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Article

Hilfer–Hadamard Fractional Boundary Value Problems with Nonlocal Mixed Boundary Conditions

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Abstract: This paper is concerned with the existence and uniqueness of solutions for a Hilfer–Hadamard fractional differential equation, supplemented with mixed nonlocal (multi-point, fractional integral multi-order and fractional derivative multi-order) boundary conditions. The existence of a unique solution is obtained via Banach contraction mapping principle, while the existence results are established by applying the fixed point theorems due to Krasnoselskiĭ and Schaefer and Leray–Schauder nonlinear alternatives. We demonstrate the application of the main results by presenting numerical examples. We also derive the existence results for the cases of convex and non-convex multifunctions involved in the multi-valued analogue of the problem at hand.

Keywords: Hilfer–Hadamard fractional derivative; Riemann–Liouville fractional derivative; Caputo fractional derivative; fractional differential equations; inclusions; nonlocal boundary conditions; existence and uniqueness; fixed point

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1. Introduction

Fractional calculus is regarded as the generalization of the integer-order integration and differentiation in the sense that it deals with derivative and integral operators of an arbitrary real or complex order. This branch of mathematical analysis gained much importance during the last few decades owing to its widespread applications in a variety of disciplines, such as mechanical engineering, bioengineering, biology, physics, chemistry, economics, viscoelasticity, acoustics, optics, robotics, control theory, electronics, etc. The main reason for the popularity of fractional calculus is that mathematical models based on fractional-order operators are considered to be more realistic than the ones relying on classical calculus as such operators are nonlocal in nature and can trace the history of the phenomena under consideration. For the theoretical development of the subject, we refer the reader to the monographs [1–9] and the references therein. For some recent applications of fractional calculus concerning structural mechanics and, more specifically, nonlocal elasticity, see [10–12].

Fractional-order boundary value problems constitute an important and interesting area of research. It reflects from the literature on the topic that a good deal of work on fractional differential equations involve either Caputo or Riemann–Liouville fractional derivatives. However, these derivatives are found to be inappropriate in the study of some engineering problems. In order to tackle such inaccuracies, some new fractional-order derivative operators such as Hadamard, Erdeyl–Kober, Katugampola, etc., were proposed. In [13], Hilfer introduced a new derivative, which is known as the Hilfer fractional derivative and can generalize both Riemann–Liouville and Caputo derivatives. For some applications of this derivative, we refer the interested reader to the investigations [14,15]. For some recent results on initial and boundary value problems involving Hilfer fractional derivative, for instance, see [16–22] and the references cited therein.

The fractional derivative presented by Hadamard in 1892 [23] differs from the well known Caputo derivative in two significant ways: (i) its kernel involves a logarithmic function with an arbitrary exponent and (ii) the Hadamard derivative of a constant is not zero. One can find applications of the Hadamard derivative and integral operators in the paper [24] and the monograph [2]. The Hadamard calculus can be obtained by changing $d/dt \to td/dt$, $(t-s)^{(\cdot)} \to (\log_e t - \log_e s)^{(\cdot)}$ and $ds \to (1/s)ds$ in Riemann–Liouville and Caputo fractional derivatives. Later, the modification of Hilfer fractional derivative resulted in the concept of the Hilfer–Hadamard derivative.

Existence results for Hilfer–Hadamard fractional differential equations of order in $(0,1]$ were studied by several researchers, for instance, see [25–27]. To the best of our knowledge, only a few results are available in the literature concerning boundary value problems for Hilfer–Hadamard fractional differential equations of order in $(1,2]$. Recently, in [28], the authors applied the tools of the fixed-point theory to study the existence and uniqueness of solutions for a boundary value problem of Hilfer–Hadamard fractional differential equations with nonlocal integro-multi-point boundary conditions:

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta}x(t) = f(t,x(t)), \quad t \in [1,T], \\ x(1) = 0, \quad \displaystyle\sum_{i=1}^{m} \theta_i x(\xi_i) = \lambda\, {}^{H}I_1^{\delta} x(\eta), \end{cases}$$

where ${}^{HH}D_1^{\alpha,\beta}$ denote the Hilfer–Hadamard fractional derivative operator of order $\alpha \in (1,2]$ and type $\beta \in [0,1]$, $\theta_i, \lambda \in \mathbb{R}$ and $i = 1,2,\ldots,m$, are given constants, $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and ${}^{H}I^{\delta}$ is the Hadamard fractional integral of order $\delta > 0$ and $\eta, \xi_i \in (1,T)$, $i = 1,2,\ldots,m$.

In [29], the existence of solutions for the following system of sequential fractional differential equations involving Hilfer–Hadamard type differential operators ${}_H\mathfrak{D}^{\cdot,\cdot}$ of different orders was discussed:

$$\begin{cases} ({}_H\mathfrak{D}_{1^+}^{\alpha_1,\beta_1} + \lambda_1 {}_H\mathfrak{D}_{1^+}^{\alpha_1-1,\beta_1})u(t) = f(t,u(t),v(t)), \;\; 1 < \alpha_1 \le 2, \;\; t \in [1,e], \\ ({}_H\mathfrak{D}_{1^+}^{\alpha_2,\beta_2} + \lambda_2 {}_H\mathfrak{D}_{1^+}^{\alpha_2-1,\beta_2})v(t) = g(t,u(t),v(t)), \;\; 1 < \alpha_2 \le 2, \;\; t \in [1,e], \\ u(1) = 0, \;\; u(e) = A_1, \;\; A_1 \in \mathbb{R}_+, \\ v(1) = 0, \;\; v(e) = A_2, \;\; A_2 \in \mathbb{R}_+, \end{cases}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and $f\ g : [1,e] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions.

Motivated by the aforementioned work, our goal in this paper is to enrich the literature on boundary value problems of Hilfer–Hadamard fractional differential equations of order in $(1,2]$. In precise terms, we introduce and study a nonlocal mixed Hilfer–Hadamard boundary value problem of the following form:

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta}x(t) = f(t,x(t)), \quad t \in [1,T], \\ x(1) = 0, \quad x(T) = \displaystyle\sum_{j=1}^{m} \eta_j x(\xi_j) + \sum_{i=1}^{n} \zeta_i\, {}^{H}I_1^{\phi_i} x(\theta_i) + \sum_{k=1}^{r} \lambda_k\, {}_{H}D_1^{\omega_k} x(\mu_k), \end{cases} \tag{1}$$

where ${}^{HH}D_1^{\alpha,\beta}$ denotes the Hilfer–Hadamard fractional derivative operator of order $\alpha \in (1,2]$ and type $\beta \in [0,1]$ and $\eta_j, \zeta_i, \lambda_k \in \mathbb{R}$ are given constants, $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{H}I^{\phi_i}$ is the Hadamard fractional integral operator of order $\phi_i > 0$ and $\xi_j, \theta_i, \mu_k \in (1,T)$, $j = 1,2,\ldots,m$, $i = 1,2,\ldots,n$, $k = 1,2,\ldots,r$. We also study the multi-valued analogue of the problem (1).

Concerning the significance of problem (1), we recall that the Hilfer fractional derivative interpolates between the Riemann–Liouville and Caputo derivatives [13]. Analogously, the Hilfer–Hadamard type fractional derivative covers the cases of the Riemann–Liouville–

Hadamard and Caputo–Hadamard fractional derivatives. Thus, the present study will be useful for improving the works related to glass forming materials [14], Turbulent Flow Model [30], etc. Furthermore, several results involving the Caputo–Hadamard fractional derivative [31–34] can be extended to the framework of Hilfer–Hadamard fractional derivative.

It is well known that the nonlocal condition is more appropriate than the local condition (initial and/or boundary) with respect to describing certain features of applied mathematics and physics correctly (see the survey paper [35]). More specifically, the boundary conditions arising in the study of boundary value problems of nonlocal elasticity are always nonlocal in nature. This is due to the fact that the long-range interactions within nonlocal solids give rise to nonlocal traction (force) boundary conditions.

Here, we remark that there are only two articles [28,29] in the literature (to the best of our knowledge) concerning boundary value problems for Hilfer–Hadamard fractional differential equations of the order in $(1, 2]$. Much of the known studies in the literature deals with initial value problems of Hilfer–Hadamard fractional differential equations of the order in $(0, 1]$. The two classes of problems are entirely different. The methodology employed to study the Hilfer–Hadamard fractional differential equations of the order in $(0, 1]$ is different from the one applied to such equations of the order in $(1, 2]$. Thus, our main objective in this paper is to enrich the new research area on Hilfer–Hadamard fractional differential equations of the order in $(1, 2]$. Moreover, the mixed boundary conditions introduced in the present study are of a more general type and include multi-point, fractional integral multi-order and fractional derivative multi-order contributions.

