - \frac{\rho_n^{1-\nu_n}}{\Gamma(\nu_n + 1)} \int_0^{t_1} \frac{s^{\rho_n-1}}{(t_1^{\rho_n} - s^{\rho_n})^{1-\nu_n}} \Phi_q \left({}^{\rho_n} I^{\mu_n} f_n(s, u(s), {}^{\rho} D^{\alpha} u(s)) \right) ds \right| \\ & \leq \frac{\Omega \rho_n^{1-\nu_n}}{\Gamma(\nu_n + 1)} \left(\int_0^{t_1} \left(\frac{s^{\rho_n-1}}{(t_2^{\rho_n} - s^{\rho_n})^{1-\nu_n}} - \frac{s^{\rho_n-1}}{(t_1^{\rho_n} - s^{\rho_n})^{1-\nu_n}} \right) ds + \int_{t_1}^{t_2} \frac{s^{\rho_n-1}}{(t_2^{\rho_n} - s^{\rho_n})^{1-\nu_n}} ds \right) \\ & \leq \frac{\Omega \rho_n^{1-\nu_n}}{\Gamma(\nu_n + 1)} \left(\int_0^{t_2} \frac{s^{\rho_n-1}}{(t_2^{\rho_n} - s^{\rho_n})^{1-\nu_n}} ds - \int_0^{t_1} \frac{s^{\rho_n-1}}{(t_1^{\rho_n} - s^{\rho_n})^{1-\nu_n}} ds \right. \\ & \quad \left. + \int_0^{t_2} \frac{s^{\rho_n-1}}{(t_2^{\rho_n} - s^{\rho_n})^{1-\nu_n}} ds - \int_0^{t_1} \frac{s^{\rho_n-1}}{(t_2^{\rho_n} - s^{\rho_n})^{1-\nu_n}} ds \right) \\ & \leq \frac{2\Omega \rho_n^{1-\nu_n}}{\Gamma(\nu_n + 1)} \left(\int_0^{t_2} \frac{s^{\rho_n-1}}{(t_2^{\rho_n} - s^{\rho_n})^{1-\nu_n}} ds - \int_0^{t_1} \frac{s^{\rho_n-1}}{(t_1^{\rho_n} - s^{\rho_n})^{1-\nu_n}} ds \right) \\ & = \frac{2\Omega}{\rho_n^{\nu_n} \Gamma(\nu_n + 1)} (t_2^{\rho_n \nu_n} - t_1^{\rho_n \nu_n}) \end{aligned}$$

where $\Omega = \Phi(|f_n|)$, which tends to zero uniformly as $t_1 \rightarrow t_2$. Similarly, we have

$$|{}^{\rho} D^{\alpha} \mathbf{F}_p u(t_2) - {}^{\rho} D^{\alpha} \mathbf{F}_p u(t_1)| = \frac{2\Omega}{\rho_n^{\nu_n - \alpha_n} \Gamma(\nu_n - \alpha_n + 1)} (t_2^{\rho_n(\nu_n - \alpha_n)} - t_1^{\rho_n(\nu_n - \alpha_n)}).$$

Thus, the operator $F_p u(t)$ is continuous uniformly on $[0, 1]$. The measure of noncompactness due to Hausdorff is evaluated, as above, by using (12) as

$$\begin{aligned}\chi_{\mathbb{X}}(F\mathcal{B}) &= \lim_{m \rightarrow \infty} \sup_{n \geq m} \left\{ \max_{t \in [0,1]} \{\beta_n |F_p u_n(t)|\} + \max_{t \in [0,1]} \{\beta_n | {}^{\rho_n} D^{\alpha_n} F_p u_n(t) |\}, u \in \mathcal{B} \right\} \\ &\leq 4^{(q-2)H(q-2)} C(\Delta_v + \Delta_{v-\alpha}) \lim_{m \rightarrow \infty} \sup_{n \geq m} \max_{t \in [0,1]} \beta_n (|u_n(t)| + | {}^{\rho_n} D^{\alpha_n} u_n(t) |) \\ &= 4^{(q-2)H(q-2)} C(\Delta_v + \Delta_{v-\alpha}) \chi_{\mathbb{X}}(u_n(t)) = \Delta \chi_{\mathbb{X}}(u_n(t)).\end{aligned}$$

Since $\Delta < 1$, according Darbo fixed point theorem, then our system (1)–(2) has at least a solution in the set \mathcal{B} on the unit interval $[0, 1]$. \square

4. Illustrated Numerical Example

Let us provide the example below:

$${}^{\frac{1}{2}} D^{\frac{3}{4}} \Phi_3({}^{\frac{1}{2}} D^{\frac{3}{2}} u_n(t)) = f_n(t, u(t), t^{\frac{1}{2}} u'(t)), \quad t \in [0, 1], n \in \mathbb{N}. \quad (14)$$

