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Certain Recurrence Relations of Two Parametric Mittag-Leffler Function and Their Application in Fractional Calculus

Dheerandra Shanker Sachan [1], Shailesh Jaloree [2] and Junesang Choi [3,*]

[1] Department of Mathematics, St.Mary's PG College, Vidisha 464001, India; sachan.dheerandra17@gmail.com
[2] Department of Mathematics, Samrat Ashok Technological Institute, Vidisha 464001, India; shailesh_jaloree@rediffmail.com
[3] Department of Mathematics, Dongguk University, Gyeongju 38066, Korea
[*] Correspondence: junesangchoi@gmail.com; Tel.: +82-010-6525-2262

Abstract: The purpose of this paper is to develop some new recurrence relations for the two parametric Mittag-Leffler function. Then, we consider some applications of those recurrence relations. Firstly, we express many of the two parametric Mittag-Leffler functions in terms of elementary functions by combining suitable pairings of certain specific instances of those recurrence relations. Secondly, by applying Riemann–Liouville fractional integral and differential operators to one of those recurrence relations, we establish four new relations among the Fox–Wright functions, certain particular cases of which exhibit four relations among the generalized hypergeometric functions. Finally, we raise several relevant issues for further research.

Keywords: gamma function; Beta function; Mittag-Leffler function; Generalized Mittag-Leffler functions; generalized hypergeometric function; Fox–Wright function; recurrence relations; Riemann–Liouville fractional calculus operators

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1. Introduction and Preliminaries

Magnus Gösta Mittag-Leffler (1846–1927), a Swedish mathematician, invented the function $E_\mu(z)$ (1) in conjunction with the summation technique for divergent series, which is eponymously referred to as the Mittag-Leffler (M-L) function and represented by the following convergent power series across the whole complex plane (see [1–4]):

$$E_\mu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)} \quad (\Re(\mu) > 0,\, z \in \mathbb{C}), \tag{1}$$

where $\Gamma(\cdot)$ is the familiar Gamma function (see, e.g., ([5], [Section 1.1])). Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ represent the sets of complex numbers, real numbers, positive real numbers, integers, and positive integers, respectively, in this and subsequent sections. Further, for some $\ell \in \mathbb{Z}$, let $\mathbb{Z}_{\leq \ell}$ denote the set of integers less than or equal to ℓ. The function (1) is an entire function of order $1/\Re(\mu)$ and type 1 (see, e.g., ([6], [Section 4.1])). Numerous mathematicians have investigated the properties of this entire function (see, e.g., [7–16]). This function (1) reduces to a number of elementary and special functions such as (see, e.g., ([6], [Section 3.2]))

$$E_1(\pm z) = e^{\pm z}, \quad E_2(-z^2) = \cos z, \quad E_2(z^2) = \cosh z, \tag{2}$$

and

$$E_{\frac{1}{2}}\left(\pm z^{\frac{1}{2}}\right) = e^z \left[1 + \operatorname{erf}\left(\pm z^{\frac{1}{2}}\right)\right] = e^z \operatorname{erfc}\left(\mp z^{\frac{1}{2}}\right), \tag{3}$$

where erf (erfc) represents the error function (complementary error function)

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2}\, \mathrm{d}u, \quad \operatorname{erfc} := 1 - \operatorname{erf}(z) \quad (z \in \mathbb{C}), \tag{4}$$

and $z^{\frac{1}{2}}$ denotes the principal branch of the associated multi-valued function. Numerous generalizations of the function (1) have been developed, including (5) and (7). In the 1960s, the Mittag-Leffler function began to be classified as a particular instance of the broader class of Fox H-function, which may have an arbitrary number of parameters in their Mellin–Barnes contour integral representation. Among the special instances of the H-function is the Fox–Wright function (8) (see, e.g., [10,17–19]). This function (1) and its numerous generalizations are significant because they are involved in a large number of applied problems (see, e.g., [20–27]). The Mittag-Leffler function (1) and its numerous extensions have been used, particulary, in conjunction with fractional calculus such as: The resolvant for a particular case of Volterra's equation (see ([28], [Theorem 4.2])) is explicitly expressed in terms of the Mittag-Leffler function (1); The two parametric Mittag-Leffler function $E_{\mu,\varrho}(z)$ (5) and its slight extension are used as solutions of the linear integral equation of the Abel–Volterra type (see ([29], [Theorem 5.3])); Kilbas and Saigo ([30], [Theorems 6 and 7]) employed an extension of the Mittag-Leffler function as solutions of the Abel–Volterra integral equations (see also [31,32]). The monograph ([33], [pages 132, 143, 144, 206]) demonstrated in detail that the Mittag-Leffler function (1) and its modest modification are solutions to a certain class of fractional differential equations; the monograph ([34], [p. 21]) referred to the two parametric Mittag-Leffler function (5) and (1), as well as their fascinating Laplace transform formula. Choi et al. [35] proposed an extension of the Prabhakar function (7) that is used to determine a number of its features and formulae, including higher-order differential equations, many integral transformations, and several fractional derivative and integral formulas. Due to the breadth of applications in connection with fractional calculus, the Mittag-Leffler function has been dubbed the "Queen Function of the Fractional Calculus" by certain academics in the past (see [36]; see also [17]). Haubold et al. [24] provided a concise overview of the Mittag-Leffler function, extended Mittag-Leffler functions, Mittag-Leffler type functions, and their intriguing and useful features.

The two parametric Mittag-Leffler function $E_{\mu,\varrho}(z)$ is defined by (see, e.g., [8,9], ([6], [Chapter 4]))

$$E_{\mu,\varrho}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \varrho)} \quad (\Re(\mu) > 0,\ \varrho, z \in \mathbb{C}). \tag{5}$$

The function (5) is an entire function of order $1/\Re(\mu)$ and type 1 with the argument z under the constraints $\mu, \varrho \in \mathbb{R}^+$ (see, e.g., [37–39]), whose fundamental properties, such as asymptotic behavior and zero distribution, have been explored by a number of researchers (see, e.g., [8–11,40,41]). The function (5) reduces to some special functions such as (see, e.g., ([6], [Equation 4.3.13]))

$$E_{1/2,1}(z) = e^{z^2}(1 + \operatorname{erf} z) = e^{z^2} \operatorname{erfc}(-z). \tag{6}$$

The interested reader may refer to ([6], [Chapter 4]) for further basic properties and many other relations, representations and applications of the two parametric Mittag-Leffler function (5).

The three parametric Mittag-Leffler function (also known as the Prabhakar function [42]) $E^{\xi}_{\mu,\varrho}(z)$ is defined by (see [43]; see also ([6], [Chapter 5]), [42])

$$E^{\xi}_{\mu,\varrho}(z) = \sum_{n=0}^{\infty} \frac{(\xi)_n\, z^n}{n!\, \Gamma(\mu n + \varrho)} \quad (\Re(\mu) > 0, \Re(\varrho) > 0,\ \xi, z \in \mathbb{C}), \tag{7}$$

which is also an entire function of order $1/\Re(\mu)$ and type 1. Giusti et al. [42] offered an excellent survey paper in which they covered important findings and applications of the Prabhakar function (7), together with major historical events leading to its discovery and subsequent development, its capacity of introducing an upgraded scheme for fractional calculus, an overview of the advances made in applying this new general framework to physics and renewal processes, a collection of results on its numerical evaluation, and as many as 159 references.