One can notice that the boundary conditions considered in problem (1) reduce to several special cases such as (i) nonlocal multi-point boundary conditions if we choose all $\zeta_i = 0, i = 1, 2, \ldots, n$ and $\lambda_k = 0, k = 1, 2, \ldots, r$; (ii) nonlocal Hadamard fractional integral boundary conditions when all $\eta_j = 0, j = 1, 2, \ldots, m$ and $\lambda_k = 0, k = 1, 2, \ldots, r$; and (iii) nonlocal Hadamard fractional boundary conditions if we take all $\eta_j = 0, j = 1, 2, \ldots, m$ and $\zeta_i = 0, i = 0, k = 1, 2, \ldots, n$. Likewise, we can consider the combination of nonlocal multipoint and Hadamard fractional integral conditions when we fix all $\lambda_k = 0$, $k = 1, 2, \ldots, r$ and so on. Thus, the results presented in this paper are significant as they specialize to the ones associated with several interesting boundary conditions. Another novelty in the present work is concerned with the derivation of the existence results for the Hilfer–Hadamard fractional differential inclusions of the order in $(1, 2]$ supplemented with the mixed boundary conditions. Thus, the investigation of single-valued and multi-valued nonlocal nonlinear Hilfer–Hadamard fractional boundary value problems of the order in $(1, 2]$ enhances the scope of the literature on the topic.

The remaining part of this manuscript is arranged as follows. Section 2 contains some basic notions and known results of fractional differential calculus. In Section 3, we first prove an auxiliary result that plays a key role in transforming the given problem into a fixed point problem. Then, based on Banach's contraction mapping principle, we establish the existence of a unique solution for the problem (1). By using the fixed point theorems due to Krasnoselskiĭ and Schaefer and nonlinear alternative of Leray–Schauder type, we prove some existence results for problem (1). Examples illustrating the applicability of the main results are also presented in this section. The existence results for the multi-valued analogue of the problem (1) are obtained in Section 4. Some interesting observations are presented in the last section of the paper.

2. Preliminaries

In this section, we recall some basic concepts.

Definition 1. *(Hadamard fractional integral [2]). Let $f : [a, \infty) \to \mathbb{R}$. Then, the Hadamard fractional integral of order $\alpha > 0$ is defined as follows:*

$$ {}^{H}I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{z} \right)^{\alpha - 1} \frac{f(z)}{z} dz, \quad t > a, \tag{2} $$

provided that the integral exists, where $\log(.) = \log_e(.)$.

Definition 2. *(Hadamard fractional derivative [2]). For a function* $f : [a, \infty) \to \mathbb{R}$, *the Hadamard fractional derivative of order* $\alpha > 0$ *is defined as follows:*

$$ {}_H D_a^\alpha f(t) = \delta^n \left({}^H I_a^{n-\alpha} f \right)(t), \qquad n = [\alpha] + 1, \tag{3}$$

where $\delta^n = (t\frac{d}{dt})^n$ *and* $[\alpha]$ *denote the integer part of the real number* α.

Lemma 1. *[2] If* $\alpha > 0, \beta > 0$ *and* $0 < a < b < \infty$, *then*

(i) $$\left({}^H I_{a^+}^\alpha \left(\log \frac{t}{a} \right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\log \frac{x}{a} \right)^{\beta+\alpha-1};$$

(ii) $$\left({}_H D_{a^+}^\alpha \left(\log \frac{t}{a} \right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{x}{a} \right)^{\beta-\alpha-1}.$$

In particular, if $\beta = 1$, *then the following is the case:*

$$({}_H D_{a^+}^\alpha)(1) = \frac{1}{\Gamma(1-\alpha)} \left(\log \frac{x}{a} \right)^{-\alpha} \neq 0, \ \ 0 < \alpha < 1.$$

Definition 3. *(Hilfer–Hadamard fractional derivative [15]). Let* $f \in L^1(a,b)$ *and* $n-1 < \alpha < n$, $0 \le \beta \le 1$. *We define the Hilfer–Hadamard fractional derivative of order* α *and type* β *for* f *as follows:*

$$\begin{aligned} ({}^{HH}D_a^{\alpha,\beta} f)(t) &= ({}^H I_a^{\beta(n-\alpha)} \delta^n \, {}^H I_a^{(n-\alpha)(1-\beta)} f)(t) \\ &= ({}^H I_a^{\beta(n-\alpha)} \delta^n \, {}^H I_a^{(n-\gamma)} f)(t) \\ &= ({}^H I_a^{\beta(n-\alpha)} \, {}_H D_a^\gamma f)(t), \quad \gamma = \alpha + n\beta - \alpha\beta, \end{aligned}$$

where ${}^H I_a^{(.)}$ *and* ${}_H D_a^{(.)}$ *are defined by (2) and (3), respectively.*

Here, we remark that the Hilfer–Hadamard fractional derivative reduces to the Hadamard fractional derivative for $\beta = 0$ and corresponds to the Caputo–Hadamard derivative for $\beta = 1$ given in the following equation:

$$ {}_H^C D_a^\alpha f(t) = \left({}^H I_a^{n-\alpha} \delta^n f \right)(t), \qquad n = [\alpha] + 1.$$

Next, we recall the following known theorem that will be used in the sequel.

Theorem 1. *([5]). Let* $\alpha > 0$, $0 \le \beta \le 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n = [\alpha] + 1$ *and* $0 < a < b < \infty$. *If* $f \in L^1(a,b)$ *and* $({}^H I_a^{n-\gamma} f)(t) \in AC_\delta^n[a,b]$, *then the following is the case:*

$$\begin{aligned} {}^H I_a^\alpha ({}^{HH}D_a^{\alpha,\beta} f)(t) &= {}^H I_a^\gamma ({}^{HH}D_a^\gamma f)(t) \\ &= f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}^H I_a^{n-\gamma} f))(a)}{\Gamma(\gamma - j)} \left(\log \frac{t}{a} \right)^{\gamma-j-1}. \end{aligned}$$

Observe that $\Gamma(\gamma - j)$ *exists for all* $j = 1, 2, \ldots, n-1$ *for* $\gamma \in [\alpha, n]$.

3. Main Results

This section is concerned with the existence and uniqueness of solutions for the nonlinear Hilfer–Hadamard fractional boundary value problem (1). First of all, we prove an auxiliary lemma dealing with the linear variant of the boundary value problem (1), which will be used to transform the problem at hand into an equivalent fixed point problem. In the case $n = [\alpha] + 1 = 2$, we have $\gamma = \alpha + (2-\alpha)\beta$.

Lemma 2. *Let $h \in C([1,T],\mathbb{R})$ and that*

$$\begin{aligned}\Lambda &= (\log T)^{\gamma-1} - \sum_{j=1}^{m} \eta_j (\log \xi_j)^{\gamma-1} - \sum_{i=1}^{n} \zeta_i \frac{\Gamma(\gamma)}{\Gamma(\gamma+\phi_i)} (\log \theta_i)^{\gamma+\phi_i-1} \\ &\quad - \sum_{k=1}^{r} \lambda_k \frac{\Gamma(\gamma)}{\Gamma(\gamma-\omega_k)} (\log \mu_k)^{\gamma-\omega_k-1} \neq 0. \end{aligned} \tag{4}$$

Then, x is a solution of the following linear Hilfer–Hadamard fractional boundary value problem:

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta} x(t) = h(t,), \quad t \in [1,T], \\ x(1) = 0, \quad x(T) = \displaystyle\sum_{j=1}^{m} \eta_j x(\xi_j) + \sum_{i=1}^{n} \zeta_i \, {}^{H}I_1^{\phi_i} x(\theta_i) + \sum_{k=1}^{r} \lambda_k \, {}_{H}D_1^{\omega_k} x(\mu_k), \end{cases} \tag{5}$$

if and only if it satisfies the integral equation:

$$\begin{aligned} x(t) &= {}^{H}I_1^{\alpha} h(t) + \frac{(\log t)^{\gamma-1}}{\Lambda} \Bigg\{ \sum_{j=1}^{m} {}^{H}I^{\alpha} h(\xi_j) + \sum_{i=1}^{n} \zeta_i {}^{H}I^{\alpha+\phi_i} h(\theta_i) \\ &\quad + \sum_{k=1}^{r} \lambda_k {}^{H}I^{\alpha-\omega_k} h(\mu_k) - {}^{H}I^{\alpha} h(T) \Bigg\}, \; t \in [1,T]. \end{aligned} \tag{6}$$

Proof. Applying the Hadamard fractional integral operator of order α from 1 to t on both sides of Hilfer–Hadamard fractional differential equation in (5) and using Theorem 1, we find that

$$x(t) - \frac{\delta({}_{H}I_{1^+}^{2-\gamma} x)(1)}{\Gamma(\gamma)} (\log t)^{\gamma-1} - \frac{({}_{H}I_{1^+}^{2-\gamma} x)(1)}{\Gamma(\gamma-1)} (\log t)^{\gamma-2} = {}^{H}I_1^{\alpha} h(t), \tag{7}$$

which can be rewritten as follows:

$$x(t) = c_0 (\log t)^{\gamma-1} + c_1 (\log t)^{\gamma-2} + \frac{1}{\Gamma(\alpha)} \int_1^t \frac{h(s)}{s} \left(\log \frac{t}{s} \right)^{\alpha-1} ds, \tag{8}$$

where c_0 and c_1 are arbitrary constants. Using the first boundary condition ($x(1) = 0$) in (8) yields $c_1 = 0$, since $\gamma \in [\alpha, 2]$. In consequence, (8) takes the following form:

$$x(t) = c_0 (\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{h(s)}{s} ds. \tag{9}$$

