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Qualitative Analysis of Langevin Integro-Fractional Differential Equation under Mittag–Leffler Functions Power Law

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Abstract: This research paper intends to investigate some qualitative analysis for a nonlinear Langevin integro-fractional differential equation. We investigate the sufficient conditions for the existence and uniqueness of solutions for the proposed problem using Banach's and Krasnoselskii's fixed point theorems. Furthermore, we discuss different types of stability results in the frame of Ulam–Hyers by using a mathematical analysis approach. The obtained results are illustrated by presenting a numerical example.

Keywords: ϕ-ABC fractional derivative; initial conditions; existence and UH stability; fixed point theorem

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1. Introduction

Due to its relevance in simulating numerous complicated events in various and widespread disciplines of science and engineering, fractional calculus (FC) and its applications have grown in importance over the last several decades. Some researchers noticed the need to develop the idea of fractional calculus by constructing new fractional derivatives (FD) with separate singular or nonsingular kernels to satisfy the requirement to simulate diverse real-life situations in science and engineering [1–8]. In the exponential kernel, Caputo and Fabrizio in [9] introduced a new type of FD. Atangana and Baleanu investigated new type and interesting FD with Mittag–Leffler kernels in [10]. In [11], Abdeljawad expanded the Atangana and Baleanu FD types to higher arbitrary orders and created the integral operators that accompany them.

Brownian motion is the erratic motion described by some particles found in a fluid medium. One of the first reported scientific observations of this phenomenon was made in 1785 by the Dutch physicist Jan Ingenhousz who was investigating small particles of pulverized carbon on an alcohol surface. However, the discovery of Brownian movement is generally attributed to the Scottish botanist Robert Brown. Hence, the phenomenon has inherited its name from it. In 1827, Brown [12] observed under the microscope the zigzagging movement on the water of small particles derived from pollen grains. In order to verify that it was not a phenomenon restricted to particles from organic materials, he performed several systematic experiments with inorganic materials and concluded that the motion of the particles was not characteristic of living matter. In 1877, Delsaux was the first to explain Brownian phenomenon of motion, arguing that it was a consequence of the incessant collisions of fluid molecules. However, one of the first accurate investigations of Brownian motion was carried out by Leon Gouy in 1888. In this investigation, Gouy showed that motion was an intrinsic property of the fluid. Gouy also discovered that the movement of the particle was accentuated as the size of the particle decreased and also when the viscosity of the fluid decreased. These characteristics had their origin in the thermal movements of the molecules of the fluid [13].

Paul Langevin [14], a brilliant French physicist in the early twentieth century, proposed the nonlinear Langevin equation (NLE). This scientist created a thorough and accurate description of Brownian motion using the Langevin equation. The Langevin differential equation was used to explain physical processes in oscillating domains. Analyzing the stock market [15], modelling evacuation processes [16], studying fluid suspensions [17], self-organization in complex systems [18], photo-electron counting [19] and protein dynamics [20] are just a few of the applications of this equation.

The fractional nonlinear Langevin equation is a model of the generalized Langevin equation that is also an interesting topic. There are many researchers who studied generalized nonlinear Langevin equation under fractional derivatives (see [21–29]). For instance, Eab and Lim in [23] studied an application of fractional generalized nonlinear Langevin equation. The advantage of the operator Atangana–Baleanu–Caputo fractional derivative used in this study is the freedom of choice in the suitable classical differentiation operator and the suitable function ϕ for modeling some real-world problems such as various infectious diseases including Ebola virus, dynamics of smoking, Leptospirosis, etc., in a more comprehensive manner (see [30,31]). In this paper, to develop the model of single-file diffusion, we will use a fractional generalised nonlinear Langevin equation with an external force under the AB fractional derivative. It has been demonstrated that the solution for a fractional generalised Langevin equation presents the correct short and long time behaviour for the mean square displacement of single-file diffusion for an external force that changes with power-law if an appropriate choice is made of parameters of fractional generalised Langevin equation. Recently, Baleanu et al. in [32] studied the existence and uniqueness of solution for the following nonlinear fractional Langevin equation:

$$\begin{cases} {}^{ML}\mathbf{D}^{r}\left({}^{ML}\mathbf{D}^{q}+\lambda\right)\mu(\sigma)=g(\sigma,\mu(\sigma)),\sigma\in(0,1), \\ \mu(0)=a_1,\mu'(0)=a_2, \end{cases}$$

where ${}^{ML}\mathbf{D}^{r}, {}^{ML}\mathbf{D}^{q}$ are the Mittag–Leffler fractional derivatives of order r and q, respectively, such that $r,q\in(0,1]$.

Motivated by the aforesaid arguments, in this paper, we consider a more general nonlinear fractional integrodifferential Langevin equation with the φ-Atangana–Baleanu–Caputo fractional derivative of the following type:

$$\begin{cases} {}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi}+\lambda\right)\mu(\sigma)=g(\sigma,\mu(\sigma),{}^{AB}I_{0^+}^{p,\phi}\mu(\sigma)),\sigma\in(0,b), \\ \mu(0)=a_1,\mu'_{\phi}(0)=a_2, \end{cases} \tag{1}$$

where ${}^{ABC}\mathbf{D}^{p;\phi}$ and ${}^{ABC}\mathbf{D}^{\varrho;\phi}$ are the ϕ-Atangana–Baleanu–Caputo fractional derivatives of order p and ϱ, respectively, such that $p,\varrho\in(0,1]$, ${}^{AB}I_{0^+}^{p,\phi}$ is ϕ–Atangana–Baleanu-fractional integral of order p, ϕ is an increasing function having a continuous derivative ϕ' on $(0,b)$ such that $\phi'(\sigma)\neq 0$, for all $\sigma\in(0,b)$ and $g:\mho\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and differentiable function such that $g(0,\mu(0),{}^{AB}I_{0^+}^{p,\phi}\mu(0))=0$ and $g'_{\phi}(0,\mu(0),{}^{AB}I_{0^+}^{p,\phi}\mu(0))=0$. In fractional nonlinear Langevin Equation (1), $\mu(\sigma)$ is the particle position, function g is the force acting on the particle from molecules of the fluid encircling the fractional Brownian particle, λ is a damping or viscosity term and a_1 and a_2 are the initial positions. To the best of our knowledge, this is the first work considering fractional-order Langevin equations under AB fractional derivative concerning another function ϕ.

The target of this paper is to extend the previous work studied by [32] under a new fractional operator with respect to another function. We will combine the definition of the Atangana–Baleanu derivative and an increasing function. In order to prove the existence, uniqueness and UH stability results, we introduce some auxiliary conditions in order to apply a fixed point theorem due to Banach-type and Krasnoselskii-type.

The proposed problem (1) for different values of a function ϕ includes the study of problems involving the results in [32] and many other results that are not studied yet.

The following is a breakdown of the structure of the paper. We provide notations and some preliminary facts in Section 2 that will be used throughout the study. The existence and uniqueness results for the problem (1) are discussed in Section 3. Section 4 discusses the stability analysis in the context of UH. Section 5 provides an example to demonstrate the validity of our findings. Some concluding remarks on our findings are provided in Section 6.

2. Auxiliary Results

Let $\mho = [0, b] \subset \mathbb{R}$ and $\mathcal{X} = C(\mho, \mathbb{R})$ be the space of continuous functions $\mu : \mho \to \mathbb{R}$ with the norm $\|\mu\| = \max\{|\mu(\sigma)| : \sigma \in \mho\}$. Then, $(\mathcal{X}, \|\cdot\|)$ is a Banach space.