The Mittag-Leffler function (1), its slight generalization (5), the Prabhakar function (7), and a number of other parameterized extensions are found to be particular instances of the following Fox–Wright function defined by (see [44–46]; see also [47], ([48], [p. 21]))

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,\, A_1),\, \ldots,\, (\alpha_p,\, A_p)\,; \\ (\beta_1,\, B_1),\, \ldots,\, (\beta_q,\, B_q)\,;\end{matrix}\; z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}, \tag{8} $$

where the coefficients $A_j \in \mathbb{R}$ $(j = 1, \ldots, p)$ and $B_j \in \mathbb{R}$ $(j = 1, \ldots, q)$; $\alpha_j \in \mathbb{C}$ $(j = 1, \ldots, p)$ and $\beta_j \in \mathbb{C}$ $(j = 1, \ldots, q)$. The convergence constraints of (8) are given as follows (see ([49], [Theorem 1.5])): Let

$$ \begin{aligned} \Omega &:= \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \\ \omega &:= \prod_{j=1}^{p} |A_j|^{-A_j} \prod_{j=1}^{q} |B_j|^{B_j}, \\ \nu &:= \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j + \frac{p-q}{2}. \end{aligned} \tag{9} $$

Then

(i) If $\Omega > -1$, then the series (8) is absolutely convergent for all $z \in \mathbb{C}$;
(ii) If $\Omega = -1$, the series (8) is absolutely convergent for $|z| < \omega$;
(iii) If $\Omega = -1$ and $\Re(\nu) > \frac{1}{2}$, the series (8) is absolutely convergent for $|z| = \omega$.

The Fox–Wright function ${}_p\Psi_q$ is an extension of generalized hypergeometric function ${}_pF_q$ and a particular case of the H-function (see, e.g., ([19], [Equation (1.140)]), [50]):

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,\, 1),\, \ldots,\, (\alpha_p,\, 1)\,; \\ (\beta_1,\, 1),\, \ldots,\, (\beta_q,\, 1)\,;\end{matrix}\; z\right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)}\, {}_pF_q\left[\begin{matrix}\alpha_1,\, \ldots,\, \alpha_p\,; \\ \beta_1,\, \ldots,\, \beta_q\,;\end{matrix}\; z\right] \tag{10} $$

and

$$ \begin{aligned} &{}_p\Psi_q\left[\begin{matrix}(\alpha_1,\, A_1),\, \ldots,\, (\alpha_p,\, A_p)\,; \\ (\beta_1,\, B_1),\, \ldots,\, (\beta_q,\, B_q)\,;\end{matrix}\; z\right] \\ &= H^{1,p}_{p,q+1}\left[-z \,\middle|\, \begin{matrix}(1-\alpha_1, A_1),\, \ldots,\, (1-\alpha_p, A_p) \\ (0,1),\, (1-\beta_1, B_1),\, \ldots,\, (1-\beta_q, B_q)\end{matrix}\right]. \end{aligned} \tag{11} $$

There have been many introductions and investigations of fractional integrals and derivatives. We recall the left-sided and right-sided Riemann–Liouville fractional integrals $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}$ defined as (see, e.g., [33,34,49])

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau \quad (x > a,\ \Re(\alpha) > 0), \tag{12}$$

and

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau)\, d\tau \quad (b > x,\ \Re(\alpha) > 0), \tag{13}$$

respectively. The Riemann–Liouville fractional derivatives $D^{\alpha}_{a+}f$ and $D^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) \geq 0)$ are defined by

$$(D^{\alpha}_{a+}f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\mathsf{n}} (I^{\mathsf{n}-\alpha}_{a+}f)(x) \quad (x > a) \tag{14}$$

and

$$(D^{\alpha}_{b-}f)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\mathsf{n}} \left(I^{\mathsf{n}-\alpha}_{b-}f\right)(x) \quad (x < b), \tag{15}$$

where $\mathsf{n} = [\Re(\alpha)] + 1$. Here and elsewhere, $[x]$ denotes the largest integer less than or equal to $x \in \mathbb{R}$.

We recall some of the recurrence relations for the two parametric Mittag-Leffler function (5) and the three parametric Mittag-Leffler function (7).

1.1. *Two Parametric Mittag-Leffler Function*:

$$E_{\mu,\varrho}(z) = z\, E_{\mu,\mu+\varrho}(z) + \frac{1}{\Gamma(\varrho)} \tag{16}$$

(see, e.g., ([10], [Equation (23)]), ([51], [Equation (5)])).

Further, for all $\mu,\ \varrho \in \mathbb{R}^+$,

$$E_{\mu,\varrho}(z) = z^2\, E_{\mu,\varrho+2\mu}(z) + \frac{1}{\Gamma(\varrho)} + \frac{z}{\Gamma(\varrho+\mu)}, \tag{17}$$

$$E_{\mu,\varrho}(z) = z^3\, E_{\mu,\varrho+3\mu}(z) + \frac{1}{\Gamma(\varrho)} + \frac{z}{\Gamma(\varrho+\mu)} + \frac{z^2}{\Gamma(\varrho+2\mu)}, \tag{18}$$

$$E_{\mu,\varrho}(z) = z^4\, E_{\mu,\varrho+4\mu}(z) + \frac{1}{\Gamma(\varrho)} + \frac{z}{\Gamma(\varrho+\mu)} + \frac{z^2}{\Gamma(\varrho+2\mu)} + \frac{z^3}{\Gamma(\varrho+3\mu)} \tag{19}$$

(see, e.g., ([6], [Lemma 4.1]), [51]).

Generally, for $\Re(\mu) > 0$, $\Re(\varrho) > 0$, and $\ell \in \mathbb{N}$,

$$E_{\mu,\varrho}(z) = z^{\ell}\, E_{\mu,\varrho+\ell\mu}(z) + \sum_{k=0}^{\ell-1} \frac{z^k}{\Gamma(\varrho+k\mu)} \tag{20}$$

(see [52]; see also ([24], [Theorem 5.2])).

Gupta and Debnath (see ([51], [Equation (30)])) presented the following interesting differential recurrence relation for the two parametric Mittag-Leffler function (5):

$$\begin{aligned} E_{\mu,n+1}(z) =\ & n(n+2)\, E_{\mu,n+3}(z) + \mu z[\mu + 2(n+1)]\, E'_{\mu,n+3}(z) \\ & + z^2\, E''_{\mu,n+3}(z) + E_{\mu,n+2}(z) \quad (n \in \mathbb{N}), \end{aligned} \tag{21}$$

where
$$E^{(\ell)}_{\mu,\varrho}(z) = \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} E_{\mu,\varrho}(z) \quad (\ell \in \mathbb{N}_0). \tag{22}$$

The following recurrence relation reveals that, under the restrictions, computation of $E_{\mu,\varrho}(z)$ for the case $\mu > 1$ may be reduced to the case $0 < \mu \leq 1$ (see, e.g., ([53], [Equation (6)]), ([10], [Chapter XVIII]), ([54], [p. 24]), ([55], [Equation (2.2)]), [56]):

$$E_{\mu,\varrho}(z) = \frac{1}{2m+1} \sum_{k=-m}^{m} E_{\mu/(2m+1),\varrho}\Big(z^{1/(2m+1)} e^{2\pi i k/(2m+1)}\Big) \tag{23}$$

$$\big(\mu \in \mathbb{R}^+,\ \varrho \in \mathbb{R},\ z \in \mathbb{C},\ m = [(\mu-1)/2]+1\big).$$

1.2. Three Parametric Mittag-Leffler Function and Its Various Extensions

Giusti et al. pointed out the following interesting recurrence relations for the Prabhakar function (7) (see ([42], [Equations (4.2) and (4.3)])):

$$E^{\xi+1}_{\mu,\varrho}(z) = \frac{E^{\xi}_{\mu,\varrho-1}(z) + (1-\varrho+\mu\xi)E^{\xi}_{\mu,\varrho}(z)}{\mu\xi} \tag{24}$$

(see [43]) and

$$E^{\xi+1}_{\mu,\varrho}(z) = \frac{E^{\xi}_{\mu,\varrho-\mu-1}(z) + (1-\varrho+\mu)E^{\xi}_{\mu,\varrho-\mu}(z)}{\mu\xi z} \quad (z \in \mathbb{C}\setminus\{0\}) \tag{25}$$

(see [57]).