Now, inserting (9) into the second boundary condition:

$$x(T) = \sum_{j=1}^{m} \eta_j x(\xi_j) + \sum_{i=1}^{n} \zeta_i \, {}^{H}I_1^{\phi_i} x(\theta_i) + \sum_{k=1}^{r} \lambda_k \, {}_{H}D_1^{\omega_k} x(\mu_k),$$

and using notation (4), we obtain the following:

$$c_0 = \frac{1}{\Lambda} \Bigg\{ \sum_{j=1}^{m} {}^{H}I^{\alpha} h(\xi_j) + \sum_{i=1}^{n} \zeta_i {}^{H}I^{\alpha+\phi_i} h(\theta_i) + \sum_{k=1}^{r} \lambda_k {}^{H}I^{\alpha-\omega_k} h(\mu_k) - {}^{H}I^{\alpha} h(T) \Bigg\}.$$

Substituting the value of c_0 in (9) results in Equation (6) as desired. By direct computation, one can obtain the converse of the lemma. The proof is completed. □

Let $X = C\big([1,T],\mathbb{R}\big)$ be the Banach space endowed with the norm

$$\|x\| := \max_{t \in [1,T]} |x(t)|.$$

In view of Lemma 2 and Definition 1, we introduce an operator $\mathcal{F} : X \to X$ associated with the problem (1) as follows:

$$\begin{aligned}
\mathcal{F}(x)(t) \quad = \quad & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{z}\right)^{\alpha-1} \frac{f(z, x(z))}{z} dz \\
& + \frac{(\log t)^{\gamma-1}}{\Lambda} \Bigg\{ \sum_{j=1}^{m} \frac{\eta_j}{\Gamma(\alpha)} \int_1^{\xi_j} \left(\log \frac{\xi_j}{z}\right)^{\alpha-1} \frac{f(z, x(z))}{z} dz \\
& + \sum_{i=1}^{n} \frac{\zeta_i}{\Gamma(\alpha+\phi_i)} \int_1^{\theta_i} \left(\log \frac{\theta_i}{z}\right)^{\alpha+\phi_i-1} \frac{f(z, x(z))}{z} dz \\
& + \sum_{k=1}^{r} \frac{\lambda_k}{\Gamma(\alpha-\omega_k)} \int_1^{\mu_k} \left(\log \frac{\mu_k}{z}\right)^{\alpha-\omega_k-1} \frac{f(z, x(z))}{z} dz \\
& - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{z}\right)^{\alpha-1} \frac{f(z, x(z))}{z} dz \Bigg\}, \; t \in [1, T]. \qquad (10)
\end{aligned}$$

In the sequel, we use the following notation:

$$\begin{aligned}
\Omega \quad = \quad & \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \Bigg\{ \sum_{i=j}^{m} \frac{|\eta_j|(\log \xi_j)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{n} \frac{|\zeta_i|(\log \theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} \\
& + \sum_{k=1}^{r} \frac{|\lambda_k|(\log \mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \Bigg\}. \qquad (11)
\end{aligned}$$

3.1. Uniqueness Result

Here, by applying Banach's contraction mapping principle [36], we prove the existence of a unique solution for problem (1).

Theorem 2. *Suppose that the following condition holds:*
(H_1) *There exists a constant $l > 0$ such that for all $t \in [1, T]$ and $u_i \in \mathbb{R}$, $i = 1, 2,$.*

$$|f(t, u_1) - f(t, u_2)| \;\le\; l|u_1 - u_2|.$$

Then, the nonlinear Hilfer–Hadamard fractional boundary value problem (1) has a unique solution on $[1, T]$ if $l\Omega < 1$, where Ω is defined by (11).

Proof. We will verify that the operator $\mathcal{F}$ defined by (10) satisfies the hypotheses of Banach's contraction mapping principle. Fixing $N = \max_{t\in[1,T]} |f(t, 0)| < \infty$ and using the assumption (H_1), we obtain the following:

$$|f(t, x(t))| \le l|x(t)| + |f(t, 0)| \le l\|x\| + N. \qquad (12)$$

The proof is divided into two steps.
Step I : We show that $\mathcal{F}(B_r) \subset B_r$, where $B_r = \{x \in X : \|x\| < r\}$ with $r \ge N\Omega/(1 - l\Omega)$. Let $x \in B_r$. Then, we have the following:

$$
\begin{aligned}
|\mathcal{F}(x)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|f(z,x(z))|}{z}dz \\
&\quad+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{|\eta_j|}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{|f(z,x(z))|}{z}dz\right. \\
&\quad+\sum_{i=1}^{n}\frac{|\zeta_i|}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{|f(z,x(z))|}{z}dz \\
&\quad+\sum_{k=1}^{r}\frac{|\lambda_k|}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{|f(z,x(z))|}{z}dz \\
&\quad\left.+\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{|f(z,x(z))|}{z}dz\right\} \\
&\leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}(l\|x\|+N)+\frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right. \\
&\quad\left.+\sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\}(l\|x\|+N) \\
&\leq \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right.\right. \\
&\quad\left.\left.+\sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\}\right](lr+N) \\
&= \Omega(lr+N)\leq r.
\end{aligned}
$$

Thus, the following is the case:

$$
\|\mathcal{F}(x)\| = \max_{t\in[1,T]}|\mathcal{F}(u)(t)| \leq r,
$$

which means that $\mathcal{F}(B_r)\subset B_r$.

Step II: To show that the operator $\mathcal{F}$ is a contraction, let $x_1, x_2 \in X$. Then, for any $t\in[1,T]$, we have the following:

$$
\begin{aligned}
&|\mathcal{F}(x_2)(z)-\mathcal{F}(x_1)(z)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|f(z,x_2(z))-f(z,x_1(z))|}{z}dz \\
&+\frac{(\log t)^{\gamma-1}}{|\Lambda|}\left\{\sum_{j=1}^{m}\frac{|\eta_j|}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{|f(z,x_2(z))-f(z,x_1(z))|}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{|\zeta_i|}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{|f(z,x_2(z))-f(z,x_1(z))|}{z}dz \\
&+\sum_{k=1}^{r}\frac{|\lambda_k|}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{|f(z,x_2(z))-f(z,x_1(z))|}{z}dz \\
&\left.+\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|f(z,x_2(z))-f(z,x_1(z))|}{z}dz\right\} \\
\leq\ & \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right.\right. \\
&\left.\left.+\sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\}\right]l\|x_2-x_1\|.
\end{aligned}
$$

Thus, the following is the case:

$$\|\mathcal{F}(x_2)-\mathcal{F}(x_1)\| = \max_{t\in[1,T]} |\mathcal{F}(x_2)(t)-\mathcal{F}(x_1)(t)| \leq l\Omega\|x_2-x_1\|,$$

which, in view of $l\Omega < 1$, shows that the operator $\mathcal{F}$ is a contraction. Hence, the operator $\mathcal{F}$ has a unique fixed point by Banach's contraction mapping principle. Therefore, the problem (1) has a unique solution on $[1,T]$. The proof is completed. □

3.2. Existence Results

In this subsection, we present different criteria for the existence of solutions for the problem (1). First, we prove an existence result based on Krasnoselskiĭ's fixed point theorem [37].

Theorem 3. *Let $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (H_1). In addition, we assume that the following condition is satisfied:*
(H_2) There exists a continuous function $\phi\in C([1,T],\mathbb{R}^+)$ such that

$$|f(t,x)| \leq \phi(t), \ \textit{for each} \ (t,u)\in[1,T]\times\mathbb{R}.$$

Then, the nonlinear Hilfer–Hadamard fractional boundary value problem (1) has at least one solution on $[1,T]$, provided that the following condition holds:

$$\begin{aligned}\frac{(\log T)^{\gamma-1}}{|\Lambda|}\Bigg\{&\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\\ &+\sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Bigg\}l<1. \end{aligned}\tag{13}$$

Proof. By assumption (H_2), we can fix $\rho\geq\Omega\|\phi\|$ and consider a closed ball $B_\rho=\{x\in C([1,T],\mathbb{R}):\|x\|\leq\rho\}$, where $\|\phi\|=\sup_{t\in[1,T]}|\phi(t)|$ and Ω is given by (11). We verify the hypotheses of Krasnoselskiĭ's fixed point theorem [37] by splitting the operator $\mathcal{F}$ defined by (10) on B_ρ to $C([1,T],\mathbb{R})$ as $\mathcal{F}=\mathcal{F}_1+\mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined by the following:

$$(\mathcal{F}_1x)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{f(z,x(z))}{z}dz, \ \ t\in[1,T],$$

$$\begin{aligned}(\mathcal{F}_2x)(t) = \frac{(\log t)^{\gamma-1}}{\Lambda}\Bigg\{&\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{f(z,x(z))}{z}dz\\ &+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{f(z,x(z))}{z}dz\\ &+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{f(z,x(z))}{z}dz\\ &-\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{f(z,x(z))}{z}dz\Bigg\}, \ \ t\in[1,T].\end{aligned}$$

