fractal and fractional

Article

Hermite–Jensen–Mercer-Type Inequalities via Caputo–Fabrizio Fractional Integral for *h*-Convex Function

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Abstract: Integral inequalities involving many fractional integral operators are used to solve various fractional differential equations. In the present paper, we will generalize the Hermite–Jensen–Mercer-type inequalities for an *h*-convex function via a Caputo–Fabrizio fractional integral. We develop some novel Caputo–Fabrizio fractional integral inequalities. We also present Caputo–Fabrizio fractional integral identities for differentiable mapping, and these will be used to give estimates for some fractional Hermite–Jensen–Mercer-type inequalities. Some familiar results are recaptured as special cases of our results.

Keywords: convex function; *h*-convex function; Hermite–Hadamard inequality; Caputo–Fabrizio fractional integral; Hermite–Hadamard inequality; Jensen inequality; Jensen–Mercer inequality

Citation: Vivas-Cortez, M.; Saleem, M.S.; Sajid, S.; Zahoor, M.S.; Kashuri, A. Hermite–Jensen–Mercer-Type Inequalities via Caputo–Fabrizio Fractional Integral for *h*-Convex Function. *Fractal Fract.* **2021**, *5*, 269. https://doi.org/10.3390/fractalfract5040269

Academic Editors: Savin Treanţă and Carlo Cattani

Received: 3 September 2021
Accepted: 25 November 2021
Published: 10 December 2021

1. Introduction

Fractional calculus has undergone rapid development in both applied and pure mathematics because of its enormous use in image processing, physics, machine learning, networking, and other branches. For more on fractional calculus identities, see [1–3]. The fractional derivative has received rapid attention among experts from different branches of science. Most of the applied problems can not be modeled by classical derivations. The complications in real-world problems are addressed by fractional differential equations. The famous fractional integral contains Riemann–Liouville [4–6], Hadamard [6,7], Caputo–Fabrizio [8], and Katugampola [6], etc.

In this paper, we will restrict ourselves to the Caputo–Fabrizio fractional integral operator. In the current direction of fractional calculus, numerous analysts are characterizing new operators by various methods to cover most of the real-world problems. Usually, the operators are not the same as each other in terms of singularity and locality of kernels. The main aspect that makes Caputo–Fabrizio different from others is that it has a non-singular kernel, and it is useful to find exact solutions for various issues.

For convex functions, the Hermite–Hadamard inequality is a famous inequality that has been proved in many ways and has several extensions and generalizations in the literature (see [9–19]). The Hermite–Hadamard inequality for the convex function is defined as:

Let $\xi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then

$$\xi\left(\frac{v+\mu}{2}\right) \leq \frac{1}{\mu-v}\int_v^\mu \xi(\chi)d\chi \leq \frac{\xi(v)+\xi(\mu)}{2},$$

holds $\forall\ \upsilon, \mu \in I$ and $\upsilon < \mu$.

The generalization of the Hermite–Hadamard inequality for h-convex are defined as (see [20]):

Let $\xi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\xi\left(\frac{\upsilon+\mu}{2}\right) \leq \frac{1}{\mu-\upsilon}\int_{\upsilon}^{\mu}\xi(\chi)d\chi \leq [\xi(\upsilon)+\xi(\mu)]\int_{0}^{1}h(\sigma)d\sigma,$$

holds $\forall\ \upsilon, \mu \in I$ and $\upsilon < \mu$.

In the literature, some more interesting extensions and refinements of the Hermite–Hadamard integral inequality with the help of h-convex functions have been widely studied (see [21–26]).

In the literature, for the Jensen inequality, several interesting studies are given. In [27], for a convex function, a variant of Jensen's inequality is proved by Mercer within the year 2003. Later, Matković et al. presented the Jensen–Mercer inequality for operators with applications in the year 2006 (see [28]).

Vivas-Cortez et al. presented the following variant of the Jensen–Mercer inequality (see [29]).

Theorem 1 ([29])**.** *Let ξ be a h-convex function defined on interval $[\upsilon, \mu]$. Then*

$$\xi\left(\upsilon+\mu-\sum_{i=1}^{n}\chi_i x_i\right) \leq M[\xi(\upsilon)+\xi(\mu)]-\sum_{i=1}^{n}h(\chi_i)\xi(x_i), \tag{1}$$

holds $\forall\ x_i \in [\upsilon, \mu]$ and $\chi_i \in [0,1]$ with $\sum_{i=1}^{n}\chi_i = 1$, where $M = sup\ \{h(\sigma) : \sigma \in (0,1)\}$.

In 2019, the authors established the Hermite–Hadamard–Mercer-like inequalities for fractional integrals [30]. In [31], Butt et al. presented the Hermite–Jensen–Mercer type inequalities for conformable fractional integrals within the year 2020. Furthermore, they developed the Hermite–Jensen–Mercer-like inequalities for k-fractional integrals, generalized fractional integrals and ψ-Riemann–Liouville k-fractional integrals (see [32–34]). In 2020, several researchers presented Hermite–Jensen–Mercer-like inequalities in the setting of a k-Caputo fractional derivative and Caputo fractional derivative (see [35,36]). In [37], the authors developed the weighted Hermite–Hadamard–Mercer-type inequalities for convex functions within the year 2020. Chu et al. presented the new fractional estimates for Hermite–Hadamard–Mercer inequalities in the year 2020 (see [38]).

The present paper is organized as follows. First, we write definitions and preliminary material associated with our present paper. In Section 2, we will present Hermite–Jensen–Mercer-type inequalities for a Caputo–Fabrizio fractional integral operator with the help of an h-convex function. In Section 3, we will develop new Lemmas and then present some results for an h-convex function via a Caputo–Fabrizio fractional integral operator. In Section 4, some more integral inequalities for h-convex functions are established making use of the Hölder–İşcan integral inequality for an improved power mean integral inequality, and at last, we will write concluding remarks to our present paper.