Shukla and Prajapati ([58], [Theorem 1]) offered an intriguing differential recurrence relation for an extended Mittag-Leffler function, which can be made by replacing $(\xi)_n$ in (7) with $(\xi)_{qn}$ under the restriction $q \in (0,1) \cup \mathbb{N}$ (see [59]), which was further generalized and investigated by Srivastava and Tomovski [50] who substituted a $k \in \mathbb{C}$ for the $q \in (0,1) \cup \mathbb{N}$ under constraints $\Re(\mu) > \max\{0, \Re(k)-1\}$ and $\Re(k) > 0$.

Salim ([60], [Theorem 2.2]) presented two interesting differential recurrence relation for an extended Mittag-Leffler function, which can be derived by replacing $n!$ in (7) with a Pochhammer symbol, say $(\eta)_n$ under the constraint $\Re(\eta) > 0$.

Kurulay and Bayram ([61], [Theorems 4]) established an intriguing differential recurrence relation for the Prabhakar function (7).

Dhakar and Sharma ([62], [Theorem 2.1]) provided an interesting differential recurrence relation for the k-Mittag-Leffler function (see [63]), which may be obtained by substituting $(\xi)_{n,k}$ and $\Gamma_k(\mu n+\varrho)$ for $(\xi)_n$ and $\Gamma(\mu n+\varrho)$ in (7), respectively. Here the k-Pochhammer symbol is defined as follows (see [64]):

$$(\gamma)_{n,k} := \begin{cases} \dfrac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & (n \in \mathbb{N};\, k \in \mathbb{R}^+;\, \gamma \in \mathbb{C}\setminus\{0\}), \\ \gamma(\gamma+k)\cdots(\gamma+(n-1)k) & (n \in \mathbb{N}; \gamma \in \mathbb{C}), \end{cases} \tag{26}$$

where Γ_k is the k-gamma function defined by

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}}\, t^{z-1}\, dt \quad (\Re(z) > 0;\ k \in \mathbb{R}^+). \tag{27}$$

Sharma and Jain ([65], [Theorem 1]) gave an interesting recurrence relation for the q-analog (or extension) of the Prabhakar function (7).

Gehlot ([66], [Theorem 2.1]) provided an intriguing differential recurrence relation for the following p-k Mittag-Leffler function (see [67]):

$$ {}_pE^{\xi,q}_{k,\mu,\varrho}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\xi)_{nq,k}\, z^n}{n!\, {}_p\Gamma_k(\mu n + \varrho)} \tag{28} $$

$$ \big(k,\ p \in \mathbb{R}^+,\ q \in (0,1) \cup \mathbb{N},\ \min\{\Re(\mu),\, \Re(\varrho),\, \Re(\xi)\} > 0\big). $$

Here the p-k Pochhammer symbol ${}_p(\alpha)_{n,k}$ and the p-k Gamma function ${}_p\Gamma_k$ are defined by

$$ {}_p(\alpha)_{n,k} = \left(\frac{\alpha p}{k}\right)\left(\frac{\alpha p}{k} + p\right) \cdots \left(\frac{\alpha p}{k} + (n-1)p\right) \tag{29} $$

$$ \big(k,\ p \in \mathbb{R}^+,\ n \in \mathbb{N},\ \Re(\alpha) > 0\big) $$

and ${}_p(\alpha)_{0,k} := 1$;

$$ {}_p\Gamma_k(z) = \int_0^{\infty} e^{-\frac{t^k}{p}}\, t^{z-1}\, \mathrm{d}t \quad \big(\Re(z) > 0;\ p,\, k \in \mathbb{R}^+\big). \tag{30} $$

Further Gehlot ([67], [Theorem 2.3]) presented an interesting recurrence relation for the p-k Mittag-Leffler function.

Choi et al. ([35], [Theorem 3.1]) established a differential recurrence relation for the extended Mittag-Leffler function, which may be obtained by replacing $(\xi)_n$ in (7) with the generalized Pochhammer symbol $(\xi;p)_n$. Here the generalized Pochhammer symbol $(\xi;p)_\nu$ $(\xi,\, \nu \in \mathbb{C})$ is defined by

$$ (\xi;p)_\nu := \begin{cases} \frac{\Gamma_p(\xi+\nu)}{\Gamma(\xi)} & (\Re(p) > 0), \\ (\xi)_\nu & (p = 0), \end{cases} \tag{31} $$

where $\Gamma_p(z)$ is the generalized gamma function given as follows:

$$ \Gamma_p(z) := \begin{cases} \int_0^{\infty} t^{z-1}\, e^{-t-\frac{p}{t}}\, \mathrm{d}t & (\Re(p) > 0,\, z \in \mathbb{C}), \\ \Gamma(z) & (p = 0,\, \Re(z) > 0). \end{cases} \tag{32} $$

The aim of this article is to explore some new recurrence relations for the two parametric Mittag-Leffler function. Then, we discuss several applications of such recurrence relations. To begin, we express a number of the two parametric Mittag-Leffler functions in terms of elementary functions by combining appropriate pairings of particular instances of those recurrence relations. Second, we establish four new relations among the Fox–Wright functions by applying Riemann–Liouville fractional integral and differential operators to one of those recurrence relations. Certain particular cases of the Fox–Wright function relations exhibit four relations among the generalized hypergeometric functions. Finally, we propose several pertinent research questions.

Further, for our purpose, we recall the classical Beta function (see, e.g., ([5], [p. 8]))

$$ B(\alpha,\, \beta) = \begin{cases} \displaystyle\int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1}\, \mathrm{d}\tau & (\Re(\alpha) > 0,\ \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\, \beta \in \mathbb{C} \setminus \mathbb{Z}_{\le 0}). \end{cases} \tag{33} $$

The following formula is one of a number of definite integrals that may be expressed in terms of the Beta function (see, e.g., ([5], [p. 9, Equation (49)])):

$$ \int_a^b (\tau - a)^{\alpha-1}(b-\tau)^{\beta-1}\mathrm{d}\tau = (b-a)^{\alpha+\beta-1}\, B(\alpha,\beta) \tag{34} $$

$$(b > a,\ \Re(\alpha) > 0,\ \Re(\beta) > 0).$$

2. Recurrence Relations

This section explores some new recurrence relations for the two parametric Mittag-Leffler function (5).