Definition 1 ([2])**.** *Let $p > 0$, $f \in L_1(J)$. Then, the ϕ-RL fractional integral and ϕ-RL fractional derivatives of f of order p are defined by the following:*

$$^{RL}\mathbf{I}_{0^+}^{p,\phi} f(\sigma) = \int_0^\sigma \phi'(\theta) \frac{(\phi(\sigma) - \phi(\theta))^{p-1}}{\Gamma(p)} f(\theta) d\theta,$$

and the following, respectively.

$$^{RL}\mathbf{D}_{0^+}^{p,\phi} f(\sigma) = \left(\frac{1}{\phi'(\sigma)} \frac{d}{d\sigma} \right)^n \left(^{RL}\mathbf{I}_{0^+}^{n-p,\phi} f(\sigma) \right),$$

There is an important model of fractional calculus in the kernel of Mittag–Leffler (ML) functions, namely the Atangana–Baleanu [10].

Definition 2. *Let $0 < p \le 1$ and $\mu \in H^1(\mho)$. Then, the left-sided ABC fractional derivatives of order p for a function μ with respect to another function $\phi(\sigma)$ is defined by the following:*

$$^{ABC}\mathbf{D}_{0^+}^{p,\phi} \mu(\sigma) = \frac{\mathfrak{B}(p)}{1-p} \sum_{n=0}^{\infty} \left(\frac{-p}{1-p} \right)^n \mathbf{D}_{0^+}^{-np-1} \frac{\mu'(\sigma)}{\phi'(\sigma)},$$

where $\mathfrak{B}(p)$ is the normalization function such that the following is the case. $\mathfrak{B}(0) = \mathfrak{B}(1) = 1$.

Definition 3. *Let $0 < p \le 1$ and $\mu \in H^1(\mho)$. Then, the correspondent AB fractional integral of ABC fractional derivatives of order p for a function μ with respect to another function $\phi(\sigma)$ is defined by the following.*

$$^{AB}\mathbf{I}_{0^+}^{p,\phi} \mu(\sigma) = \frac{1-p}{\mathfrak{B}(p)} \mu(\sigma) + \frac{p}{\mathfrak{B}(p)} {}^{RL}\mathbf{I}_{0^+}^{p,\phi} \mu(\sigma).$$

Lemma 1 ([33])**.** *Let $0 < p \le 1$ and $\mu \in H^1(\mho)$. If ϕ-ABC fractional derivative exists, then we have the following.*

$$^{AB}\mathbf{I}_{0^+}^{p,\phi} \, {}^{ABC}\mathbf{D}_{0^+}^{p,\phi} \mu(\sigma) = \mu(\sigma) - \mu(a).$$

Lemma 2 ([11])**.** *Let $\mu(\sigma)$ be a function defined on $\mho$ and $n < p \le n+1$. Then, for some $n \in \mathbb{N}_0$, we have the following:*

$$\left(^{ABC}\mathbf{D}_{0^+}^{p} \, {}^{AB}\mathbf{I}_{0^+}^{p} \mu \right)(\sigma) = \mu(\sigma),$$

and the following is the case.

$$\left({}^{AB}\mathbf{I}^{p}_{0^+}{}^{ABC}\mathbf{D}^{p}_{0^+}\mu\right)(\sigma)=\mu(\sigma)-\sum_{i=0}^{n}\frac{\mu^{(i)}(0)}{i!}\sigma^{i}.$$

Theorem 1 ([34])**.** *Let* $\mathbf{K}$ *be a closed subspace from a Banach space* $\mathbf{X}$ *and* $\mathbf{G}$ *be a strict contraction defined as* $\mathbf{G}:\mathbf{K}\to\mathbf{K}$*, i.e.,* $\|\mathbf{G}(x)-\mathbf{G}(y)\|\le\mathbf{L}\|x-y\|$ *for some* $0<\mathbf{L}<1$ *and all* $x,y\in\mathbf{K}$*. Then,* $\mathbf{G}$ *has a fixed point in* $\mathbf{K}$*.*

Theorem 2 ([35])**.** *Let* K *be a nonempty, closed convex and bounded subset of Banach space* $\mathbf{X}$ *and* Φ^1,Φ^2 *be two operators. If the following is the case:*

(1) $\Phi^1\mu+\Phi^2 v\in\mathbf{X}$ *for all* $\mu,v\in\mathbf{X}$*;*
(2) Φ^1 *is compact and continuous;*
(3) Φ^2 *is a contraction mapping;*
then there exists a function $z\in K$ *such that* $z=\Phi^1 z+\Phi^2 z$*.*

Lemma 3. *Let* $p,\varrho\in(0,1]$ *and* $h\in\mathcal{X}$ *with* $h(0)=0$, $h'_\phi(0)=0$*. Then, the following* ϕ*-ABC-problem*

$$\begin{cases} {}^{ABC}\mathbf{D}^{p;\phi}_{0^+}\left({}^{ABC}\mathbf{D}^{\varrho;\phi}+\lambda\right)\mu(\sigma)=h(\sigma),\\ \mu(0)=a_1,\mu'_\phi(0)=a_2,\end{cases}\tag{2}$$

is equivalent to the following equation:

$$\begin{aligned}\mu(\sigma) &= Q_1\int_0^\sigma\phi'(\theta)\tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}h(\theta)d\theta\\ &\quad+Q_2\int_0^\sigma\phi'(\theta)\tfrac{(\phi(\sigma)-\phi(\theta))^{p-1}}{\Gamma(p)}h(\theta)d\theta\\ &\quad+Q_3\int_0^\sigma\phi'(\theta)\tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)}h(\theta)d\theta\\ &\quad-Q_4\int_0^\sigma\phi'(\theta)\tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}\mu(\theta)d\theta+\mathcal{S}(\sigma),\end{aligned}\tag{3}$$

where

$$Q_1=\frac{\varrho(1-p)}{M\mathfrak{B}(\varrho)\mathfrak{B}(p)},Q_2=\frac{1-\varrho}{M\mathfrak{B}(\varrho)},Q_3=\frac{\varrho}{M\mathfrak{B}(\varrho)},Q_4=\frac{\lambda\varrho}{M\mathfrak{B}(\varrho)},$$

and the following is the case.

$$\begin{aligned}\mathcal{S}(\sigma) &= \frac{\left(\lambda+\frac{\varrho}{\mathfrak{B}(\varrho)}\right)a_1+a_2}{M\mathfrak{B}(\varrho)}\left(\frac{\varrho(\phi(\sigma)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)}+1-\varrho\right)+\frac{\frac{1}{M}\left(a_1-c_1\frac{\varrho}{\mathfrak{B}(\varrho)}\right)}{M},\\ M &= 1+\lambda\frac{1-\varrho}{\mathfrak{B}(\varrho)}.\end{aligned}$$

Proof. Assume that μ is the solution to first equation of (2). Applying the operator ${}^{AB}\mathbf{I}^{p,\phi}_{0^+}$ on both sides of Equation (2). Then, by Definition 3 and Lemma 2, we have the following:

$${}^{ABC}\mathbf{D}^{\varrho;\phi}\mu(\sigma)=\frac{1-p}{\mathfrak{B}(p)}h(\sigma)+\frac{p}{\mathfrak{B}(p)}{}^{RL}\mathbf{I}^{p,\phi}_{0^+}h(\sigma)-\lambda\mu(\sigma)+c_1,\tag{4}$$

where c_1 ia an arbitrary constant. Next, we apply again the operator ${}^{AB}\mathbf{I}_{0^+}^{\varrho;\phi}$ on both sides of Equation (4). Then, by Definition 3, we have the following.