For any $x,y\in B_\rho$, we have the following:

$$
\begin{aligned}
|(\mathcal{F}_1x)(t)+(\mathcal{F}_2y)(t)| \;\leq\;& \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|f(z,x(z))|}{z}dz \\
&+\frac{(\log t)^{\gamma-1}}{|\Lambda|}\left\{\sum_{j=1}^{m}\frac{|\eta_j|}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{|f(z,x(z))|}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{|\zeta_i|}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{|f(z,x(z))|}{z}dz \\
&+\sum_{k=1}^{r}\frac{|\lambda_k|}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{|f(z,x(z))|}{z}dz \\
&\left.+\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|f(z,x(z))|}{z}dz\right\} \\
\leq\;& \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right.\right. \\
&\left.\left.+\sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\}\right]\|\phi\| \\
=\;& \Omega\|\phi\|\leq\rho.
\end{aligned}
$$

Hence, $\|\mathcal{F}_1x+\mathcal{F}_2y\|\leq\rho$, which shows that $\mathcal{F}_1x+\mathcal{F}_2y\in B_\rho$. By condition (13), it is easy to prove that the operator $\mathcal{F}_2$ is a contraction mapping. The operator $\mathcal{F}_1$ is continuous by the continuity of f. Moreover, $\mathcal{F}_1$ is uniformly bounded on B_ρ, since

$$
\|\mathcal{F}_1x\|\leq\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|\phi\|.
$$

Finally, we prove the compactness of the operator $\mathcal{F}_1$. For $t_1,t_2\in[1,T], t_1<t_2$, we have the following case:

$$
\begin{aligned}
&|\mathcal{F}_1x(t_2)-\mathcal{F}_1x(t_1)| \\
\leq\;& \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{z}\right)^{\alpha-1}-\left(\log\frac{t_1}{z}\right)^{\alpha-1}\right]\frac{|f(z,x(z))|}{z}dz \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{z}\right)^{\alpha-1}\frac{|f(z,x(z))|}{z}dz \\
\leq\;& \frac{\|\phi\|}{\Gamma(\alpha+1)}\Big[2(\log t_2-\log t_1)^{\alpha}+|(\log t_2)^{\alpha}-(\log t_1)^{\alpha}|\Big],
\end{aligned}
$$

which tends to zero independently of $x\in B_\rho$, as $t_1\to t_2$. Thus, $\mathcal{F}_1$ is equicontinuous. By the application of the Arzelá–Ascoli theorem, we deduce that operator $\mathcal{F}_1$ is compact on B_ρ. Thus, the hypotheses of Krasnoselskiĭ's fixed point theorem [37] hold true. In consequence, there exists at least one solution for the nonlinear Hilfer–Hadamard fractional boundary value problem (1) on $[1,T]$, which completes the proof. □

Our next existence result is based on Schaefer's fixed point theorem [38].

Theorem 4. *Let $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the following assumption:*
(H_3) There exists a real constant $M>0$ such that for all $t\in[1,T], u\in\mathbb{R}$,

$$
|f(t,u)|\;\leq M.
$$

Then, there exists at least one solution for the nonlinear Hilfer–Hadamard fractional boundary value problem (1) on $[1,T]$.

Proof. We will prove that the operator $\mathcal{F}$, defined by (10), has a fixed point by using Schaefer's fixed point theorem [38]. The proof is given in two steps.
Step I. We show that the operator $\mathcal{F} : X \to X$ is completely continuous.
Let us first establish that $\mathcal{F}$ is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in X. Then, for each $t \in [1, T]$, we have the following:

$$
\begin{aligned}
& |\mathcal{F}(x_n)(t) - \mathcal{F}(x)(t)| \\
\le\ & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{z}\right)^{\alpha-1} \frac{|f(z, x_n(z)) - f(z, x(z))|}{z} dz \\
& + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \sum_{j=1}^{m} \frac{|\eta_j|}{\Gamma(\alpha)} \int_1^{\xi_j} \left(\log \frac{\xi_j}{z}\right)^{\alpha-1} \frac{|f(x, x_n(z)) - f(z, x(z))|}{z} dz \right. \\
& + \sum_{i=1}^{n} \frac{|\zeta_i|}{\Gamma(\alpha+\phi_i)} \int_1^{\theta_i} \left(\log \frac{\theta_i}{z}\right)^{\alpha+\phi_i-1} \frac{|f(z, x_n(z)) - f(z, x(z))|}{z} dz \\
& + \sum_{k=1}^{r} \frac{|\lambda_k|}{\Gamma(\alpha-\omega_k)} \int_1^{\mu_k} \left(\log \frac{\mu_k}{z}\right)^{\alpha-\omega_k-1} \frac{|f(z, x_n(z)) - f(z, x(z))|}{z} dz \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{t}{z}\right)^{\alpha-1} \frac{|f(z, x_n(z)) - f(z, x(z))|}{z} dz \right\}.
\end{aligned}
$$

Taking into account the fact that f is continuous, that is, $|f(s, x_n(s)) - f(s, x(s))| \to 0$ *as* $x_n \to x$, we obtain from the foregoing inequality that the following is the case:

$$\|\mathcal{F}(x_n) - \mathcal{F}(x)\| \to 0 \ as\ x_n \to x.$$

Hence, $\mathcal{F}$ is continuous.

Now we show that the operator $\mathcal{F}$ which maps bounded sets into bounded sets in X. For $R > 0$, let $B_R = \{x \in X : \|x\| \le R\}$. Then, for $t \in [1, T]$, we have the following case:

$$
\begin{aligned}
|\mathcal{F}(x)(t)| \ \le\ & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{z}\right)^{\alpha-1} \frac{|f(z, x(z))|}{z} dz \\
& + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \left\{ \sum_{j=1}^{m} \frac{|\eta_j|}{\Gamma(\alpha)} \int_1^{\xi_j} \left(\log \frac{\xi_j}{z}\right)^{\alpha-1} \frac{|f(z, x(z))|}{z} dz \right. \\
& + \sum_{i=1}^{n} \frac{|\zeta_i|}{\Gamma(\alpha+\phi_i)} \int_1^{\theta_i} \left(\log \frac{\theta_i}{z}\right)^{\alpha+\phi_i-1} \frac{|f(z, x(z))|}{z} dz \\
& + \sum_{k=1}^{r} \frac{|\lambda_k|}{\Gamma(\alpha-\omega_k)} \int_1^{\mu_k} \left(\log \frac{\mu_k}{z}\right)^{\alpha-\omega_k-1} \frac{|f(z, x(z))|}{z} dz \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{t}{z}\right)^{\alpha-1} \frac{|f(z, x(z))|}{z} dz \right\} \\
\le\ & \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} M + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \sum_{i=j}^{m} \frac{|\eta_j|(\log \xi_j)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{n} \frac{|\zeta_i|(\log \theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} \right. \\
& \left. + \sum_{k=1}^{r} \frac{|\lambda_k|(\log \mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \right\} M,
\end{aligned}
$$

which, after taking the norm for $t \in [1, T]$, results in the following inequality:

$$\begin{aligned} \|\mathcal{F}(x)\| &\leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}M + \frac{(\log T)^{\gamma-1}}{|\Lambda|}\Bigg\{\sum_{i=j}^{m} \frac{|\eta_j|(\log \xi_j)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{n} \frac{|\zeta_i|(\log \theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} \\ &\quad + \sum_{k=1}^{r} \frac{|\lambda_k|(\log \mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Bigg\}M. \end{aligned}$$

Next, we show that bounded sets are mapped into equicontinuous sets by $\mathcal{F}$. For $t_1, t_2 \in [1, T]$, $t_1 < t_2$ and $u \in B_R$, we obtain the following:

$$\begin{aligned} &|\mathcal{F}(x)(t_2) - \mathcal{F}(x)(t_1)| \\ \leq\ & \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{z} \right)^{\alpha-1} - \left(\log \frac{t_1}{z} \right)^{\alpha-1} \right] \frac{|f(z, x(z))|}{z} dz \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{z} \right)^{\alpha-1} \frac{|f(z, x(z))|}{z} dz \\ &+ \frac{|(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|\Lambda|} \Bigg\{ \sum_{j=1}^{m} \frac{|\eta_j|}{\Gamma(\alpha)} \int_1^{\xi_j} \left(\log \frac{\xi_j}{z} \right)^{\alpha-1} \frac{|f(z, x(z))|}{z} dz \\ &+ \sum_{i=1}^{n} \frac{|\zeta_i|}{\Gamma(\alpha+\phi_i)} \int_1^{\theta_i} \left(\log \frac{\theta_i}{z} \right)^{\alpha+\phi_i-1} \frac{|f(z, x(z))|}{z} dz \\ &+ \sum_{k=1}^{r} \frac{|\lambda_k|}{\Gamma(\alpha-\omega_k)} \int_1^{\mu_k} \left(\log \frac{\mu_k}{z} \right)^{\alpha-\omega_k-1} \frac{|f(z, x(z))|}{z} dz \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^{T} \left(\log \frac{t}{z} \right)^{\alpha-1} \frac{|f(z, x(z))|}{z} dz \Bigg\} \\ \leq\ & \frac{M}{\Gamma(\alpha+1)} \Big[2(\log t_2 - \log t_1)^{\alpha} + |(\log t_2)^{\alpha} - (\log t_1)^{\alpha}| \Big] \\ &+ \frac{|(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|\Lambda|} \Bigg\{ \sum_{i=j}^{m} \frac{|\eta_j|(\log \xi_j)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{n} \frac{|\zeta_i|(\log \theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} \\ &+ \sum_{k=1}^{r} \frac{|\lambda_k|(\log \mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \Bigg\} M, \end{aligned}$$

which tends to zero independently of $x \in B_R$, as $t_1 \to t_2$. Thus, the operator $\mathcal{F} : X \to X$ is completely continuous by applying the Arzelá–Ascoli theorem.