Theorem 1. *Let μ, ϱ, $z \in \mathbb{C}$ with $\Re(\mu) > 0$. Then*

$$E_{\mu,\varrho}(z) = \varrho(\varrho+1)E_{\mu,\varrho+2}(z) - \varrho(\varrho+1)zE_{\mu,\mu+\varrho+2}(z) + zE_{\mu,\mu+\varrho}(z) \quad (\Re(\varrho) > 0); \tag{35}$$

$$\begin{aligned} E_{\mu,\varrho}(z) =& z^3 E_{\mu,3\mu+\varrho}(z) - z^2(\mu+\varrho)E_{\mu,2\mu+\varrho+1}(z) + z(\mu+\varrho)E_{\mu,\mu+\varrho+1}(z) \\ &+ \frac{z^2}{\Gamma(2\mu+\varrho)} + \frac{1}{\Gamma(\varrho)} \quad (\Re(\varrho) > 0); \end{aligned} \tag{36}$$

$$\begin{aligned} (\varrho-1)E_{\mu,\varrho}(z) =& z^3 E_{\mu,3\mu+\varrho-1}(z) - zE_{\mu,\mu+\varrho-1}(z) + z(\varrho-1)E_{\mu,\mu+\varrho}(z) \\ &+ \frac{z^2}{\Gamma(2\mu+\varrho-1)} + \frac{z}{\Gamma(\mu+\varrho-1)} + \frac{1}{\Gamma(\varrho-1)} \quad (\Re(\varrho) > 1); \end{aligned} \tag{37}$$

$$\begin{aligned} (\varrho-1)E_{\mu,\varrho}(z) =& z(\varrho-1)E_{\mu,\mu+\varrho}(z) - z^2E_{\mu,2\mu+\varrho-1}(z) + \frac{1}{z}E_{\mu,\varrho-\mu-1}(z) \\ &- \frac{z}{\Gamma(\mu+\varrho-1)} - \frac{1}{z\Gamma(\varrho-\mu-1)} \quad (\Re(\varrho) > 1,\ \Re(\varrho-\mu) > 1); \end{aligned} \tag{38}$$

$$\begin{aligned} (\varrho-1)E_{\mu,\varrho}(z) =& (\varrho-1)z^2E_{\mu,2\mu+\varrho}(z) - z^2E_{\mu,2\mu+\varrho-1}(z) + \frac{1}{z}E_{\mu,\varrho-\mu-1}(z) \\ &- \frac{1}{z\Gamma(\varrho-\mu-1)} - \frac{\mu z}{\Gamma(\mu+\varrho)} \quad (\Re(\varrho) > 0,\ \Re(\varrho-\mu) > 1); \end{aligned} \tag{39}$$

$$\begin{aligned} (\varrho-2)(\varrho-1)E_{\mu,\varrho}(z) =& z(\varrho-1)(\varrho-2)E_{\mu,\mu+\varrho}(z) + z^3E_{\mu,3\mu+\varrho-2}(z) \\ &- zE_{\mu,\mu+\varrho-2}(z) + \frac{z^2}{\Gamma(2\mu+\varrho-2)} + \frac{z}{\Gamma(\mu+\varrho-2)} + \frac{1}{\Gamma(\varrho-2)} \quad (\Re(\varrho) > 2); \end{aligned} \tag{40}$$

$$\begin{aligned} (\mu+\varrho-2)(\mu+\varrho-1)E_{\mu,\varrho}(z) =& (\mu+\varrho-2)(\mu+\varrho-1)z^2E_{\mu,2\mu+\varrho}(z) \\ &+ z^3E_{\mu,3\mu+\varrho-2}(z) - z^2E_{\mu,2\mu+\varrho-2}(z) \\ &+ \frac{z^2}{\Gamma(2\mu+\varrho-2)} + \frac{z}{\Gamma(\mu+\varrho-2)} + \frac{(\mu+\varrho-2)(\mu+\varrho-1)}{\Gamma(\varrho)} \\ &(\Re(\varrho) > 0,\ \Re(\varrho+\mu) > 2). \end{aligned} \tag{41}$$

Proof. We establish only (35). Let $\mathcal{R}_1$ be the right-handed member of (35). By using (5), we obtain

$$\begin{aligned} \mathcal{R}_1 =& \varrho(\varrho+1)\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\varrho+2)} - \varrho(\varrho+1)\sum_{k=0}^{\infty}\frac{z^{k+1}}{\Gamma(\mu k+\mu+\varrho+2)} \\ &+ \sum_{k=0}^{\infty}\frac{z^{k+1}}{\Gamma(\mu k+\mu+\varrho)}. \end{aligned} \tag{42}$$

Setting $k+1=k'$ on the second and third summations in (42) and dropping the prime on k, we obtain

$$\begin{aligned}\mathcal{R}_1 =& \varrho(\varrho+1)\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\varrho+2)} - \varrho(\varrho+1)\sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\mu k+\varrho+2)} + \sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\mu k+\varrho)} \\ =& \frac{\varrho(\varrho+1)}{\Gamma(\varrho+2)} + \sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\mu k+\varrho)} = \frac{1}{\Gamma(\varrho)} + \sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\mu k+\varrho)} \\ =& \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\varrho)} = E_{\mu,\varrho}(z).\end{aligned} \tag{43}$$

For the third equality of (43), the following fundamental relation for the Gamma function

$$\Gamma(z+1) = z\,\Gamma(z) \tag{44}$$

is used. The proof of (35) is complete.

Likewise, the remaining relations (36)–(41) may be established. We omit specifics. □

Taking $\varrho = 1$ in (35), (36), and (41), we obtain some relations between the Mittag-Leffler function (1) and the two parametric Mittag-Leffler function (5) in the following corollary.

Corollary 1. *Let $\mu, z \in \mathbb{C}$ with $\Re(\mu) > 0$. Then*

$$E_\mu(z) = 2E_{\mu,3}(z) - 2zE_{\mu,\mu+3}(z) + zE_{\mu,\mu+1}(z); \tag{45}$$

$$\begin{aligned}E_\mu(z) =& z^3E_{\mu,3\mu+1}(z) - z^2(\mu+1)E_{\mu,2\mu+2}(z) \\ &+ z(\mu+1)E_{\mu,\mu+2}(z) + \frac{z^2}{\Gamma(2\mu+1)} + 1;\end{aligned} \tag{46}$$

$$\begin{aligned}\mu(\mu-1)E_\mu(z) =& \mu(\mu-1)z^2E_{\mu,2\mu+1}(z) + z^3E_{\mu,3\mu-1}(z) \\ &- z^2E_{\mu,2\mu-1}(z) + \frac{z^2}{\Gamma(2\mu-1)} + \frac{z}{\Gamma(\mu-1)} + \mu(\mu-1).\end{aligned} \tag{47}$$

Similar to (2), the following corollary provides certain interesting expressions of several elementary functions in terms of the two parametric functions (5).

Corollary 2. *Let $z \in \mathbb{C}$. Then*

$$e^z = E_1(z) = zE_{1,2}(z) + 2E_{1,3}(z) - 2zE_{1,4}(z); \tag{48}$$

$$e^z = E_1(z) = 2zE_{1,3}(z) + \left(z^3 - 2z^2\right)E_{1,4}(z) + \frac{z^2}{2} + 1; \tag{49}$$

$$\cos z = E_2(-z^2) = \left(2 - z^2\right)E_{2,3}(-z^2) + 2z^2E_{2,5}(-z^2); \tag{50}$$

$$\begin{aligned}\cos z = E_2(-z^2) =& -3z^2E_{2,4}(-z^2) - 3z^4E_{2,6}(-z^2) \\ &- z^6E_{2,7}(-z^2) + \frac{z^4}{24} + 1;\end{aligned} \tag{51}$$

$$\begin{aligned}\cos z = E_2(-z^2) =& -\frac{z^4}{2}E_{2,3}(-z^2) + \left(z^4 - \frac{z^6}{2}\right)E_{2,5}(-z^2) \\ &+ \frac{z^4}{4} - \frac{z^2}{2} + 1;\end{aligned} \tag{52}$$