$$\begin{aligned}
\mu(\sigma) &= \tfrac{1-\varrho}{\mathfrak{B}(\varrho)}\Big(\tfrac{1-p}{\mathfrak{B}(p)}h(\sigma)+\tfrac{p}{\mathfrak{B}(p)}\,{}^{RL}\mathbf{I}_{0^+}^{p,\phi}h(\sigma)-\lambda\mu(\sigma)+c_1\Big)\\
&\quad+\tfrac{\varrho}{\mathfrak{B}(\varrho)}\,{}^{RL}\mathbf{I}_{0^+}^{\varrho,\phi}\Big(\tfrac{1-p}{\mathfrak{B}(p)}h(\sigma)+\tfrac{p}{\mathfrak{B}(p)}\,{}^{RL}\mathbf{I}_{0^+}^{p,\phi}h(\sigma)-\lambda\mu(\sigma)+c_1\Big)+c_2\\
&= \tfrac{(1-\varrho)(1-p)}{\mathfrak{B}(\varrho)\mathfrak{B}(p)}h(\sigma)+\tfrac{(1-\varrho)p}{\mathfrak{B}(\varrho)\mathfrak{B}(p)}\,{}^{RL}\mathbf{I}_{0^+}^{p,\phi}h(\sigma)\\
&\quad-\tfrac{\lambda(1-\varrho)}{\mathfrak{B}(\varrho)}\mu(\sigma)+\tfrac{1-\varrho}{\mathfrak{B}(\varrho)}c_1\\
&\quad+\tfrac{\varrho(1-p)}{\mathfrak{B}(\varrho)\mathfrak{B}(p)}\,{}^{RL}\mathbf{I}_{0^+}^{\varrho,\phi}h(\sigma)+\tfrac{\varrho p}{\mathfrak{B}(\varrho)\mathfrak{B}(p)}\,{}^{RL}\mathbf{I}_{0^+}^{\varrho+p,\phi}h(\sigma)\\
&\quad-\tfrac{\lambda\varrho}{\mathfrak{B}(\varrho)}\,{}^{RL}\mathbf{I}_{0^+}^{\varrho,\phi}\mu(\sigma)+\tfrac{\varrho}{\mathfrak{B}(\varrho)}\,{}^{RL}\mathbf{I}_{0^+}^{\varrho,\phi}c_1+c_2.
\end{aligned} \tag{5}$$

Now, by conditions $\mu(0)=a_1$, $h(0)=0$ and $\mu'_\phi(0)=a_2$, $h'_\phi(0)=0$, we obtain the following.

$$\begin{aligned}
c_1 &= \left(\lambda+\frac{\varrho}{\mathfrak{B}(\varrho)}\right)a_1+a_2,\\
c_2 &= \frac{1}{M}\left(a_1-c_1\frac{\varrho}{\mathfrak{B}(\varrho)}\right).
\end{aligned}$$

Substituting the values of c_1 and c_2 in Equation (5), we obtain Equation (3). □

3. Existence and Uniqueness of Solutions

In order to obtain our results, the following hypotheses must be satisfied.

Hypothesis 1 (H1). *There exists a non-negative constant $\mathfrak{L}>0$ such that for any $\mu,\widehat{\mu},v,\widehat{v}\in\mathbb{R}^+$, we have the following.*

$$|g(\sigma,\mu(\sigma),v(\sigma))-g(\sigma,\widehat{\mu}(\sigma),\widehat{v}(\sigma))|\le\mathfrak{L}(|\mu-\widehat{\mu}|+|v-\widehat{v}|).$$

Hypothesis 2 (H2). *The function $g:\mho\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous and differentiable function such that the following is the case:*

$$\left|g(\sigma,\mu(\sigma),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\sigma))\right|\le\eta_g(\sigma)+\Upsilon_g(\sigma)|\mu(\sigma)|,$$

where $\eta_g,\Upsilon_g\in\mathcal{X}$ are nonnegative functions, and $\eta_g^*=\max_{\sigma\in\mho}|\eta_g(\sigma)|$ and $\Upsilon_g^*=\max_{\sigma\in\mho}|\Upsilon_g(\sigma)|$.

In order to simplify our analysis, we set the following notation:

$$\begin{aligned}
\mathcal{G} &= \Bigg[|Q_1|\mathfrak{L}\left(1+\left[\frac{1-p}{\mathfrak{B}(p)}+\frac{p}{\mathfrak{B}(p)}\frac{(\phi(b)-\phi(0))^p}{\Gamma(p+1)}\right]\right)\frac{(\phi(b)-\phi(0))^\varrho}{\Gamma(\varrho+1)}\\
&\quad+|Q_2|\mathfrak{L}\left(1+\left[\frac{1-p}{\mathfrak{B}(p)}+\frac{p}{\mathfrak{B}(p)}\frac{(\phi(b)-\phi(0))^p}{\Gamma(p+1)}\right]\right)\frac{(\phi(b)-\phi(0))^p}{\Gamma(p+1)}\\
&\quad+|Q_3|\mathfrak{L}\left(1+\left[\frac{1-p}{\mathfrak{B}(p)}+\frac{p}{\mathfrak{B}(p)}\frac{(\phi(b)-\phi(0))^p}{\Gamma(p+1)}\right]\right)\frac{(\phi(b)-\phi(0))^{\varrho+p}}{\Gamma(\varrho+p+1)}\\
&\quad+|Q_4|\frac{(\phi(b)-\phi(0))^\varrho}{\Gamma(\varrho+1)}\Bigg],
\end{aligned}$$

and the following is the case:

$$\begin{aligned}\mathcal{G}_1 &= \Bigg[|Q_1|\omega_g\frac{(\phi(b)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)}+|Q_2|\omega_g\frac{(\phi(b)-\phi(0))^{p}}{\Gamma(p+1)}\\ &\quad +|Q_3|\omega_g\frac{(\phi(b)-\phi(0))^{\varrho+p}}{\Gamma(\varrho+p+1)}+\sup_{\sigma\in\mho}|\mathcal{S}(\sigma)|\Bigg],\end{aligned}$$

where $\omega_g = \max_{\sigma\in\mathcal{J}}|g(\sigma,0,0)|$.

Theorem 3. *Assume that (H1) holds. Then, ϕ-ABC problem (1) has a unique solution provided that $\mathcal{G} < 1$.*

Proof. By Lemma 3, we define the operator $\Xi : \mathcal{X} \to \mathcal{X}$ by the following.

$$\begin{aligned}\Xi\mu(\sigma) &= Q_1\int_0^{\sigma}\phi'(\theta)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}g(\theta,\mu(\theta),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta\\ &\quad +Q_2\int_0^{\sigma}\phi'(\theta)\frac{(\phi(\sigma)-\phi(\theta))^{p-1}}{\Gamma(p)}g(\theta,\mu(\theta),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta\\ &\quad +Q_3\int_0^{\sigma}\phi'(\theta)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)}g(\theta,\mu(\theta),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta\\ &\quad -Q_4\int_0^{\sigma}\phi'(\theta)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}\mu(\theta)d\theta+\mathcal{S}(\sigma).\end{aligned}$$

Define a closed ball Π_δ as the following:

$$\Pi_\delta = \{\mu\in\mathcal{X} : \|\mu\|\le\delta\},$$

with radius $\delta \ge \frac{\mathcal{G}_1}{1-\mathcal{G}}$. Now, we divide the proof into two steps as follows.