Step II.: We show that the set $\mathcal{E} = \{x \in X \mid x = \nu\mathcal{F}(x), 0 \leq \nu \leq 1\}$ is bounded. Let $x \in \mathcal{E}$, then $x = \nu\mathcal{F}(x)$. For any $t \in [1, T]$, we have $x(t) = \nu\mathcal{F}(x)(t)$. Then, in view of the hypothesis (H_3), as in Step I, we obtain

$$\begin{aligned} |x(t)| &\leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}M + \frac{(\log T)^{\gamma-1}}{|\Lambda|}\Bigg\{\sum_{i=j}^{m} \frac{|\eta_j|(\log \xi_j)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{n} \frac{|\zeta_i|(\log \theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} \\ &\quad + \sum_{k=1}^{r} \frac{|\lambda_k|(\log \mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Bigg\}M. \end{aligned}$$

Thus, the following is the case:

$$\begin{aligned}\|x\| \quad &\leq \quad \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}M + \frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right.\\ &\quad \left. + \sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\}M,\end{aligned}$$

which shows that the set $\mathcal{E}$ is bounded. Thus, it follows by Schaefer's fixed point theorem [38] that the operator $\mathcal{F}$ has at least one fixed point. Therefore, there exists at least one solution for the nonlinear Hilfer–Hadamard fractional boundary value problem (1) on $[1, T]$. This completes the proof. $\square$

We apply the Leray–Schauder nonlinear alternative [39] to prove our last existence result.

Theorem 5. *Let $f \in C([1, T] \times \mathbb{R}, \mathbb{R})$. In addition, it is assumed that the following conditions are satisfied:*

(H_4) *There exist $p \in C([1, T], \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t, u)| \leq p(t)\psi(\|u\|)$ for each $(t, u) \in [1, T] \times \mathbb{R}$;*

(H_5) *There exists a constant $K > 0$ such that*

$$\frac{K}{\Omega\|p\|\psi(K)} > 1,$$

where Ω is defined by (11).

Then, the nonlinear Hilfer–Hadamard fractional boundary value problem (1) has at least one solution on $[1, T]$.

Proof. As argued in Theorem 4, one can obtain that the operator $\mathcal{F}$ is completely continuous. Next, we establish that we can find an open set $U \subseteq C([1, T], \mathbb{R})$ with $x \neq \mu\mathcal{F}(x)$ for $\mu \in (0, 1)$ and $x \in \partial U$.

Let $x \in C([1, T], \mathbb{R})$ be such that $x = \mu\mathcal{F}(x)$ for some $0 < \mu < 1$. Then, for each $t \in [1, T]$, we have the following case:

$$\begin{aligned}|x(t)| \quad &\leq \quad \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|f(z, x(z))|}{z}dz\\ &\quad + \frac{(\log t)^{\gamma-1}}{|\Lambda|}\left\{\sum_{j=1}^{m}\frac{|\eta_j|}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{|f(z, x(z))|}{z}dz\right.\\ &\quad + \sum_{i=1}^{n}\frac{|\zeta_i|}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{|f(z, x(z))|}{z}dz\\ &\quad + \sum_{k=1}^{r}\frac{|\lambda_k|}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{|f(z, x(z))|}{z}dz\\ &\quad \left. + \frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|f(z, x(z))|}{z}dz\right\}\\ &\leq \quad \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right.\right.\\ &\quad \left.\left. + \sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\}\right]\|p\|\psi(\|x\|).\end{aligned}$$

Consequently, we obtain

$$\frac{\|x\|}{\Omega\|p\|\psi(\|x\|)} \leq 1.$$

In view of (H_5), there is no solution x such that $\|x\| \neq K$. Let us set the following:

$$U = \{x \in C([1,T],\mathbb{R}) : \|x\| < K\}.$$

The operator $\mathcal{F} : \overline{U} \to C([1,T],\mathbb{R})$ is continuous and completely continuous. Note that there is no $u \in \partial U$ such that $x = \mu\mathcal{F}(x)$ for some $\mu \in (0,1)$, by the choice of U. Thus, it follows by the Leray–Schauder nonlinear alternative [39] that $\mathcal{F}$ has a fixed point $x \in \overline{U}$ which is a solution of the nonlinear Hilfer–Hadamard fractional boundary value problem (1). This ends the proof. □

3.3. Examples

Consider the following Hilfer–Hadamard fractional boundary value problem:

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta}x(t) = f(t,x(t)), \quad t \in [1,T], \\ x(1) = 0, \quad x(T) = \sum\limits_{j=1}^{m} \eta_j x(\xi_j) + \sum\limits_{i=1}^{n} \zeta_i\, {}^{H}I_1^{\phi_i}x(\theta_i) + \sum\limits_{k=1}^{r} \lambda_k\, {}_{H}D_1^{\omega_k}x(\mu_k), \end{cases} \tag{14}$$

with $\alpha = 5/3, \beta = 3/4, T = 5, m = 4, n = 3, r = 2, \eta_1 = 1/15, \eta_2 = 1/10, \eta_3 = 2/15, \eta_4 = 1/6, \xi_1 = 5/4, \xi_2 = 3/2, \xi_3 = 7/4, \xi_4 = 2, \zeta_i = 1/18, \zeta_2 = 1/9, \zeta_3 = 1/6, \phi_1 = 1/2, \phi_2 = 1, \phi_3 = 3/2, \theta_1 = 5/2, \theta_2 = 3, \theta_3 = 7/2, \lambda_1 = 1/28, \lambda_2 = 1/14, \omega_1 = 1/4, \omega_2 = 2/3, \mu_1 = 4, \mu_2 = 9/2$ and $f(t,x(t))$ to be fixed later. Using the given data, it is found that $\gamma = 23/12, \Lambda \approx 0.662923$ (Λ is given by (4)), $\Omega \approx 39.388095$ (Ω is given by (11)).

(a). For illustrating Theorem 2, we take

$$f(t,x) = \frac{ae^{-(t-1)^2}}{15}\arctan x + \frac{\sin t + 1}{\sqrt{t^3+1}}, \ t \in [1,5], \tag{15}$$

where a is a positive real constant. Obviously, the nonlinear function $f(t,x)$ satisfies the assumption (H_1) with $l = a/15$ and the condition $l\Omega < 1$ holds for $a < 0.3808257$. Thus, the hypothesis of Theorem 2 is satisfied; hence, the problem (14) with $f(t,x)$ given by (15) has a unique solution on $[1,5]$.

(b). In order to illustrate Theorem 3, we take the following function:

$$f(t,x) = \frac{|x|}{[(t-1)^2+100](1+|x|)} + \cos t, \ t \in [1,5]. \tag{16}$$

It is easy to verify that the nonlinear function $f(t,x)$ satisfies the assumption (H_1) with $l = 1/100$ and that

$$\frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right.$$
$$\left. + \sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\} l \approx 0.379190 < 1.$$

Thus, the assumptions of Theorem 3 hold true. Therefore, by the conclusion of Theorem 3, there exists at least one solution for the problem (14) with $f(t,x)$ given by (16) on $[1,5]$.

(c). Now, we illustrate Theorem 4 with the aid of the following nonlinear function:

$$f(t,x) = \frac{e^{1-t}}{t^2+4}\cos^4\left(\frac{3+2|x|}{1+|x|^2}\right) + \frac{1}{10}\sqrt{(t^2+11)}, \ t \in [1,5]. \tag{17}$$

It is easy to obtain that $|f(t,x)| \leq 4/5$. Thus, by the conclusion of Theorem 4, the problem (14) with $f(t,x)$ given by (17) has at least one solutions on $[1,5]$.

(d). Finally, we demonstrate the application of Theorem 5 by considering the nonlinear function:

$$f(t,x) = \frac{2}{[(t-1)^2+100]}\left(|x|\cos(5+2|x|)+\frac{1}{2}\right),\ t \in [1,5]. \tag{18}$$

Note that $|f(t,x)| \leq p(t)\psi(\|x\|)$, where $p(t) = 2[(t-1)^2+100]^{-1}$ ($\|p\| = 1/50$) and $\psi(\|x\|) = \|x\| + 1/2$. By the condition (H_5), we find that $K > 1.85584$. Thus, all the assumptions of Theorem 5 hold true; hence, its conclusion ensures the existence of at least one solution for the problem (14) with $f(t,x)$ given by (18) on $[1,5]$.