$$\sin z = z\,E_{2,2}(-z^2) = \left(6z - z^3\right)E_{2,4}(-z^2) + 6z^3 E_{2,6}(-z^2); \tag{53}$$

$$\begin{aligned}\sin z = z\,E_{2,2}(-z^2) = &-4z^3 E_{2,5}(-z^2) - 4z^5 E_{2,7}(-z^2)\\ &- z^7 E_{2,8}(-z^2) + \frac{z^5}{120} + z;\end{aligned} \tag{54}$$

$$\begin{aligned}\sin z = z\,E_{2,2}(-z^2) = &z^3 E_{2,3}(-z^2) - z^3 E_{2,4}(-z^2)\\ &- z^7 E_{2,7}(-z^2) + \frac{z^5}{24} - \frac{z^3}{2} + z;\end{aligned} \tag{55}$$

$$\begin{aligned}\sin z = z\,E_{2,2}(-z^2) = &-\frac{z^5}{6}E_{2,4}(-z^2) + \left(z^5 - \frac{z^7}{6}\right)E_{2,6}(-z^2)\\ &+ \frac{z^5}{36} - \frac{z^3}{6} + z;\end{aligned} \tag{56}$$

$$\cosh z = E_2(z^2) = \left(2 + z^2\right)E_{2,3}(z^2) - 2z^2 E_{2,5}(z^2); \tag{57}$$

$$\begin{aligned}\cosh z = E_2(z^2) = &3z^2 E_{2,4}(z^2) - 3z^4 E_{2,6}(z^2)\\ &+ z^6 E_{2,7}(z^2) + \frac{z^4}{24} + 1;\end{aligned} \tag{58}$$

$$\begin{aligned}\cosh z = E_2(z^2) = &-\frac{z^4}{2}E_{2,3}(z^2) + \left(z^4 + \frac{z^6}{2}\right)E_{2,5}(z^2)\\ &+ \frac{z^4}{4} + \frac{z^2}{2} + 1;\end{aligned} \tag{59}$$

$$\sinh z = z\,E_{2,2}(z^2) = \left(6z + z^3\right)E_{2,4}(z^2) - 6z^3 E_{2,6}(z^2); \tag{60}$$

$$\begin{aligned}\sinh z = z\,E_{2,2}(z^2) = &4z^3 E_{2,5}(z^2) - 4z^5 E_{2,7}(z^2)\\ &+ z^7 E_{2,8}(z^2) + \frac{z^5}{120} + z;\end{aligned} \tag{61}$$

$$\begin{aligned}\sinh z = z\,E_{2,2}(z^2) = &-z^3 E_{2,3}(z^2) + z^3 E_{2,4}(z^2) + \frac{z^5}{24}\\ &+ z^7 E_{2,7}(z^2) + \frac{z^3}{2} + z;\end{aligned} \tag{62}$$

$$\begin{aligned}\sinh z = z\,E_{2,2}(z^2) = &-\frac{z^5}{6}E_{2,4}(z^2) + \left(z^5 + \frac{z^7}{6}\right)E_{2,6}(z^2)\\ &+ \frac{z^5}{36} + \frac{z^3}{6} + z.\end{aligned} \tag{63}$$

Proof. Setting $\mu = 1$ in (45) and (46), respectively, yields (48) and (49).

Putting $\mu = 2$ and replacing z by $-z^2$ in (45), (46), and (47), respectively, gives (50), (51), and (52).

Taking $\mu = 2, \varrho = 2$ and replacing z by $-z^2$ in (35), (36), (37), and (41), respectively, produces (53), (54), (55), and (56).

Setting $\mu = 2$ and replacing z by z^2 in (45), (46), and (47), respectively, offers (57), (58), and (59).

Putting $\mu = 2, \varrho = 2$ and replacing z by z^2 in (35), (36), (37), and (41), respectively, affords (60), (61), (62), and (63). □

By combining appropriate pairings of the identities in Corollary 2, such as (2), we may express several of the two parametric Mittag-Leffler functions in terms of elementary functions, as stated in the following corollary.

Corollary 3. *The following formulas hold.*

$$E_{2,3}(-z^2) = -\frac{1}{z^2}(\cos z - 1) \quad (z \in \mathbb{C} \setminus \{0\}); \tag{64}$$

$$E_{2,5}(-z^2) = \frac{1}{z^4}\Big(\cos z + \frac{z^2}{2} - 1\Big) \quad (z \in \mathbb{C} \setminus \{0\}); \tag{65}$$

$$E_{2,4}(-z^2) = \frac{1}{z^2}\Big(1 - \frac{\sin z}{z}\Big) \quad (z \in \mathbb{C} \setminus \{0\}); \tag{66}$$

$$E_{2,6}(-z^2) = \frac{1}{z^4}\Big(-1 + \frac{z^2}{6} + \frac{\sin z}{z}\Big) \quad (z \in \mathbb{C} \setminus \{0\}); \tag{67}$$

$$E_{2,3}(z^2) = \frac{1}{z^2}(\cosh z - 1) \quad (z \in \mathbb{C} \setminus \{0\}); \tag{68}$$

$$E_{2,5}(z^2) = \frac{1}{z^4}(\cosh z - 1) - \frac{1}{2z^2} \quad (z \in \mathbb{C} \setminus \{0\}); \tag{69}$$

$$E_{2,4}(z^2) = \frac{1}{z^3}\sinh z - \frac{1}{z^2} \quad (z \in \mathbb{C} \setminus \{0\}); \tag{70}$$

$$E_{2,6}(z^2) = \frac{1}{z^5}\sinh z - \frac{1}{z^4} - \frac{1}{6\,z^2} \quad (z \in \mathbb{C} \setminus \{0\}). \tag{71}$$

Proof. From (50) and (52), we can obtain (64) and (65).

From (53) and (56), we can derive (66) and (67).

From (57) and (59), we can obtain (68) and (69).

From (60) and (63), we can find (70) and (71). □

3. Certain Relations among the Fox–Wright Functions

In this section, by applying the Riemann–Liouville fractional integrals and derivatives to the recurrence relation (35), we obtain four relations among the Fox–Wright functions ${}_p\Psi_q$, as stated in the following theorems. We also consider some particular cases of our main results.

Theorem 2. *Let μ, ϱ, $x \in \mathbb{R}^+$, and $a \in \mathbb{C}$. Then*

$$\begin{aligned}
&{}_1\Psi_1\left[\begin{matrix}(1,1);\\ (\mu+\varrho,\mu);\end{matrix}\; ax^\mu\right] - ax^\mu\,{}_1\Psi_1\left[\begin{matrix}(1,1);\\ (2\mu+\varrho,\mu);\end{matrix}\; ax^\mu\right]\\
&\quad = \varrho(\varrho+1)\,{}_2\Psi_2\left[\begin{matrix}(\varrho,\mu),(1,1);\\ (\varrho+2,\mu),(\mu+\varrho,\mu);\end{matrix}\; ax^\mu\right]\\
&\qquad - a\varrho(\varrho+1)x^\mu\,{}_2\Psi_2\left[\begin{matrix}(\mu+\varrho,\mu),(1,1);\\ (\mu+\varrho+2,\mu),(2\mu+\varrho,\mu);\end{matrix}\; ax^\mu\right].
\end{aligned} \tag{72}$$

Proof. A particular case of (34) is

$$\begin{aligned}
&\int_0^x \tau^{\alpha-1}(x-\tau)^{\beta-1}\mathrm{d}\tau = x^{\alpha+\beta-1}\,B(\alpha,\beta)\\
&\qquad (x>0,\ \Re(\alpha)>0,\ \Re(\beta)>0).
\end{aligned} \tag{73}$$