Step (1): We will show that $\Xi\Pi_\delta \subset \Pi_\delta$. By (H1), we have the following.

$$\begin{aligned}&\left|g(\sigma,\mu(\sigma),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\sigma))\right|\\ \le\ &\left|g(\sigma,\mu(\sigma),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\sigma))-g(\sigma,0,0)\right|+|g(\sigma,0,0)|\\ \le\ &\mathfrak{L}\left[|\mu(\sigma)|+\left|{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\sigma)\right|\right]+\omega_g\\ \le\ &\mathfrak{L}\left(|\mu(\sigma)|+\left[\frac{1-p}{\mathfrak{B}(p)}+\frac{p}{\mathfrak{B}(p)}\frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)}\right]|\mu(\sigma)|\right)+\omega_g.\end{aligned}$$

Now, for all $\mu \in \Pi_\delta$ and $\sigma \in \mho$, we have the following.

$$\begin{aligned}
|(\Xi\mu)(\sigma)| \leq & \sup_{\sigma\in\mho}\Big\{ |Q_1| \int_0^\sigma \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} \Big| g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) \Big| d\theta \\
& + |Q_2| \int_0^\sigma \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{p-1}}{\Gamma(p)} \Big| g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) \Big| d\theta \\
& + |Q_3| \int_0^\sigma \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)} \Big| g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) \Big| d\theta \\
& + |Q_4| \int_0^\sigma \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} |\mu(\theta)| d\theta + |\mathcal{S}(\sigma)| \Big\} \\
\leq & |Q_1| \frac{(\phi(\sigma)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)} \\
& \left[\mathfrak{L}\left(\delta + \left[\frac{1-p}{\mathfrak{B}(p)} + \frac{p}{\mathfrak{B}(p)} \frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} \right] \delta \right) + \omega_g \right] \\
& + |Q_2| \frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} \\
& \left[\mathfrak{L}\left(\delta + \left[\frac{1-p}{\mathfrak{B}(p)} + \frac{p}{\mathfrak{B}(p)} \frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} \right] \delta \right) + \omega_g \right] \\
& + |Q_3| \frac{(\phi(\sigma)-\phi(s))^{\varrho+p}}{\Gamma(\varrho+p+1)}
\end{aligned}$$

$$\begin{aligned}
& \left[\mathfrak{L}\left(\delta + \left[\frac{1-p}{\mathfrak{B}(p)} + \frac{p}{\mathfrak{B}(p)} \frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} \right] \delta \right) + \omega_g \right] \\
& + |Q_4| \delta \frac{(\phi(\sigma)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)} + \sup_{\sigma\in\mho} |\mathcal{S}(\sigma)| \\
\leq & \Bigg[|Q_1| \mathfrak{L}\left(1 + \left[\frac{1-p}{\mathfrak{B}(p)} + \frac{p}{\mathfrak{B}(p)} \frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} \right] \right) \frac{(\phi(\sigma)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)} \\
& + |Q_2| \mathfrak{L}\left(1 + \left[\frac{1-p}{\mathfrak{B}(p)} + \frac{p}{\mathfrak{B}(p)} \frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} \right] \right) \frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} \\
& + |Q_3| \mathfrak{L}\left(1 + \left[\frac{1-p}{\mathfrak{B}(p)} + \frac{p}{\mathfrak{B}(p)} \frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} \right] \right) \frac{(\phi(\sigma)-\phi(0))^{\varrho+p}}{\Gamma(\varrho+p+1)} \\
& + |Q_4| \frac{(\phi(\sigma)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)} \Bigg] \delta \\
& + \Bigg[|Q_1| \omega_g \frac{(\phi(\sigma)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)} + |Q_2| \omega_g \frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} \\
& + |Q_3| \omega_g \frac{(\phi(\sigma)-\phi(s))^{\varrho+p}}{\Gamma(\varrho+p+1)} + \sup_{\sigma\in\mho} |\mathcal{S}(\sigma)| \Bigg] \\
\leq & \mathcal{G}\delta + \mathcal{G}_1 \leq \delta.
\end{aligned}$$

Thus, $\Xi\Pi_\delta \subset \Pi_\delta$.

Step (2): We will show that Ξ is a contraction mapping. Let $\mu, \widehat{\mu} \in \Pi_\delta$ and $\sigma \in \mho$. Then, the following is the case.

$$\begin{aligned}
& |(\Xi\mu)(\sigma) - (\Xi\widehat{\mu})(\sigma)| \\
\leq \;& Q_1 \int_0^\sigma \phi'(\theta) \tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} \Big| g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) - g(\theta, \widehat{\mu}(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\widehat{\mu}(\theta)) \Big| d\theta \\
& + Q_2 \int_0^\sigma \phi'(\theta) \tfrac{(\phi(\sigma)-\phi(\theta))^{p-1}}{\Gamma(p)} \Big| g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) - g(\theta, \widehat{\mu}(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\widehat{\mu}(\theta)) \Big| d\theta \\
& + Q_3 \int_0^\sigma \phi'(\theta) \tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)} \Big| g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) - g(\theta, \widehat{\mu}(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\widehat{\mu}(\theta)) \Big| d\theta \\
& + Q_4 \int_0^\sigma \phi'(\theta) \tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} |\mu(\theta) - \widehat{\mu}(\theta)| d\theta.
\end{aligned}$$

From (H1), we obtain the following.

$$\begin{aligned}
& \Big| g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) - g(\theta, \widehat{\mu}(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\widehat{\mu}(\theta)) \Big| \\
\leq \;& \mathfrak{L}\Big[|\mu(\theta) - \widehat{\mu}(\theta)| + {}^{AB}\mathbf{I}_{0^+}^{p,\phi} |\mu(\theta) - \widehat{\mu}(\theta)| \Big] \\
\leq \;& \mathfrak{L}\|\mu - \widehat{\mu}\| + \Big[\tfrac{1-p}{\mathfrak{B}(p)} + \tfrac{p}{\mathfrak{B}(p)} \tfrac{(\phi(\sigma)-\phi(0))^p}{\Gamma(p+1)} \Big] \|\mu - \widehat{\mu}\| \\
\leq \;& \mathfrak{L}\Big(1 + \Big[\tfrac{1-p}{\mathfrak{B}(p)} + \tfrac{p}{\mathfrak{B}(p)} \tfrac{(\phi(\sigma)-\phi(0))^p}{\Gamma(p+1)} \Big] \Big) \|\mu - \widehat{\mu}\|.
\end{aligned} \tag{6}$$

Hence, the following is the case.

$$\begin{aligned}
\|\Xi\mu - \Xi\widehat{\mu}\| &= \sup_{\sigma \in \mathcal{J}} |\Xi\mu(\sigma) - \Xi\widehat{\mu}(\sigma)| \\
&\leq \mathcal{G}\|\mu - \widehat{\mu}\|.
\end{aligned}$$

Due to $\mathcal{G} < 1$, we conclude that Ξ is a contraction operator. Hence, Theorem 1 implies that Ξ has a unique fixed point. □

Theorem 4. *Assume that (H1)–(H2) hold. Then, the ϕ-ABC problem (1) has at least one solution provided that the following is the case:*

$$\Sigma_2 = \left(|Q_2| \frac{(\phi(b) - \phi(\theta))^p}{\Gamma(p+1)} + |Q_3| \frac{(\phi(b) - \phi(\theta))^{\varrho+p}}{\Gamma(\varrho+p+1)} \right) \Upsilon_g^* < 1, \tag{7}$$

and the following is obtained.