This system subject to the assumptions

$$u_n(0) = 0, \quad {}^{\frac{1}{2}} D^{\frac{3}{4}} u_n(0) = 0, \quad \lim_{t \rightarrow 0} t^{\frac{1}{2}} u'_n(t) = 0 \quad (15)$$

$\rho_n = 1/2, \mu_n = 3/4, \nu_n = 3/2, p = 3, q = 3/2, \alpha = 1$. Furthermore, we take the tempered sequence $(\beta_n) = (1 + 1/n)$ and

$$f_n(t, y, z) = \frac{e^{-nt} \cos^2(nt)}{(t+1)^2} + \frac{t^2 e^{-nt}}{2(2\pi - t)^2(t+n)^2} \sum_{i=n}^{\infty} \frac{\sin^2(i\pi t)}{(i+t)^2} (y_i^2 + z_i^2).$$

It is obvious that the tempered sequence $(\beta_n) = (1 + 1/n)$ is a positive decreasing sequence for all $n \in \mathbb{N}$ with $\beta_{\infty} = 1 \neq 0$. Furthermore, one can check that $h_n \in c_0$ and $h_n \in c_0^{\beta}$. In order to verify the assumption (\mathcal{H}_1) . Let $y, y', z, z' \in c_0^{\beta}$. Then,

$$\begin{aligned}|f_n(t, y, z) - f_n(t, y', z')| &\leq \frac{t^2 e^{-nt}}{2(2\pi - t)^2(t+n)^2} \sum_{i=n}^{\infty} \frac{1}{(i+t)^2} (|y_i^2 - y_i'^2| + |z_i^2 - z_i'^2|) \\ &\leq \frac{1}{2(2\pi - 1)^2} \sum_{i=1}^{\infty} \frac{1}{i^2} (|y + y'| |y - y'| + |z + z'| |z - z'|) \\ &\leq L(|y - y'| + |z - z'|)\end{aligned}$$

where $L = \frac{\pi^2 r}{6(2\pi - 1)^2}$ and r is the radius of the closed ball \mathcal{B} which is fully compatible with the assumption (\mathcal{H}_1) . It is easy to see that

$$\begin{aligned}|f_n(t, y, z)| &\leq \frac{e^{-nt} \cos^2(nt)}{(t+1)^2} + \frac{t^2 e^{-nt}}{2(2\pi - t)^2(t+n)^2} \sum_{i=n}^{\infty} \frac{1}{(i+t)^2} (|y_n|^2 + |z_n|^2) \\ &= g_n(t) + h_n(t)(|y_n|^2 + |z_n|^2)\end{aligned}$$

which is fully coincident with the assumption (\mathcal{H}_2) , where

$$g_n(t) = \frac{e^{-nt} \cos^2(nt)}{(t+1)^2}, \quad h_n(t) = \frac{t^2 e^{-nt}}{2(2\pi - t)^2(t+n)^2} \sum_{i=n}^{\infty} \frac{1}{(i+t)^2}.$$

According the third assumption (\mathcal{H}_3) , we find that

$$C = \sup_{n \in \mathbb{N}} \max_{t \in [0,1]} |h_n(t)|^{q-1} = \left(\frac{\pi^2}{12(2\pi-1)^2} \right)^{\frac{1}{2}} = \frac{\pi}{\sqrt{12}(2\pi-1)} \simeq 0.171658.$$

Therefore,

$$\Delta = 4^{(q-2)H(q-2)} C(\Delta_\nu + \Delta_{\nu-\alpha}) \simeq (0.5)(0.171658)(1.90248 + 1.78358) = 0.316371 < 1$$

These lead to all assumptions of the theorem are satisfied. Hence, the infinite system (1) and (2) has at least one solution in $[0, 1]$.

5. Conclusions

In the present study, we investigated an infinite system of fractional order with p -Laplacian operator. We used the Caputo–Katugampola derivative in our model, which related to several well-known fractional derivatives. The existence of solution to our infinite system is discussed by using the Darbo’s fixed point theorem through applying the Hausdorff measure of noncompactness technique. A new sequence space related to c_0 space is presented to be our domain. An illustrated numerical example is provided to show that the applicability of our idea in practice.

Author Contributions: A.S., L.A. and F.A. contributed equally and significantly in writing this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, grant number (G:001-130-1442). The authors, therefore, gratefully acknowledge DSR technical and financial support.

Informed Consent Statement: Not applicable

Data Availability Statement: Not applicable.

Acknowledgments: We are very grateful to the anonymous referees for their suggestions and valuable advices.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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