Replacing z by az^μ in (35) and multiplying both sides of the resulting identity by $z^{\varrho-1}$, we obtain

$$\begin{aligned}
z^{\varrho-1}E_{\mu,\varrho}(az^\mu) =& \varrho(\varrho+1)z^{\varrho-1}E_{\mu,\varrho+2}(az^\mu) - a\varrho(\varrho+1)z^{\mu+\varrho-1}E_{\mu,\mu+\varrho+2}(az^\mu)\\
&+ az^{\mu+\varrho-1}E_{\mu,\mu+\varrho}(az^\mu).
\end{aligned} \tag{74}$$

Taking the left-sided Riemann–Liouville fractional integral (12) on both sides of (74), we obtain

$$\begin{aligned}\left(I^{\mu}_{0+}\left[z^{\varrho-1}E_{\mu,\varrho}(az^{\mu})\right]\right)(x) =&\varrho(\varrho+1)\left(I^{\mu}_{0+}\left[z^{\varrho-1}E_{\mu,\varrho+2}(az^{\mu})\right]\right)(x)\\ &-a\varrho(\varrho+1)\left(I^{\mu}_{0+}\left[z^{\mu+\varrho-1}E_{\mu,\mu+\varrho+2}(az^{\mu})\right]\right)(x)\\ &+a\left(I^{\mu}_{0+}\left[z^{\mu+\varrho-1}E_{\mu,\mu+\varrho}(az^{\mu})\right]\right)(x).\end{aligned} \tag{75}$$

Let $\mathcal{L}_1$ be the left-handed member of (75). Interchanging the integral and summation in (5), which may be verified under restrictions, we obtain

$$\mathcal{L}_1=\frac{1}{\Gamma(\mu)}\sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\mu k+\varrho)}\int_0^x (x-\tau)^{\mu-1}\,\tau^{\mu k+\varrho-1}\,\mathrm{d}\tau. \tag{76}$$

Using (73) in the integral in (76), we obtain

$$\mathcal{L}_1=x^{\mu+\varrho-1}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(\mu k+\mu+\varrho)}\frac{(ax^{\mu})^k}{k!}=x^{\mu+\varrho-1}\,{}_1\Psi_1\left[\begin{matrix}(1,1);\\(\mu+\varrho,\mu);\end{matrix}\;ax^{\mu}\right]. \tag{77}$$

Now, let $\mathcal{R}_1$ be the right-handed member of (75). Similarly, as in obtaining (77), we derive

$$\begin{aligned}\mathcal{R}_1=&\varrho(\varrho+1)x^{\mu+\varrho-1}{}_2\Psi_2\left[\begin{matrix}(\varrho,\mu),(1,1);\\(\varrho+2,\mu),(\mu+\varrho,\mu);\end{matrix}\;ax^{\mu}\right]\\ &-a\varrho(\varrho+1)x^{2\mu+\varrho-1}{}_2\Psi_2\left[\begin{matrix}(\mu+\varrho,\mu),(1,1);\\(\mu+\varrho+2,\mu),(2\mu+\varrho,\mu);\end{matrix}\;ax^{\mu}\right]\\ &+ax^{2\mu+\varrho-1}{}_1\Psi_1\left[\begin{matrix}(1,1);\\(2\mu+\varrho,\mu);\end{matrix}\;ax^{\mu}\right].\end{aligned} \tag{78}$$

Finally, equating the two identities in (77) and (78), we have (72). □

Theorem 3. *Let $x>0$, $0<\mu<1-\varrho$, and $a\in\mathbb{C}$. Then*

$$\begin{aligned}&x^{\mu}{}_2\Psi_2\left[\begin{matrix}(1-\mu-\varrho,\mu),(1,1);\\(\varrho,\mu),(1-\varrho,\mu);\end{matrix}\;ax^{-\mu}\right]\\ &=\varrho(\varrho+1)x^{\mu}{}_2\Psi_2\left[\begin{matrix}(1-\mu-\varrho,\mu),(1,1);\\(\varrho+2,\mu),(1-\varrho,\mu);\end{matrix}\;ax^{-\mu}\right]\\ &-a\varrho(\varrho+1){}_2\Psi_2\left[\begin{matrix}(1-\varrho,\mu),(1,1);\\(\mu+\varrho+2,\mu),(\mu-\varrho+1,\mu);\end{matrix}\;ax^{-\mu}\right]\\ &+a{}_2\Psi_2\left[\begin{matrix}(1-\varrho,\mu),(1,1);\\(\mu+\varrho,\mu),(\mu-\varrho+1,\mu);\end{matrix}\;ax^{-\mu}\right].\end{aligned} \tag{79}$$

Proof. The following formula is readily derived:

$$\begin{gathered}\int_x^{\infty}(\tau-x)^{\alpha-1}\,\tau^{\beta-1}\mathrm{d}\tau=x^{\alpha+\beta-1}\,B(\alpha,1-\alpha-\beta)\\ (x>0,\;0<\Re(\alpha)<1-\Re(\beta)).\end{gathered} \tag{80}$$

Replacing z by $az^{-\mu}$ in (35) and multiplying both sides of the resulting identity by $z^{\varrho-1}$, we obtain

$$\begin{aligned}z^{\varrho-1}E_{\mu,\varrho}(az^{-\mu}) =&\varrho(\varrho+1)z^{\varrho-1}E_{\mu,\varrho+2}(az^{-\mu})-a\varrho(\varrho+1)z^{-\mu+\varrho-1}E_{\mu,\mu+\varrho+2}(az^{-\mu})\\ &+az^{-\mu+\varrho-1}E_{\mu,\mu+\varrho}(az^{-\mu}).\end{aligned} \tag{81}$$

Taking the right-sided Riemann–Liouville fractional integral (13) on both sides of (81), we obtain

$$\begin{aligned}\left(I_{\infty-}^{\mu}\left[z^{\varrho-1}E_{\mu,\varrho}(az^{-\mu})\right]\right)(x) =&\varrho(\varrho+1)\left(I_{\infty-}^{\mu}\left[z^{\varrho-1}E_{\mu,\varrho+2}(az^{-\mu})\right]\right)(x)\\&-a\varrho(\varrho+1)\left(I_{\infty-}^{\mu}\left[z^{-\mu+\varrho-1}E_{\mu,\mu+\varrho+2}(az^{-\mu})\right]\right)(x)\\&+a\left(I_{\infty-}^{\mu}\left[z^{-\mu+\varrho-1}E_{\mu,\mu+\varrho}(az^{-\mu})\right]\right)(x).\end{aligned}\tag{82}$$

Now, similarly as in the proof of Theorem 2, applying (13) to each term of (82) and using (80), we can obtain the desired identity (79). □

Theorem 4. *Let μ, ϱ, $x \in \mathbb{R}^+$. Further, let $a \in \mathbb{C}$ and $n = [\mu] + 1 \in \mathbb{N}$. Then*

$$\begin{aligned}&x^{-\mu}{}_1\Psi_1\left[\begin{matrix}(1,1);\\(\varrho-\mu,\mu);\end{matrix}\ ax^{\mu}\right]-a\cdot{}_1\Psi_1\left[\begin{matrix}(1,1);\\(\varrho,\mu);\end{matrix}\ ax^{\mu}\right]\\&=\varrho(\varrho+1)x^{-\mu}{}_2\Psi_2\left[\begin{matrix}(\varrho,\mu),(1,1);\\(\varrho+2,\mu),(\varrho-\mu,\mu);\end{matrix}\ ax^{\mu}\right]\\&\quad-a\varrho(\varrho+1){}_2\Psi_2\left[\begin{matrix}(\mu+\varrho,\mu),(1,1);\\(\mu+\varrho+2,\mu),(\varrho,\mu);\end{matrix}\ ax^{\mu}\right].\end{aligned}\tag{83}$$