$$|Q_4| \frac{(\phi(b) - \phi(0))^\varrho}{\Gamma(\varrho+1)} < 1. \tag{8}$$

Proof. Let us consider the operator Ξ defined in Theorem 3 such that the following is the case:

$$(\Xi\mu)(\sigma) = (\Xi_1\mu)(\sigma) + (\Xi_2\mu)(\sigma),$$

where

$$(\Xi_1\mu)(\sigma) = \mathcal{S}(\sigma) - Q_4 \int_0^\sigma \phi'(\theta) \frac{(\phi(\sigma) - \phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} \mu(\theta) d\theta,$$

and

$$\begin{aligned}
(\Xi_2\mu)(\sigma) \;=\;& Q_1 \int_0^\sigma \phi'(\theta) \frac{(\phi(\sigma) - \phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) d\theta \\
& + Q_2 \int_0^\sigma \phi'(\theta) \frac{(\phi(\sigma) - \phi(\theta))^{p-1}}{\Gamma(p)} g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) d\theta \\
& + Q_3 \int_0^\sigma \phi'(\theta) \frac{(\phi(\sigma) - \phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)} g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta)) d\theta.
\end{aligned}$$

Define a closed ball Π_r by $\Pi_r = \{\mu \in \mathcal{X} : \|\mu\| \leq r\}$, with $r \geq \frac{\Sigma_1}{\Sigma_2}$, where the following is the case.

$$\begin{aligned}\Sigma_1 &= \max_{\sigma\in\mho}|\mathcal{S}(\sigma)| + (|Q_4| + |Q_1|)\tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho}}{\Gamma(\varrho+1)}\eta_g^* \\ &\quad+\Big(|Q_2|\tfrac{(\phi(b)-\phi(\theta))^{p}}{\Gamma(p+1)} + |Q_3|\tfrac{(\phi(b)-\phi(\theta))^{\varrho+p}}{\Gamma(\varrho+p+1)}\Big)\eta_g^*.\end{aligned} \tag{9}$$

In order to apply Theorem 2, we will divide the proof into three steps as follows.

Step1: $\Xi_1\mu + \Xi_2\widetilde{\mu} \in \Pi_\delta$ for all $\mu, \widetilde{\mu} \in \Pi_r$. First, for Ξ_1. Let $\mu \in \Pi_r, \sigma \in \mho$. Then, we have the following.

$$\begin{aligned}\|\Xi_1\mu\| &= \max_{\sigma\in\mho}|(\Xi_1\mu)(\sigma)| \\ &\leq \max_{\sigma\in\mho}|\mathcal{S}(\sigma)| + \max_{\sigma\in\mho}|Q_4|\int_0^\sigma \phi'(\theta)\tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}\Big|g(\theta,\mu(\theta),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta \\ &\leq \max_{\sigma\in\mho}|\mathcal{S}(\sigma)| + |Q_4|\tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho}}{\Gamma(\varrho+1)}\eta_g^* + \Upsilon_g^* r.\end{aligned} \tag{10}$$

Next, for Ξ_2. Let $\widetilde{\mu} \in \Pi_r, \sigma \in \mho$. Then, we have the following.

$$\begin{aligned}\|\Xi_2\mu\| &= \max_{\sigma\in\mho}|(\Xi_2\mu)(\sigma)| \\ &\leq \max_{\sigma\in\mho}|Q_1|\int_0^\sigma \phi'(\theta)\tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}\Big|g(\theta,\mu(\theta),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta \\ &\quad+\max_{\sigma\in\mho}|Q_2|\int_0^\sigma \phi'(\theta)\tfrac{(\phi(\sigma)-\phi(\theta))^{p-1}}{\Gamma(p)}\Big|g(\theta,\mu(\theta),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta \\ &\quad+\max_{\sigma\in\mho}|Q_3|\int_0^\sigma \phi'(\theta)\tfrac{(\phi(\sigma)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)}\Big|g(\theta,\mu(\theta),{}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta \\ &\leq \Big(|Q_1|\tfrac{(\phi(b)-\phi(\theta))^{\varrho}}{\Gamma(\varrho+1)} + |Q_2|\tfrac{(\phi(b)-\phi(\theta))^{p}}{\Gamma(p+1)} + |Q_3|\tfrac{(\phi(b)-\phi(\theta))^{\varrho+p}}{\Gamma(\varrho+p+1)}\Big) \\ &\quad\Big(\eta_g^* + \Upsilon_g^* r\Big).\end{aligned} \tag{11}$$

From (10) and (11), we obtain the following:

$$\begin{aligned}\|\Xi\mu\| &\leq \|\Xi_1\mu\| + \|\Xi_2\mu\| \\ &\leq \max_{\sigma\in\mho}|\mathcal{S}(\sigma)| + (|Q_4| + |Q_1|)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho}}{\Gamma(\varrho+1)}\Big(\eta_g^* + \Upsilon_g^* r\Big) \\ &\quad+\left(|Q_2|\frac{(\phi(b)-\phi(\theta))^{p}}{\Gamma(p+1)} + |Q_3|\frac{(\phi(b)-\phi(\theta))^{\varrho+p}}{\Gamma(\varrho+p+1)}\right)\Big(\eta_g^* + \Upsilon_g^* r\Big) \\ &\leq \max_{\sigma\in\mho}|\mathcal{S}(\sigma)| + (|Q_4| + |Q_1|)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho}}{\Gamma(\varrho+1)}\eta_g^* \\ &\quad+\left(|Q_2|\frac{(\phi(b)-\phi(\theta))^{p}}{\Gamma(p+1)} + |Q_3|\frac{(\phi(b)-\phi(\theta))^{\varrho+p}}{\Gamma(\varrho+p+1)}\right)\eta_g^* \\ &\quad+(|Q_4| + |Q_1|)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho}}{\Gamma(\varrho+1)}\Upsilon_g^* r \\ &\quad+\left(|Q_2|\frac{(\phi(b)-\phi(\theta))^{p}}{\Gamma(p+1)} + |Q_3|\frac{(\phi(b)-\phi(\theta))^{\varrho+p}}{\Gamma(\varrho+p+1)}\right)\Upsilon_g^* r \\ &\leq \Sigma_1 + \Sigma_2 r < r,\end{aligned}$$

where Σ_1 and Σ_2 defined in (7) and (9), respectively, $\Sigma_2 < 1$. Hence, $\Xi_1\mu + \Xi_2\widetilde{\mu} \in \Pi_r$.

Step2: Ξ_1 is a contraction map. Let $\mu, \widehat{\mu} \in \Pi_\delta$ and $\sigma \in \mho$, then we have the following.

$$\begin{aligned}|(\Xi_1\mu)(\sigma) - (\Xi_1\widehat{\mu})(\sigma)| &\leq |Q_4|\int_0^\sigma \phi'(\theta)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}|\mu(\theta) - \widehat{\mu}(\theta)|d\theta \\ &\leq |Q_4|\frac{(\phi(b)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)}\|\mu - \widehat{\mu}\|.\end{aligned}$$

Due to (8), we conclude that Ξ_1 is contraction.