4. Multi-Valued Case

This section is devoted to the study of the multi-valued case of the boundary value problem (1) given as follows:

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta}x(t) \in F(t,x(t)), \quad t \in [1,T], \\ x(1) = 0, \quad x(T) = \sum_{j=1}^{m}\eta_j x(\xi_j) + \sum_{i=1}^{n}\zeta_i\, {}^{H}I_1^{\phi_i}x(\theta_i) + \sum_{k=1}^{r}\lambda_k\, {}_{H}D_1^{\omega_k}x(\mu_k), \end{cases} \tag{19}$$

where the symbols are the same as defined in problem (1) and $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map. By $\mathcal{P}(\mathbb{R})$, we denote the family of all nonempty subsets of $\mathbb{R}$.

Next, we define

$$\mathcal{P}_p = \{Y \in \mathcal{P}(X) : Y \neq \emptyset \text{ and has the property } p\},$$

where $(X, \|\cdot\|)$ is a normed space. Thus, $\mathcal{P}_{cl}$, $\mathcal{P}_b$, $\mathcal{P}_{cp}$ and $\mathcal{P}_{cp,c}$ respectively, denote the classes of all closed, bounded, compact and compact and convex sets in X.

Define the set of selections of F for each $\omega \in C([1,T],\mathbb{R})$ as

$$S_{F,\omega} := \{z \in L^1([1,T],\mathbb{R}) : z(t) \in F(t,\omega(t)) \text{ for a.e. } t \in [1,T]\}.$$

4.1. Existence Results for the Problem (19)

Let us first define the solution for Hilfer–Hadamard inclusions fractional boundary value problem (19).

Definition 4. *A function $x \in C([1,T],\mathbb{R})$ is called a solution of the multi-valued problem (19) if we can find a function $v \in L^1([1,T],\mathbb{R})$ with $v(t) \in F(t,x)$ almost everywhere on $[1,T]$ such that*

$$\begin{aligned} x(t) \;=\; & \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz + \frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz\right. \\ & + \sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v(z)}{z}dz \\ & + \sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v(z)}{z}dz \\ & \left. -\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz\right\}, \end{aligned}$$

where Λ is given by (4).

4.1.1. Case 1: Convex-Valued Multifunctions

In the first theorem, dealing with convex-valued multifunctions, we assume that the multifunction F is L^1-Carathéodory and apply the nonlinear alternative for Kakutani maps [39] and closed graph operator theorem [40] to prove it.

Theorem 6. *Assume that the following conditions are satisfied:*

(A_1) *The multifunction* $F : [1, T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is* L^1*-Carathéodory;*

(A_2) *There exist a nondecreasing function* $\chi \in C([0, \infty)(0, \infty))$ *and a continuous function* $q : [1, T] \to \mathbb{R}^+$ *such that*
$\|F(t, \omega)\|_{\mathcal{P}} := \sup\{|z| : z \in F(t, \omega)\} \leq q(t)\chi(\|\omega\|)$ *for each* $(t, \omega) \in [1, T] \times \mathbb{R}$*;*

(A_3) *There exists* $M > 0$ *satisfying the following inequality:*

$$\frac{M}{\chi(M)\|q\|\Omega} > 1,$$

where Ω *is given by (11).*

Then, there exists at least one solution for the inclusions problem (19) on $[1, T]$.

Proof. Associated with the Hilfer–Hadamard fractional inclusions boundary value problem (19), we introduce a multi-valued operator, $N : C([1, T], \mathbb{R}) \to \mathcal{P}(C([1, T], \mathbb{R}))$, as follows:

$$N(x) = \left\{ \begin{array}{l} h \in C([1,T],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1} \frac{v(z)}{z} dz \\ + \dfrac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \displaystyle\sum_{j=1}^m \frac{\eta_j}{\Gamma(\alpha)} \int_1^{\xi_j} \left(\log\frac{\xi_j}{z}\right)^{\alpha-1} \frac{v(z)}{z} dz \right. \\ + \displaystyle\sum_{i=1}^n \frac{\zeta_i}{\Gamma(\alpha+\phi_i)} \int_1^{\theta_i} \left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1} \frac{v(z)}{z} dz \\ + \displaystyle\sum_{k=1}^r \frac{\lambda_k}{\Gamma(\alpha-\omega_k)} \int_1^{\mu_k} \left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1} \frac{v(z)}{z} dz \\ \left. - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^T \left(\log\frac{T}{z}\right)^{\alpha-1} \frac{v(z)}{z} dz \right\}, \quad v \in S_{F,x}. \end{array} \right. \end{array} \right\}$$

In what follows, we will prove in several steps that the operator N satisfies the hypotheses of the Leray–Schauder nonlinear alternative for Kakutani maps [39].

Step 1. *N is bounded on bounded sets of $C([1, T], \mathbb{R})$.*

For a fixed $r > 0$, let $B_r = \{x \in C[1, T], \mathbb{R}) : \|x\| \leq r\}$ be a bounded set in $C([1, T], \mathbb{R})$. For each $h \in N(x)$ and $x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) \;=\; & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1} \frac{v(z)}{z} dz + \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \sum_{j=1}^m \frac{\eta_j}{\Gamma(\alpha)} \int_1^{\xi_j} \left(\log\frac{\xi_j}{z}\right)^{\alpha-1} \frac{v(z)}{z} dz \right. \\ & + \sum_{i=1}^n \frac{\zeta_i}{\Gamma(\alpha+\phi_i)} \int_1^{\theta_i} \left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1} \frac{v(z)}{z} dz \\ & + \sum_{k=1}^r \frac{\lambda_k}{\Gamma(\alpha-\omega_k)} \int_1^{\mu_k} \left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1} \frac{v(z)}{z} dz \\ & \left. - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log\frac{T}{z}\right)^{\alpha-1} \frac{v(z)}{z} dz \right\}, \quad t \in [1, T]. \end{aligned}$$

For $t \in [1,T]$, using the assumption (A_2), we obtain the following inequality:

$$\begin{aligned}
|h(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|v(z)|}{z}dz \\
&+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{|\eta_j|}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{|v(z)|}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{|\zeta_i|}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{|v(z)|}{z}dz \\
&+\sum_{k=1}^{r}\frac{|\lambda_k|}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{|v(z)|}{z}dz \\
&\left.+\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|v(z)|}{z}dz\right\} \\
\leq\ & \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right.\right. \\
&\left.\left.+\sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\}\right]\|p\|\chi(\|x\|),
\end{aligned}$$

which yields

$$\|h\| \leq \|p\|\chi(r)\Omega.$$

Step 2. *Bounded sets are mapped by N into equicontinuous sets of $C([1,T],\mathbb{R})$.* Let $x \in B_r$ and $h \in N(x)$. Then, there exists $v \in S_{F,x}$ such that

$$\begin{aligned}
h(t) =\ & \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v(z)}{z}dz \\
&+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v(z)}{z}dz \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz\right\},\ t\in[1,T].
\end{aligned}$$

Let $t_1,\ t_2 \in [1,T],\ t_1 < t_2$. Then, we obtain

$$\begin{aligned}
& |\mathcal{F}(x)(t_2)-\mathcal{F}(x)(t_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{z}\right)^{\alpha-1}-\left(\log\frac{t_1}{z}\right)^{\alpha-1}\right]\frac{|f(z,x(z))|}{z}dz \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{z}\right)^{\alpha-1}\frac{|v(z)|}{z}dz \\
&+\frac{|(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}|}{|\Lambda|}\left\{\sum_{j=1}^{m}\frac{|\eta_j|}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{|v(z)|}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{|\zeta_i|}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{|v(z)|}{z}dz \\
&+\sum_{k=1}^{r}\frac{|\lambda_k|}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{|v(z)|}{z}dz \\
&\left.+\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|v(z)|}{z}dz\right\} \\
\leq\ & \frac{\|p\|\chi(r)}{\Gamma(\alpha+1)}\left[2(\log t_2-\log t_1)^{\alpha}+|(\log t_2)^{\alpha}-(\log t_1)^{\alpha}|\right] \\
&+\frac{|(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}|}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right. \\
&\left.+\sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\}\|p\|\chi(r)\to 0,
\end{aligned}$$

as $t_1 \to t_2$ independently of $x \in B_r$. Hence, $N : C([1,T],\mathbb{R}) \to \mathcal{P}(C[1,T],\mathbb{R}))$ is completely continuous by Arzelá–Ascoli theorem.