Proof. Similar to (75), we have

$$\begin{aligned}\left(D_{0+}^{\mu}\left[z^{\varrho-1}E_{\mu,\varrho}(az^{\mu})\right]\right)(x)=&\varrho(\varrho+1)\left(D_{0+}^{\mu}\left[z^{\varrho-1}E_{\mu,\varrho+2}(az^{\mu})\right]\right)(x)\\&-a\varrho(\varrho+1)\left(D_{0+}^{\mu}\left[z^{\mu+\varrho-1}E_{\mu,\mu+\varrho+2}(az^{\mu})\right]\right)(x)\\&+a\left(D_{0+}^{\mu}\left[z^{\mu+\varrho-1}E_{\mu,\mu+\varrho}(az^{\mu})\right]\right)(x).\end{aligned}\tag{84}$$

Let $\mathcal{L}$ be the left-handed member of (84). Using (14), (12), and (5), we obtain

$$\begin{aligned}\mathcal{L}&=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left\{\frac{1}{\Gamma(n-\mu)}\int_0^x(x-\tau)^{n-\mu-1}\tau^{\varrho-1}\sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\mu k+\varrho)}\tau^{\mu k}\,\mathrm{d}\tau\right\}\\&=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left\{\frac{1}{\Gamma(n-\mu)}\sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\mu k+\varrho)}\int_0^x(x-\tau)^{n-\mu-1}\tau^{\varrho+\mu k-1}\mathrm{d}\tau\right\},\end{aligned}\tag{85}$$

under the restrictions of which integral and summation can be interchanged. Using (73) to evaluate the integral in (85) and interchanging differentiation and summation, we obtain

$$\mathcal{L}=\sum_{k=0}^{\infty}\frac{a^k}{\Gamma(n-\mu+\varrho+\mu k)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(x^{n-\mu+\varrho+\mu k-1}\right).\tag{86}$$

Employing the following easily derivable formula

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(x^{\lambda}\right)=(-1)^n\,(-\lambda)_n\,x^{\lambda-n}=(-1)^n\,\frac{\Gamma(n-\lambda)}{\Gamma(-\lambda)}\,x^{\lambda-n}\tag{87}$$

in (86), we find

$$\mathcal{L}=x^{\varrho-\mu-1}\sum_{k=0}^{\infty}\frac{(ax^{\mu})^k}{\Gamma(n-\mu+\varrho+\mu k)}\cdot(-1)^n\frac{\Gamma(1+\mu-\varrho-\mu k)}{\Gamma(1-n+\mu-\varrho-\mu k)}.\tag{88}$$

Using the following well-known formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}) \tag{89}$$

in (88), we derive

$$\begin{aligned} \mathcal{L} &= x^{\varrho-\mu-1} \sum_{k=0}^{\infty} \frac{(ax^{\mu})^k}{\Gamma(\varrho-\mu+\mu k)} \\ &= x^{\varrho-\mu-1}{}_1\Psi_1\left[\begin{matrix} (1,1); \\ (\varrho-\mu,\mu); \end{matrix}\ ax^{\mu}\right]. \end{aligned} \tag{90}$$

Further, the other three fractional derivatives in (84) can be evaluated as in (90). Finally, all of those evaluations that are used in (84) gives the desired identity (83). □

Theorem 5. *Let μ, ϱ, $x \in \mathbb{R}^+$ and $0 < n - \mu < 1 - \varrho$. Further, let $a \in \mathbb{C}$ and $n = [\mu] + 1 \in \mathbb{N}$. Then*

$$\begin{aligned} &{}_2\Psi_2\left[\begin{matrix} (\mu-\varrho+1,\mu),(1,1); \\ (\varrho,\mu),(1-\varrho,\mu); \end{matrix}\ ax^{-\mu}\right] \\ &= \varrho(\varrho+1){}_2\Psi_2\left[\begin{matrix} (1+\mu-\varrho,\mu),(1,1); \\ (\varrho+2,\mu),(1-\varrho,\mu); \end{matrix}\ ax^{-\mu}\right] \\ &\quad - a\,\varrho(\varrho+1)\,x^{-\mu}\,{}_2\Psi_2\left[\begin{matrix} (1+2\mu-\varrho,\mu),(1,1); \\ (\mu+\varrho+2,\mu),(1+\mu-\varrho,\mu); \end{matrix}\ ax^{-\mu}\right] \\ &\quad + a\,x^{-\mu}\,{}_2\Psi_2\left[\begin{matrix} (1+2\mu-\varrho,\mu),(1,1); \\ (\mu+\varrho,\mu),(1+\mu-\varrho,\mu); \end{matrix}\ ax^{-\mu}\right]. \end{aligned} \tag{91}$$

Proof. Similar to (82), we have

$$\begin{aligned} &\left(D^{\mu}_{\infty-}\left[z^{\varrho-1}E_{\mu,\varrho}(az^{-\mu})\right]\right)(x) \\ &= \varrho(\varrho+1)\left(D^{\mu}_{\infty-}\left[z^{\varrho-1}E_{\mu,\varrho+2}(az^{-\mu})\right]\right)(x) \\ &\quad - a\varrho(\varrho+1)\left(D^{\mu}_{\infty-}\left[z^{-\mu+\varrho-1}E_{\mu,\mu+\varrho+2}(az^{-\mu})\right]\right)(x) \\ &\quad + a\left(D^{\mu}_{\infty-}\left[z^{-\mu+\varrho-1}E_{\mu,\mu+\varrho}(az^{-\mu})\right]\right)(x). \end{aligned} \tag{92}$$

As in the proof of Theorem 4, by using (80), we can evaluate all of the four fractional derivatives in (92). For example,

$$\left(D^{\mu}_{\infty-}\left[z^{\varrho-1}E_{\mu,\varrho}(az^{-\mu})\right]\right)(x) = x^{\varrho-\mu-1}\,{}_2\Psi_2\left[\begin{matrix} (\mu-\varrho+1,\mu),(1,1); \\ (\varrho,\mu),(1-\varrho,\mu); \end{matrix}\ ax^{-\mu}\right]. \tag{93}$$

Hence, all the evaluations which are set in (92) yields the desired identity (91). □

Since the results presented in Theorems 2–5 are quite general, they may reduce to yield a number of interesting identities. For example, setting $\mu = 1$ in the results, with the aid of (10), we obtain several new relations among the generalized hypergeometric functions ${}_pF_q$, as stated in the following corollary (for current research of summation and reduction formulae for ${}_pF_q$, see [68,69]).