Step3: Ξ_2 is continuous and compact. Since f is continuous, Ξ_2 is continuous too. Moreover, by the first step in Theorem 3, we conclude that Ξ_2 is uniformly bounded on Π_δ. Now, we show that $\Xi_2(\Pi_r)$ is equicontinuous. For this purpose, let $\mu \in \Pi_r$. Then, for $0 \le \sigma_1 < \sigma_2 \le b$, we have the following.

$$\begin{aligned}
&|(\Xi_2\mu)(\sigma_2) - (\Xi_2\mu)(\sigma_1)|\\
\le\ & \Bigg| Q_1 \int_0^{\sigma_2} \phi'(\theta)\frac{(\phi(\sigma_2)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta\\
&+Q_2 \int_0^{\sigma_2} \phi'(\theta)\frac{(\phi(\sigma_2)-\phi(\theta))^{p-1}}{\Gamma(p)} g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta\\
&+Q_3 \int_0^{\sigma_2} \phi'(\theta)\frac{(\phi(\sigma_2)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)} g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta\\
&-Q_1 \int_0^{\sigma_1} \phi'(\theta)\frac{(\phi(\sigma_1)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta\\
&-Q_2 \int_0^{\sigma_1} \phi'(\theta)\frac{(\phi(\sigma_1)-\phi(\theta))^{p-1}}{\Gamma(p)} g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta\\
&-Q_3 \int_0^{\sigma_1} \phi'(\theta)\frac{(\phi(\sigma_1)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)} g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta \Bigg|
\end{aligned}$$

$$\begin{aligned}
\le\ & |Q_1| \textstyle\int_0^{\sigma_1} \phi'(\theta)\frac{(\phi(\sigma_2)-\phi(\theta))^{\varrho-1}-(\phi(\sigma_1)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}\Big|g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta\\
&+|Q_2| \textstyle\int_0^{\sigma_1} \phi'(\theta)\frac{(\phi(\sigma_2)-\phi(\theta))^{p-1}-(\phi(\sigma_1)-\phi(\theta))^{p-1}}{\Gamma(p)}\Big|g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta\\
&+|Q_3| \textstyle\int_0^{\sigma_1} \phi'(\theta)\frac{(\phi(\sigma_2)-\phi(\theta))^{\varrho+p-1}-(\phi(\sigma_1)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)}\Big|g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta\\
&+|Q_1| \textstyle\int_{\sigma_1}^{\sigma_2} \phi'(\theta)\frac{(\phi(\sigma_2)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}\Big|g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta\\
&+|Q_2| \textstyle\int_{\sigma_1}^{\sigma_2} \phi'(\theta)\frac{(\phi(\sigma_2)-\phi(\theta))^{p-1}}{\Gamma(p)}\Big|g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta\\
&+|Q_3| \textstyle\int_{\sigma_1}^{\sigma_2} \phi'(\theta)\frac{(\phi(\sigma_2)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)}\Big|g(\theta,\mu(\theta),{}^{AB}\,\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))\Big|d\theta\\
\le\ & M_g|Q_1|\tfrac{(\phi(\sigma_2)-\phi(0))^{\varrho}-(\phi(\sigma_1)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)}\\
&+M_g|Q_2|\tfrac{(\phi(\sigma_2)-\phi(0))^{p}-(\phi(\sigma_1)-\phi(0))^{p}}{\Gamma(p+1)}\\
&+M_g|Q_3|\tfrac{(\phi(\sigma_2)-\phi(0))^{\varrho+p}-(\phi(\sigma_1)-\phi(0))^{\varrho+p}}{\Gamma(\varrho+p+1)}\\
\to\ & 0 \text{ as } \sigma_2 \to \sigma_1.
\end{aligned}$$

According to the above steps and the Arzela–Ascoli theorem, we understand that $(\Xi_2\Pi_r)$ is relatively compact. Consequently, Ξ_2 is completely continuous. Thus, by Theorem 2, we infer that the ϕ-ABC problem (1) has at least one solution on $\mho$. □

4. Stability Results

Ulam's question about the stability of group homomorphisms in 1940 [36] inspired the problem of functional equation stability. Hyers [37] presented a positive interpretation of the Ulam question within Banach spaces the next year, which was the first important advance and step toward more solutions in this topic. Many studies on various generalisations of the Ulam problem and Hyers theory have been published since then. Rassias [38] was the first to extend Hyers idea of mappings across Banach spaces in 1978. Rassias' result drew the attention of many mathematicians all over the world, who began looking into the difficulties of functional equation stability. For more information about the stability of solutions, we refer the readers to papers [39,40].

In this regard, we discuss the stability results in the frame of Ulam–Hyers (UH)–Rassias (HUR). First of all, we introduce the following definitions. For $\varepsilon > 0$, we consider the following inequality.

$$\left| {}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\widehat{\mu}(\sigma) - g(\sigma, \widehat{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{\alpha,\phi}\widehat{\mu}(\sigma))\right| \leq \varepsilon, \ \ \sigma \in \mho. \tag{12}$$

Definition 4. *The ϕ-ABC problem (1) is UH stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $\widehat{\mu} \in \mathcal{X}$ of inequality (12), there exists a unique solution $\mu \in \mathcal{X}$ of ϕ-ABC problem(1) with the following.*

$$|\widehat{\mu}(\sigma) - \mu(\sigma)| \leq C_f\varepsilon.$$

Moreover, the ϕ-ABC problem (1) is GUH stable if there is a function $\varphi_f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi_f(0) = 0$ such that the following is the case.

$$|\widehat{\mu}(\sigma) - \mu(\sigma)| \leq \varphi_f\varepsilon.$$

Remark 1. *A function $\widehat{\mu} \in \mathcal{X}$ satisfies the inequality (12) if and only if there is a function $z \in \mathcal{X}$ such that the following is the case:*
(i) $|z(\sigma)| \leq \varepsilon$ for all $\sigma \in \mho$ (z depends on μ);
(ii) ${}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\widehat{\mu}(\sigma) = g(\sigma, \widehat{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{\alpha,\phi}\widehat{\mu}(\sigma)) + z(\sigma)$, $\sigma \in \mho$.

Lemma 4. *If $\mu \in \mathcal{X}$ is a solution to inequality (12), then μ satisfies the following inequality:*

$$\left|\mu(\sigma) - \Psi_\mu\right| \leq \varepsilon\Delta_{Q_1,Q_2,Q_3},$$

where the following is the case:

$$\begin{aligned}
\Psi_\mu \ = \ & Q_1\int_0^\sigma \phi'(\theta)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta \\
& + Q_2\int_0^\sigma \phi'(\theta)\frac{(\phi(\sigma)-\phi(\theta))^{p-1}}{\Gamma(p)}g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta \\
& + Q_3\int_0^\sigma \phi'(\theta)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)}g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta \\
& - Q_4\int_0^\sigma \phi'(\theta)\frac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}\mu(\theta)d\theta + \mathcal{S}(\sigma),
\end{aligned}$$

and

$$\Delta_{Q_1,Q_2,Q_3} = \left(Q_1\frac{(\phi(\sigma)-\phi(0))^{\varrho}}{\Gamma(\varrho+1)} + Q_2\frac{(\phi(\sigma)-\phi(0))^{p}}{\Gamma(p+1)} + Q_3\frac{(\phi(\sigma)-\phi(0))^{\varrho+p}}{\Gamma(\varrho+p+1)}\right).$$

Proof. In view of Remark 1, we have the following.

$$\begin{cases} {}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\widehat{\mu}(\sigma) = g(\sigma, \widehat{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{\alpha,\phi}\widehat{\mu}(\sigma)) + z(\sigma) \\ \mu(0) = \widehat{\mu}(0) = a_1, \mu'_\phi(0) = \widehat{\mu}'_\phi = a_2. \end{cases}$$

Then, by Lemma 3, we obtain the following:

$$\begin{aligned} \widehat{\mu}(\sigma) &= \Psi_{\mu} + Q_1 \int_0^{\sigma} \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} z(\theta) d\theta \\ &+ Q_2 \int_0^{\sigma} \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{p-1}}{\Gamma(p)} z(\theta) d\theta \\ &+ Q_3 \int_0^{\sigma} \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)} z(\theta) d\theta. \end{aligned}$$

which implies the following.

$$\begin{aligned} \left|\widetilde{\mu}(\sigma) - \Psi_{\widetilde{\mu}}\right| &\leq Q_1 \int_0^{\sigma} \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} |z(\theta)| d\theta \\ &+ Q_2 \int_0^{\sigma} \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{p-1}}{\Gamma(p)} |z(\theta)| d\theta \\ &+ Q_3 \int_0^{\sigma} \phi'(\theta) \frac{(\phi(\sigma)-\phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)} |z(\theta)| d\theta \\ &\leq \varepsilon \Delta_{Q_1,Q_2,Q_3}. \end{aligned}$$

□

Theorem 5. *Suppose that (H_1) holds. If $\mathcal{G} < 1$, then the following equation is the case.*

$$^{ABC}\mathbf{D}^{p;\phi}\left(^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\widehat{\mu}(\sigma) = g(\sigma, \widehat{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{\alpha,\phi}\widehat{\mu}(\sigma)), \tag{13}$$

Moreover, it is Ulam–Hyers stable.