Step 3. *For each $x \in C([1,T],\mathbb{R})$, $N(x)$ is convex.*

For $h_1, h_2 \in N(x)$, there exist $v_1, v_2 \in S_{F,x}$ such that, for each $t \in [1,T]$, we have

$$\begin{aligned} h_\varpi(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v_\varpi(z)}{z}dz \\ &+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^m \frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v_\varpi(z)}{z}dz\right. \\ &+\sum_{i=1}^n \frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v_\varpi(z)}{z}dz \\ &+\sum_{k=1}^r \frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v_\varpi(z)}{z}dz \\ &\left.-\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v_\varpi(z)}{z}dz\right\}, \quad \varpi = 1,2. \end{aligned}$$

Let $0 \le \sigma \le 1$. Then, for each $t \in [1,T]$, we have the following:

$$\begin{aligned} &[\sigma h_1 + (1-\sigma)h_2](t) \\ = &\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{[\sigma v_1(z)+(1-\sigma)v_2(z)]}{z}dz \\ &+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^m \frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{[\sigma v_1(z)+(1-\sigma)v_2(z)]}{z}dz\right. \\ &+\sum_{i=1}^n \frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\sigma_i}\left(\log\frac{\sigma_i}{z}\right)^{\alpha+\phi_i-1}\frac{[\sigma v_1(z)+(1-\sigma)v_2(z)]}{z}dz \\ &+\sum_{k=1}^r \frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{[\sigma v_1(z)+(1-\sigma)v_2(z)]}{z}dz \\ &\left.-\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{[\sigma v_1(z)+(1-\sigma)v_2(z)]}{z}dz\right\}. \end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), $\sigma h_1 + (1-\sigma)h_2 \in N(x)$. In consequence, N is convex-valued.

Next, it will be shown that the operator N is upper semicontinuous. By using the fact that a completely continuous operator, which has a closed graph, is upper semicontinuous ([41] (Proposition 1.2)), it is enough to prove that the operator N has a closed graph. This will be established in the following step.

Step 4. *The graph of N is closed.*

Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. Then, we show that $h_* \in N(x_*)$. Observe that $h_n \in N(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [1,T]$, we have

$$\begin{aligned}
h_n(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v_n(z)}{z}dz \\
&+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v_n(z)}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v_n(z)}{z}dz \\
&+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v_n(z)}{z}dz \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v_n(z)}{z}dz\right\}.
\end{aligned}$$

For each $t \in [1,T]$, we must have $v_* \in S_{F,x_*}$ and the following expression:

$$\begin{aligned}
h_*(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v_*(z)}{z}dz \\
&+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v_*(z)}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v_*(z)}{z}dz \\
&+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v_*(z)}{z}dz \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v_*(z)}{z}dz\right\}.
\end{aligned}$$

Consider the continuous linear operator $\Phi : L^1([1,T],\mathbb{R}) \to C([1,T],\mathbb{R})$ defined as follows:

$$\begin{aligned}
v \to \Phi(v)(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz \\
&+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v(z)}{z}dz \\
&+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v(z)}{z}dz \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz\right\}.
\end{aligned}$$

Clearly $\|h_n - h_*\| \to 0$ as $n \to \infty$, and consequently, by the closed graph operator theorem [40], $\Phi \circ S_{F,x}$ is a closed graph operator. Moreover, we have $h_n \in \Phi(S_{F,x_n})$ and the following:

$$
\begin{aligned}
h_*(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{z} \right)^{\alpha-1} \frac{v_*(z)}{z} dz \\
&+ \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \sum_{j=1}^m \frac{\eta_j}{\Gamma(\alpha)} \int_1^{\xi_j} \left(\log \frac{\xi_j}{z} \right)^{\alpha-1} \frac{v_*(z)}{z} dz \right. \\
&+ \sum_{i=1}^n \frac{\zeta_i}{\Gamma(\alpha+\phi_i)} \int_1^{\theta_i} \left(\log \frac{\theta_i}{z} \right)^{\alpha+\phi_i-1} \frac{v_*(z)}{z} dz \\
&+ \sum_{k=1}^r \frac{\lambda_k}{\Gamma(\alpha-\omega_k)} \int_1^{\mu_k} \left(\log \frac{\mu_k}{z} \right)^{\alpha-\omega_k-1} \frac{v_*(z)}{z} dz \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{z} \right)^{\alpha-1} \frac{v_*(z)}{z} dz \right\},
\end{aligned}
$$

for some $v_* \in S_{F,x_*}$. Thus, N has a closed graph, which implies that the operator N is upper semicontinuous.

Step 5. *There exists an open set $U \subseteq C([1,T],\mathbb{R})$ such that, for any $k \in (0,1)$ and all $x \in \partial U$, $x \notin kN(x)$.*

Let $x \in kN(x)$, $k \in (0,1)$. Then, there exists $v \in L^1([1,T],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [1,T]$, we have the following case.

$$
\begin{aligned}
x(t) &= k\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{z} \right)^{\alpha-1} \frac{v(z)}{z} dz \\
&+ k\frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \sum_{j=1}^m \frac{\eta_j}{\Gamma(\alpha)} \int_1^{\xi_j} \left(\log \frac{\xi_j}{z} \right)^{\alpha-1} \frac{v(z)}{z} dz \right. \\
&+ \sum_{i=1}^n \frac{\zeta_i}{\Gamma(\alpha+\phi_i)} \int_1^{\theta_i} \left(\log \frac{\theta_i}{z} \right)^{\alpha+\phi_i-1} \frac{v(z)}{z} dz \\
&+ \sum_{k=1}^r \frac{\lambda_k}{\Gamma(\alpha-\omega_k)} \int_1^{\mu_k} \left(\log \frac{\mu_k}{z} \right)^{\alpha-\omega_k-1} \frac{v(z)}{z} dz \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{z} \right)^{\alpha-1} \frac{v(z)}{z} dz \right\}.
\end{aligned}
$$

Following the computation as in Step 2, for each $t \in [1,T]$, we have the following inequality:

$$
\begin{aligned}
|x(t)| &\le \left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \sum_{i=j}^m \frac{|\eta_j|(\log \xi_j)^\alpha}{\Gamma(\alpha+1)} + \sum_{i=1}^n \frac{|\zeta_i|(\log \theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)} \right.\right. \\
&\left.\left. + \sum_{k=1}^r \frac{|\lambda_k|(\log \mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \right\} \right] \|p\|\chi(\|x\|) \\
&= \|p\|\chi(\|x\|)\Omega.
\end{aligned}
$$

In consequence, we obtain

$$
\frac{\|x\|}{\chi(\|x\|)\|p\|\Omega} \le 1.
$$

By assumption (A_3), we can find M such that $\|x\| \neq M$. Let us set the following:

$$U = \{x \in C([1,T],\mathbb{R}) : \|x\| < M\}.$$

Notice that $N : \overline{U} \to \mathcal{P}(C([1,T],\mathbb{R}))$ is a compact multi-valued map with convex closed values, which is upper semicontinuous; moreover, from the choice of U, there is no $x \in \partial U$ such that $x \in kN(x)$ for some $k \in (0,1)$. Hence, we deduce by the Leray–Schauder nonlinear alternative for Kakutani maps [39] that N has a fixed point $x \in \overline{U}$. This implies the existence of at least one solution for the inclusions problem (19) on $[1,T]$. The proof is complete. □

4.1.2. Case 2: Nonconvex Valued Multifunctions

Here, we prove an existence result for the Hilfer–Hadamard fractional inclusions boundary value problem (19) with a non-convex valued multi-valued map via a fixed point theorem for multivalued maps due to Covitz and Nadler [42].

Definition 5. *([43]) Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$ and $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ be defined as follows:*

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$.

Theorem 7. *Let the following conditions hold:*

(B_1) *$F : [1,T] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,x) : [1,T] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*

(B_2) *$H_d(F(t,x),F(t,\bar{x})) \leq \varrho(t)|x-\bar{x}|$ for almost all $t \in [1,T]$ and $x,\bar{x} \in \mathbb{R}$ with $\varrho \in C([1,T],\mathbb{R}^+)$ and $d(0,F(t,0)) \leq \varrho(t)$ for almost all $t \in [1,T]$.*

Then, the Hilfer–Hadamard inclusions fractional boundary value problem (19) has at least one solution on $[1,T]$ if

$$\Omega\|\varrho\| < 1,$$

where Ω is given by (11).

Proof. We verify that the operator $N : C([1,T],\mathbb{R}) \to \mathcal{P}(C([1,T],\mathbb{R}))$, defined at the beginning of the proof of Theorem 6, satisfies the hypotheses of the fixed point theorem for multivalued maps due to Covitz and Nadler [42].

Step I. N is nonempty and closed for every $v \in S_{F,x}$.

It follows by the measurable selection theorem ([44], (Theorem III.6) that the set-valued map $F(\cdot,x(\cdot))$ is measurable; hence, it admits a measurable selection $v : [1,T] \to \mathbb{R}$. In view of the assumption (B_2), we obtain $|v(t)| \leq \varrho(t)(1+|x(t)|)$, that is, $v \in L^1([1,T],\mathbb{R})$; hence, F is integrably bounded. In consequence, we deduce that $S_{F,x} \neq \emptyset$.

Now, we show that $N(x) \in \mathcal{P}_{cl}(C([1,T],\mathbb{R}))$ for each $x \in C([1,T],\mathbb{R})$. For that, let $\{u_n\}_{n\geq 0} \in N(x)$ with $u_n \to u$ $(n \to \infty)$ in $C([1,T],\mathbb{R})$. Then, $u \in C([1,T],\mathbb{R})$, and we can find $v_n \in S_{F,x_n}$ satisfying the following equation:

$$
\begin{aligned}
u_n(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v_n(z)}{z}dz \\
&+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v_n(z)}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v_n(z)}{z}dz \\
&+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v_n(z)}{z}dz \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v_n(z)}{z}dz\right\},
\end{aligned}
$$

for each $t \in [1,T]$. Then, we can obtain a sub-sequence (if necessary) v_n converging to v in $L^1([1,T],\mathbb{R})$, as F has compact values. Thus, $v \in S_{F,x}$ and for each $t \in [1,T]$, we have the following:

$$
\begin{aligned}
u_n(t) \to v(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz \\
&+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v(z)}{z}dz \\
&+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v(z)}{z}dz \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v(z)}{z}dz\right\}.
\end{aligned}
$$

Thus, $u \in N(x)$.