Corollary 4. *Let $a \in \mathbb{C}$. Then*

$$\begin{aligned} &{}_1F_1\left[\begin{matrix} 1; \\ \varrho+1; \end{matrix}\ ax\right] - \frac{ax}{\varrho+1}\,{}_1F_1\left[\begin{matrix} 1; \\ \varrho+2; \end{matrix}\ ax\right] \\ &= {}_2F_2\left[\begin{matrix} \varrho,1; \\ \varrho+2,\varrho+1; \end{matrix}\ ax\right] - \frac{\varrho\,ax}{(\varrho+2)(\varrho+1)}\,{}_2F_2\left[\begin{matrix} \varrho+1,1; \\ \varrho+3,\varrho+2; \end{matrix}\ ax\right] \end{aligned} \tag{94}$$

$$(x \in \mathbb{C},\ \varrho \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1});$$

$$\begin{aligned} &x\,{}_2F_2\left[\begin{array}{r} -\varrho, 1;\ \dfrac{a}{x} \\ \varrho+2, 1-\varrho;\ \end{array}\right] - x\,{}_2F_2\left[\begin{array}{r} -\varrho, 1;\ \dfrac{a}{x} \\ \varrho, 1-\varrho;\ \end{array}\right] \\ &= \frac{a\varrho}{(\varrho-1)(\varrho+2)}\,{}_2F_2\left[\begin{array}{r} 1-\varrho, 1;\ \dfrac{a}{x} \\ \varrho+3, 2-\varrho;\ \end{array}\right] - \frac{a}{\varrho-1}\,{}_2F_2\left[\begin{array}{r} 1-\varrho, 1;\ \dfrac{a}{x} \\ \varrho+1, 2-\varrho;\ \end{array}\right] \end{aligned} \tag{95}$$

$$(x \in \mathbb{C} \setminus \{0\},\ \varrho \in \mathbb{C} \setminus \mathbb{Z});$$

$$\begin{aligned} &{}_1F_1\left[\begin{array}{r} 1;\ ax \\ \varrho-1;\ \end{array}\right] - \frac{ax}{\varrho-1}\,{}_1F_1\left[\begin{array}{r} 1;\ ax \\ \varrho;\ \end{array}\right] \\ &= {}_2F_2\left[\begin{array}{r} \varrho, 1;\ ax \\ \varrho+2, \varrho-1;\ \end{array}\right] - \frac{\varrho\,ax}{(\varrho+2)\,(\varrho-1)}\,{}_2F_2\left[\begin{array}{r} \varrho+1, 1;\ ax \\ \varrho+3, \varrho;\ \end{array}\right]; \end{aligned} \tag{96}$$

$$(x \in \mathbb{C},\ \varrho \in \mathbb{C} \setminus \mathbb{Z}_{\leq 1});$$

$$\begin{aligned} &{}_2F_2\left[\begin{array}{r} 2-\varrho, 1;\ \dfrac{a}{x} \\ \varrho+2, 1-\varrho;\ \end{array}\right] - {}_2F_2\left[\begin{array}{r} 2-\varrho, 1;\ \dfrac{a}{x} \\ \varrho, 1-\varrho;\ \end{array}\right] \\ &= \frac{a(\varrho-2)}{x(\varrho-1)(\varrho+2)}\,{}_2F_2\left[\begin{array}{r} 3-\varrho, 1;\ \dfrac{a}{x} \\ \varrho+3, 2-\varrho;\ \end{array}\right] - \frac{a(\varrho-2)}{x\varrho(\varrho-1)}\,{}_2F_2\left[\begin{array}{r} 3-\varrho, 1;\ \dfrac{a}{x} \\ \varrho+1, 2-\varrho;\ \end{array}\right] \end{aligned} \tag{97}$$

$$(x \in \mathbb{C} \setminus \{0\},\ \varrho \in \mathbb{C} \setminus \mathbb{Z}).$$

It is worth noting that the restriction on each identity in Corollary 4 may be widened via analytic continuation. Further, each side of the identities (94) and (96) is easily checked to become 1.

4. Concluding Remarks and Posing Problems

We reviewed the birth of the Mittag-Leffler function and its several extensions (among numerous ones) together with their diverse applications to a variety of research areas, particularly, fractional calculus. We recalled many known recurrence (or differential recurrence) relations for the two parametric Mittag-Leffler function (5) and the three parametric Mittag-Leffler function (7). Then, we established a number of recurrence relations for the two parametric Mittag-Leffler function (5). Further, by using appropriate pairings of those recurrence relations presented in this paper, we demonstrated that certain particular cases of the two parametric Mittag-Leffler function (5) can be expressed in terms of elementary functions. Further, by applying the Riemann–Liouville fractional integral and derivative operators to one of the recurrence relations for the two parametric Mittag-Leffler function (5), we derived four new relations among the Fox–Wright functions. Finally, we provided four relations among the generalized hypergeometric functions ${}_pF_q$ as a particular case of those Fox–Wright function relations.

As in Section 3, by using the other formulas (36)–(41), we may derive some relations among the Fox–Wright functions ${}_p\Psi_q$ together with their corresponding relations among the generalized hypergeometric functions ${}_pF_q$ as particular cases. For example, as with Theorem 2, using (36), we obtain

$$
\begin{aligned}
&{}_1\Psi_1\left[\begin{array}{c}(1,1);\\(\mu+\varrho,\mu);\end{array} ax^{\mu}\right] - a^3x^{3\mu}\,{}_1\Psi_1\left[\begin{array}{c}(1,1);\\(4\mu+\varrho,\mu);\end{array} ax^{\mu}\right]\\
&\quad= (\mu+\varrho)\,a\,x^{\mu}\,{}_2\Psi_2\left[\begin{array}{c}(\mu+\varrho,\mu),\,(1,1);\\(\mu+\varrho+1,\mu),\,(2\mu+\varrho,\mu);\end{array} ax^{\mu}\right]\\
&\qquad-(\mu+\varrho)\,a^2\,x^{2\mu}\,{}_2\Psi_2\left[\begin{array}{c}(2\mu+\varrho,\mu),\,(1,1);\\(2\mu+\varrho+1,\mu),\,(3\mu+\varrho,\mu);\end{array} ax^{\mu}\right]\\
&\qquad+\frac{a^2x^{2\mu}}{\Gamma(3\mu+\varrho)}+\frac{1}{\Gamma(\mu+\varrho)}\\
&\qquad\qquad(\mu,\,\varrho,\,x\in\mathbb{R}^+,\,a\in\mathbb{C})
\end{aligned}
\tag{98}
$$

and

$$
\begin{aligned}
&{}_1F_1\left[\begin{array}{c}1;\\1+\varrho;\end{array} ax\right]-\frac{a^3x^3}{(\varrho+1)(\varrho+2)(\varrho+3)}\,{}_1F_1\left[\begin{array}{c}1;\\\varrho+4;\end{array} ax\right]\\
&\quad=\frac{a\,x}{\varrho+1}\,{}_2F_2\left[\begin{array}{c}\varrho+1,\,1;\\\varrho+2,\,\varrho+2;\end{array} ax\right]-\frac{a^2\,x^2}{(\varrho+2)^2}\,{}_2F_2\left[\begin{array}{c}\varrho+2,\,1;\\\varrho+3,\,\varrho+3;\end{array} ax\right]\\
&\quad+\frac{a^2x^2}{(\varrho+1)(\varrho+2)}+1\quad(\varrho\in\mathbb{C}\setminus\mathbb{Z}_{\leq-1},\;a,\,x\in\mathbb{C}).
\end{aligned}
\tag{99}
$$

Here we pose the following problems:

(i) Based on the other recurrence relations (36)–(41), by using Riemann–Liouville fractional integral and derivative operators, try to present certain relations among the Fox–Wright functions ${}_p\Psi_q$;
(ii) Demonstrate some particular cases of the identities given in (i);
(iii) Based on the recurrence relations (35)–(41), by using the other fractional integral and derivative operators, try to present certain relations among some special functions;
(iv) Consider certain particular cases of identities that will be derived in (iii).

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