Proof. Let $\varepsilon > 0$ and $\widehat{\mu} \in \mathcal{X}$ be a function that satisfies inequality (12), and let $\mu \in \mathcal{X}$ be the unique solution of the following problem

$$\begin{cases} {}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\mu(\sigma) = g(\sigma, \mu(\sigma), {}^{AB}\mathbf{I}_{0^+}^{\alpha,\phi}\mu(\sigma)) \\ \mu(0) = \widehat{\mu}(0) = a_1, \mu'_{\phi}(0) = \widehat{\mu}'_{\phi}(0) = a_2. \end{cases}$$

Then, by Lemma 3, we obtain the following.

$$\mu(\sigma) = \Psi_{\mu}.$$

Hence, by Lemma 4 and Equation (6), we have the following.

$$\begin{aligned} \|\mu - \widehat{\mu}\| = \sup_{\sigma\in\mho}\left|\mu(\sigma) - \Psi_{\widehat{\mu}}\right| &\leq \sup_{\sigma\in\mho}\left|\mu(\sigma) - \Psi_{\mu}\right| + \sup_{\sigma\in\mho}\left|\Psi_{\mu} - \Psi_{\widehat{\mu}}\right| \\ \leq \varepsilon\Delta_{Q_1,Q_2,Q_3} + \mathcal{G}\|\mu - \widehat{\mu}\|. \end{aligned}$$

Thus, the following is obtained:

$$\|\mu - \widehat{\mu}\| \leq C_f\varepsilon,$$

where the following is the case.

$$C_f = \frac{\Delta_{Q_1,Q_2,Q_3}}{1 - \mathcal{G}}.$$

Thus, the ϕ-ABC problem (13) is Ulam–Hyers stability. Now, by choosing $\varphi_f(\varepsilon) = C_f\varepsilon$ such that $\varphi_f(0) = 0$, then the ϕ-ABC problem (13) is generalized Ulam–Hyers stability. □

In order to prove Hyers–Ulam–Rassias stability, we need the following hypotheses.

(H2) There exists an increasing function $\alpha_\phi \in \mathcal{X}$ and there exists $\mathcal{R} > 0$ such that for any $\sigma \in \mho$ of the following:

$$ {}^{AB}\mathbf{I}_{0^+}^{\xi,\phi}\alpha_\phi(\sigma) \leq \mathcal{R}\alpha_\phi(\sigma), \tag{14} $$

where $\xi = \{p, \varrho, \varrho + p\}$.

Definition 5. *Let $\widehat{\mu} \in \mathcal{X}$ be a function that satisfies the following equation:*

$$ \left| {}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\widehat{\mu}(\sigma) - g(\sigma, \widehat{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\widehat{\mu}(\sigma))\right| \leq \varepsilon\alpha_\phi(\sigma), \tag{15} $$

and $\mu \in \mathcal{X}$ be a solution of (1). If there exists $0 < \mathcal{N} \in \mathbb{R}$ and non-decreasing function $\alpha_\phi(\sigma)$ such that the following is the case:

$$ |\widehat{\mu}(\sigma) - \mu(\sigma)| \leq \mathcal{N}\varepsilon\alpha_\phi(\sigma), \quad \sigma \in \mho, \ \varepsilon > 0, $$

then, the ϕ-ABC problem (1) is Hyers–Ulam–Rassias stable with respect to $\alpha_\phi(\sigma)$.

Remark 2. *A function $\widehat{\mu} \in \mathcal{X}$ satisfies the inequality (15) if and only if there is a function $z \in \mathcal{X}$ such that the following is the case:*

(i) $|z(\sigma)| \leq \varepsilon\alpha_\phi(\sigma)$ for all $\sigma \in \mho$ (z depends on μ);

(ii) ${}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\widehat{\mu}(\sigma) = g(\sigma, \widehat{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{\alpha,\phi}\widehat{\mu}(\sigma)) + z(\sigma), \sigma \in \mho$.

Lemma 5. *If $\mu \in \mathcal{X}$ is a solution of inequality (15), then μ satisfies the following inequality:*

$$ \left|\mu(\sigma) - \Psi_\mu\right| \leq \varepsilon\mathcal{R}(Q_1 + Q_2 + Q_3)\alpha_\phi(\sigma), $$

where the following is the case.

$$ \begin{aligned} \Psi_\mu \quad = \quad & Q_1\int_0^\sigma \phi'(\theta)\frac{(\phi(\sigma) - \phi(\theta))^{\varrho-1}}{\Gamma(\varrho)} g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta \\ & + Q_2\int_0^\sigma \phi'(\theta)\frac{(\phi(\sigma) - \phi(\theta))^{p-1}}{\Gamma(p)} g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta \\ & + Q_3\int_0^\sigma \phi'(\theta)\frac{(\phi(\sigma) - \phi(\theta))^{\varrho+p-1}}{\Gamma(\varrho+p)} g(\theta, \mu(\theta), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\theta))d\theta \\ & - Q_4\int_0^\sigma \phi'(\theta)\frac{(\phi(\sigma) - \phi(\theta))^{\varrho-1}}{\Gamma(\varrho)}\mu(\theta)d\theta + \mathcal{S}(\sigma). \end{aligned} $$

Proof. Indeed, by Remark 2 and Lemma 3, one can easily prove that the following is the case.

$$ \left|\mu(\sigma) - \Psi_\mu\right| \leq \varepsilon\mathcal{R}(Q_1 + Q_2 + Q_3)\alpha_\phi(\sigma). $$

□

Theorem 6. *Assume that (H1) and (H2) hold. Then, the following is the case:*

$$ {}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\widehat{\mu}(\sigma) = g(\sigma, \widehat{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{\alpha,\phi}\widehat{\mu}(\sigma)), $$

and it is HUR and generalized HUR stable.