Step II. In this step, it will be shown that there exists $0 < m_0 < 1$ ($m_0 = \Omega\|\varrho\|$) such that the following is the case.

$$
H_d(N(x), N(\bar{x})) \le m_0\|x-\bar{x}\| \text{ for each } x, \bar{x} \in C([1,T],\mathbb{R}).
$$

Let $x, \bar{x} \in C([1,T],\mathbb{R})$ and $h_1 \in N(x)$. Then, there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [1,T]$, we have

$$
\begin{aligned}
h_1(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v_1(z)}{z}dz \\
&+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v_1(z)}{z}dz\right. \\
&+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v_1(z)}{z}dz \\
&+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v_1(z)}{z}dz \\
&\left.-\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v_1(z)}{z}dz\right\}.
\end{aligned}
$$

By (B_2), we have

$$H_d(F(t,x),F(t,\bar{x})) \leq \varrho(t)|x(t)-\bar{x}(t)|.$$

Thus, there exists $\vartheta(t) \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-\vartheta| \leq \varrho(t)|x(t)-\bar{x}(t)|, \;\; t \in [1,T].$$

Let us define $\mathcal{V} : [1,T] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{V}(t) = \{\vartheta \in \mathbb{R} : |v_1(t)-\vartheta| \leq \varrho(t)|x(t)-\bar{x}(t)|\}.$$

Then, there exists a function $v_2(t)$ that is a measurable selection of $\mathcal{V}$, since the multivalued operator $\mathcal{V}(t) \cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [44]). Hence, $v_2(t) \in F(t,\bar{x}(t))$ and for each $t \in [1,T]$, we have $|v_1(t)-v_2(t)| \leq \varrho(t)|x(t)-\bar{x}(t)|$. Thus, for each $t \in [1,T]$, we have

$$\begin{aligned} h_2(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{v_2(z)}{z}dz \\ &\quad+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{v_2(z)}{z}dz\right. \\ &\quad+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{v_2(z)}{z}dz \\ &\quad+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{v_2(z)}{z}dz \\ &\quad\left.-\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{v_2(z)}{z}dz\right\}. \end{aligned}$$

Hence, we have the following:

$$\begin{aligned} |h_1(t)-h_2(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{z}\right)^{\alpha-1}\frac{|v_2(z)-v_1(z)|)}{z}dz \\ &\quad+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{j=1}^{m}\frac{\eta_j}{\Gamma(\alpha)}\int_1^{\xi_j}\left(\log\frac{\xi_j}{z}\right)^{\alpha-1}\frac{|v_2(z)-v_1(z)|}{z}dz\right. \\ &\quad+\sum_{i=1}^{n}\frac{\zeta_i}{\Gamma(\alpha+\phi_i)}\int_1^{\theta_i}\left(\log\frac{\theta_i}{z}\right)^{\alpha+\phi_i-1}\frac{|v_2(z)-v_1(z)|}{z}dz \\ &\quad+\sum_{k=1}^{r}\frac{\lambda_k}{\Gamma(\alpha-\omega_k)}\int_1^{\mu_k}\left(\log\frac{\mu_k}{z}\right)^{\alpha-\omega_k-1}\frac{|v_2(z)-v_1(z)|}{z}dz \\ &\quad\left.-\frac{1}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{z}\right)^{\alpha-1}\frac{|v_2(z)-v_1(z)|}{z}dz\right\} \\ &\leq \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=j}^{m}\frac{|\eta_j|(\log\xi_j)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=1}^{n}\frac{|\zeta_i|(\log\theta_i)^{\alpha+\phi_i}}{\Gamma(\alpha+\phi_i+1)}\right.\right. \\ &\quad\left.\left.+\sum_{k=1}^{r}\frac{|\lambda_k|(\log\mu_k)^{\alpha-\omega_k}}{\Gamma(\alpha-\omega_k+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right\}\right]\|\varrho\|\|x-\bar{x}\|, \end{aligned}$$

which yields

$$\|h_1-h_2\| \leq \Omega\|\varrho\|\|x-\bar{x}\|.$$

On switching the roles of x and $\bar{x}$, we obtain the following case:

$$H_d(N(x), N(\bar{x})) \leq \Omega \|\varrho\| \|x - \bar{x}\|,$$

which shows that N is a contraction. Consequently, the conclusion of the fixed point theorem for multivalued maps due to Covitz and Nadler [42] applies; hence, the operator N has a fixed point x which corresponds to a solution of the Hilfer–Hadamard inclusions fractional boundary value problem (19). The proof is complete. □

4.2. Examples

Let us consider the Hilfer–Hadamard fractional inclusions boundary value problem:

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta} x(t) \in F(t, x(t)), \quad t \in [1, T], \\ x(1) = 0, \quad x(T) = \sum_{j=1}^{m} \eta_j x(\xi_j) + \sum_{i=1}^{n} \zeta_i \, {}^{H}I_1^{\phi_i} x(\theta_i) + \sum_{k=1}^{r} \lambda_k \, {}_{H}D_1^{\omega_k} x(\mu_k), \end{cases} \tag{20}$$

with the values of the parameters taken in the problem (14), while the multi-valued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ will be defined below.

We first illustrate Theorem 6 by taking the following multi-valued map:

$$F(t, x(t)) = \left[\frac{e^{-(1-t)^2}}{\sqrt{(1-t)^4 + 75)}} \left(\frac{|x|^2}{(1+|x|)} + \frac{1}{5} \right), \ \frac{1}{100} \left(x + \frac{\cos x}{9} \right) \right]. \tag{21}$$

It is easy to check that $\|F(t, x)\|_{\mathcal{P}} \leq q(t)\chi(\|x\|)$, where $q(t)$ and $\chi(\|x\|)$ are given as follows:

$$q(t) = \frac{e^{-(1-t)^2}}{\sqrt{(1-t)^4 + 75)}}, \, \chi(\|x\|) = \|x\| + \frac{1}{5}.$$

Using the values $\|q\| = 1/75, \chi(\|x\|) = \|x\| + \frac{1}{5}$ and $\Omega \approx 39.388095$ (see Section 3.3) in the condition (A_3), that is, $\dfrac{M}{\chi(M)\|q\|\Omega} > 1$, we find that $M > 0.221207$. Thus, all the assumptions of Theorem 6 are satisfied; hence, its conclusion implies that problem (20) with $F(t, x(t))$ given by (21) has a solution on $[1, 5]$.

Next, we demonstrate the application of Theorem 7 by choosing the following multi-valued map:

$$F(t, x(t)) = \left[\frac{1 + \arctan x}{(20+t)^2}, \left(\frac{\sqrt{t^2 + 11}}{360} \right) \left(\frac{1 + 5|x|}{1 + 4|x|} \right) \right]. \tag{22}$$

Clearly, F is measurable for all $x \in \mathbb{R}$, satisfying the following inequality:

$$H_d(F(t, x), F(t, \bar{x})) \leq \left(\frac{\sqrt{t^2 + 11}}{360} \right) |x - \bar{x}|, \, x, \bar{x} \in \mathbb{R}, \, t \in [1, 5].$$

Fixing $\varrho(t) = \sqrt{t^2 + 11}/360$, we have $\|\varrho\| = 1/60$ and $d(0, \mathcal{F}(t, 0)) \leq \varrho(t)$, $t \in [1, 5]$. Furthermore, $\|\varrho\|\Omega \approx 0.656468 < 1$. Clearly, the hypothesis of Theorem 7 holds true; consequently, we deduce by its conclusion that there exists a solution for the problem (20) with $F(t, x(t))$ given by (22) on $[1, 5]$.

5. Conclusions

In this paper, we have presented the existence and uniqueness criteria for the solutions of a Hilfer–Hadamard fractional differential equation complemented with mixed nonlocal (multi-point, fractional integral multi-order and fractional derivative multi-order) boundary conditions. Firstly, we have converted the given nonlinear problem into a fixed point problem. Once the fixed point operator is available, we can make use of the Banach contraction mapping principle to obtain the uniqueness result. The first two existence results are proved by applying the fixed point theorems due to Schaefer and Krasnoselskiĭ, while the third existence result is based on Leray–Schauder nonlinear alternative. Next, we present the existence results for the corresponding inclusions problem. The first result for the inclusions problem deals with the convex-valued multivalued map and is obtained by applying the Leray–Schauder alternative for multivalued maps, while the non-convex valued multivalued map case relies on the Covitz–Nadler fixed point theorem for contractive multivalued maps. All the results obtained for single and multivalued maps are well illustrated by numerical examples. It is imperative to mention that our results are new in the given configuration and enrich the literature on boundary value problems involving Hilfer–Hadamard fractional differential equations and inclusions of order in $(1, 2]$.

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