Proof. Let $\varepsilon > 0$ and $\widehat{\mu} \in \mathcal{X}$ be a function that satisfies inequality (15), and let $\mu \in \mathcal{X}$ be the unique solution of the following problem.

$$ \begin{cases} {}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\mu(\sigma) = g(\sigma, \mu(\sigma), {}^{AB}\mathbf{I}_{0^+}^{\alpha,\phi}\mu(\sigma)) \\ \qquad \mu(0) = \widehat{\mu}(0) = a_1, \mu'_\phi(0) = \widehat{\mu}'_\phi = a_2. \end{cases} $$

Then, by Lemma 3, we obtain the following.

$$\mu(\sigma) = \Psi_{\widehat{\mu}}$$

Hence, by Lemma 4 and Equation (6), we have the following.

$$\begin{aligned} \|\mu - \widehat{\mu}\| &= \sup_{\sigma \in \mho}\left|\mu(\sigma) - \Psi_{\widehat{\mu}}\right| \leq \sup_{\sigma \in \mho}\left|\mu(\sigma) - \Psi_{\mu}\right| + \sup_{\sigma \in \mho}\left|\Psi_{\mu} - \Psi_{\widehat{\mu}}\right| \\ &\leq \varepsilon\mathcal{R}(Q_1 + Q_2 + Q_3)\alpha_{\phi}(\sigma) + \mathcal{G}\|\mu - \widehat{\mu}\|. \end{aligned}$$

Thus, the following is the case:

$$\|\mu - \widehat{\mu}\| \leq \mathcal{N}\varepsilon\alpha_{\phi}(\sigma),$$

where the following is obtained.

$$\mathcal{N} = \frac{\mathcal{R}(Q_1 + Q_2 + Q_3)}{1 - \mathcal{G}}.$$

Thus, the ϕ-ABC problem (13) is HUR stable as well as generalized HUR stable. □

5. An Example

Example 1. *For $p \in (2,3]$, we consider the following ϕ-ABC problem.*

$$\begin{cases} {}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\mu(\sigma) = g(\sigma, \mu(\sigma), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\sigma)), \sigma \in (0,1) \\ \mu(0) = a_1, \mu'_{\phi}(0) = a_2 \end{cases}$$

Here, $p = \varrho = \frac{1}{2} \in (0,1], b = 1, \lambda = 2, a_1 = a_2 = 1, \phi = e^{2\sigma}$ and the following are obtained.

$$g(\sigma, \mu(\sigma), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\sigma)) = \frac{\sigma}{50}\left(\mu(\sigma) + \mathbf{I}_{0^+}^{\frac{1}{5},e^{2\sigma}}\mu(\sigma)\right).$$

Clearly, $g(0, \mu(0), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(0)) = 0$. Let $\in [0,1], \mu, \overline{\mu} \in \mathbb{R}$. Then, the following is the case.

$$\left|g(\sigma, \mu(\sigma), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\mu(\sigma)) - g(\sigma, \overline{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\overline{\mu}(\sigma))\right| \leq \frac{1}{50}\left[|\mu - \overline{\mu}| + {}^{AB}\mathbf{I}_{0^+}^{p,\phi}|\mu - \overline{\mu}|\right].$$

Therefore, hypothesis (H1) holds with $\mathfrak{L} = \frac{1}{50}$. Moreover, $M = 4, Q_1 = \frac{9}{16}, Q_2 = \frac{3}{8}, Q_3 = \frac{3}{8}, Q_4 = \frac{6}{8}$. By given data, we obtain $\mathcal{G} \approx 0.76 < 1$. Then, all conditions in Theorem 3 are satisfied; hence, the ϕ-ABC problem (1) have a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $\widehat{\mu} \in \mathcal{X}$ satisfies the following.

$$\left|{}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\widehat{\mu}(\sigma) - g(\sigma, \widehat{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{\alpha,\phi}\widehat{\mu}(\sigma))\right| \leq \varepsilon$$

There exists a solution $\mu \in \mathcal{X}$ of the ϕ-ABC problem (1) with the following:

$$\|\widehat{\mu} - \mu_1\| \leq C_f\varepsilon,$$

where the following is the case.

$$C_f = \frac{\Delta_{Q_1,Q_2,Q_3}}{1 - \mathcal{G}} > 0.$$

As a result, all of the requirements in Theorem 5 are met; hence, the ϕ-ABC problem (1) is UH stable. Finally, we consider $\alpha_{\phi}(\sigma) = \phi(\sigma) - \phi(0)$ for $\sigma \in [0,1]$. Then, $\alpha_{\phi} : [0,1] \to \mathbb{R}$ is continuous non-decreasing function. Hence, by Definition 3, we obtain the following:

$$\begin{aligned}
{}^{AB}\mathbf{I}_{0^+}^{\xi,\phi}\alpha_{\phi}(\sigma) &= {}^{AB}\mathbf{I}_{0^+}^{\xi,\phi}[\phi(\sigma)-\phi(0)] \\
&= \left[\frac{1-\xi}{\mathfrak{B}(\xi)}[\phi(\sigma)-\phi(0)] + \frac{\xi}{\mathfrak{B}(\xi)}\,{}^{RL}\mathbf{I}_{0^+}^{\xi,\phi}[\phi(\sigma)-\phi(0)]\right] \\
&= \frac{1-\xi}{\mathfrak{B}(\xi)}\alpha_{\phi}(\sigma) + \frac{\xi}{\mathfrak{B}(\xi)}\frac{[\phi(\sigma)-\phi(0)]^{\xi}}{\Gamma(\xi+2)}\alpha_{\phi}(\sigma) \\
&= \left[\frac{1-\xi}{\mathfrak{B}(\xi)} + \frac{\xi}{\mathfrak{B}(\xi)}\frac{[\phi(\sigma)-\phi(0)]^{\xi}}{\Gamma(\xi+2)}\right]\alpha_{\phi}(\sigma) \\
&\leq \left[\frac{1-\xi}{\mathfrak{B}(\xi)} + \frac{\xi}{\mathfrak{B}(\xi)}\frac{[\phi(1)-\phi(0)]^{\xi}}{\Gamma(\xi+2)}\right]\alpha_{\phi}(\sigma) \\
&= \mathcal{R}\alpha_{\phi}(\sigma), \text{ for all } \varkappa \in \mathcal{J},
\end{aligned}$$

where $\mathcal{R} = \left[\frac{1-\xi}{\mathfrak{B}(\xi)} + \frac{\xi}{\mathfrak{B}(\xi)}\frac{[\phi(1)-\phi(0)]^{\xi}}{\Gamma(\xi+2)}\right] > 0$, *for* $\xi = \{p, \varrho, \varrho + p\}$. *Therefore, Theorem 6 becomes applicable. Moreover, for* $\varepsilon > 0$ *and a continuous function* $\alpha_{\phi} : \mathcal{J} \to \mathbb{R}^{+}$, *we find that the following is satisfied:*

$$\left|{}^{ABC}\mathbf{D}^{p;\phi}\left({}^{ABC}\mathbf{D}^{\varrho;\phi} + \lambda\right)\widehat{\mu}(\sigma) - g(\sigma, \widehat{\mu}(\sigma), {}^{AB}\mathbf{I}_{0^+}^{p,\phi}\widehat{\mu}(\sigma))\right| \leq \varepsilon\alpha_{\phi}(\sigma),$$

Then, Equation (13) is HUR stable with the following:

$$\|\mu - \widehat{\mu}\| \leq \mathcal{N}\varepsilon\alpha_{\phi}(\sigma)$$

where the following is the case.

$$\mathcal{N} = \frac{\mathcal{R}(Q_1 + Q_2 + Q_3)}{1 - \mathcal{G}} > 0.$$

6. Conclusion Remarks

In recent interest, the theory of fractional operators in the frame of Atangana–Baleanu is novel and significant; thus, there are some researchers who studied and developed some qualitative properties of solutions of FDEs involving such operators. We investigated sufficient conditions of the existence and uniqueness of solutions for a nonlinear fractional integro-differential Langevin equation involving ABC fractional derivative with respect to an increasing function under the nonsingular kernel.

Our approach was based on a reduction in the proposed problem into the fractional integral equation and using some standard fixed point theorems due to Banach-type and Krasnoselskii-type. Furthermore, by using mathematical analysis techniques, we analyzed the stability results in the Ulam–Hyers sense. An example was provided to justify the main results.

In fact, our outcomes generalize those in [32]. Due to the wide recent investigations and applications of Mittag–Leffler power law, we believe that the acquired results here are important for future investigations on the theory of fractional calculus and fractional inequalities.

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