Throughout the paper, we need the following assumption:

Let $\xi : I = [\upsilon, \mu] \to \mathbb{R}$ be a positive function, $0 \leq \upsilon < \mu$ and $\xi \in L_1[\upsilon, \mu]$. Furthermore, consider $h : (0,1) \to \mathbb{R}$ is a non-negative function, $h \neq 0$ and $I \subseteq \mathbb{R}$ is an interval.

Now, we begin with definitions and preliminary results, which will be used in this work.

Definition 1. *(**Convex function**) [39] The function $\xi : [\upsilon, \mu] \to \mathbb{R}$ is called convex, if*

$$\xi(\chi x_1 + (1-\chi)x_2) \leq \chi\xi(x_1) + (1-\chi)\xi(x_2),$$

holds $\forall\ x_1, x_2 \in [\upsilon, \mu]$ and $\chi \in [0,1]$.

Definition 2. *(**h-Convex function**) [40] A function $\xi : [\upsilon, \mu] \subseteq \mathbb{R} \to \mathbb{R}$ is said to be h-convex if*

$$\xi(\chi x_1 + (1-\chi)x_2) \leq h(\chi)\xi(x_1) + h(1-\chi)\xi(x_2),$$

holds $\forall\ x_1, x_2 \in [\upsilon, \mu]$ and $\chi \in [0,1]$.

Definition 3. *(**Superadditive function**) A function $h : [\upsilon, \mu] \subseteq \mathbb{R} \to \mathbb{R}$ is called superadditive function if*

$$h(x_1 + x_2) \geq h(x_1) + h(x_2),$$

holds $\forall\ x_1, x_2 \in [\upsilon, \mu]$.

Definition 4 ([8,41,42])**.** *Let $\xi \in H^1(x_1, x_2), x_1 < x_2, \theta \in [0,1]$, then the definition of the left fractional derivative in the sense of Caputo and Fabrizio is defined as*

$$\left({}^{CFC}_{x_1}D^{\theta}\xi\right)(t) = \frac{B(\theta)}{1-\theta}\int_{x_1}^{t} \xi'(z) e^{\frac{-\theta(t-z)^{\theta}}{1-\theta}}dz,$$

and the associated fractional integral is

$$\left({}^{CF}_{x_1}I^{\theta}\xi\right)(t) = \frac{1-\theta}{B(\theta)}\xi(t) + \frac{\theta}{B(\theta)}\int_{x_1}^{t}\xi(z)dz,$$

where $B(\theta) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

The right fractional derivative is defined as

$$\left({}^{CFC}D^{\theta}_{x_2}\xi\right)(t) = \frac{-B(\theta)}{1-\theta}\int_{t}^{x_2} \xi'(z) e^{\frac{-\theta(z-t)^{\theta}}{1-\theta}}dz,$$

and the associated fractional integral is

$$\left({}^{CF}I^{\theta}_{l_2}\xi\right)(t) = \frac{1-\theta}{B(\theta)}\xi(t) + \frac{\theta}{B(\theta)}\int_{t}^{l_2}\xi(z)dz.$$

In [43,44], the Hölder-İşcan integral inequality and improved power-mean integral inequality is explained as follows.

Theorem 2. *(**Hölder–İşcan integral inequality**) [43] Let ξ_1 and ξ_2 be real functions defined on $[x_1, x_2]$ and if $|\xi_1|^q$ and $|\xi_2|^q$ are integrable on $[\,x_1, x_2]$. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}\int_{x_1}^{x_2}|\xi_1(z)\xi_2(z)|dz \leq{}& \frac{1}{x_2 - x_1}\left\{\left(\int_{x_1}^{x_2}(x_2 - z)|\xi_1(z)|^p dz\right)^{\frac{1}{p}}\left(\int_{x_1}^{x_2}(x_2 - z)|\xi_2(z)|^q dz\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{x_1}^{x_2}(z - x_1)|\xi_1(z)|^p dz\right)^{\frac{1}{p}}\left(\int_{x_1}^{x_2}(z - x_1)|\xi_2(z)|^q dz\right)^{\frac{1}{q}}\right\}\\ \leq{}& \left(\int_{x_1}^{x_2}|\xi_1(z)|^p dz\right)^{\frac{1}{p}}\left(\int_{x_1}^{x_2}|\xi_2(z)|^q dz\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 3. *(**Improved power-mean integral inequality**) [44] Let ξ_1 and ξ_2 be real functions defined on $[x_1, x_2]$ and if $|\xi_1|$, $|\xi_1||\xi_2|^q$ are integrable functions on $[\,x_1, x_2]$. Let $q \geq 1$, then*

$$\begin{aligned}\int_{x_1}^{x_2}|\xi_1(z)\xi_2(z)|dz \leq{}& \frac{1}{x_2 - x_1}\left\{\left(\int_{x_1}^{x_2}(x_2 - z)|\xi_1(z)|dz\right)^{1-\frac{1}{q}}\left(\int_{x_1}^{x_2}(x_2 - z)|\xi_1(z)||\xi_2(z)|^q dz\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{x_1}^{x_2}(z - x_1)|\xi_1(z)|dz\right)^{1-\frac{1}{q}}\left(\int_{x_1}^{x_2}(z - x_1)|\xi_1(z)||\xi_2(z)|^q dz\right)^{\frac{1}{q}}\right\}\\ \leq{}& \left(\int_{x_1}^{x_2}|\xi_1(z)|dz\right)^{1-\frac{1}{q}}\left(\int_{x_1}^{x_2}|\xi_1(z)||\xi_2(z)|^q dz\right)^{\frac{1}{q}}.\end{aligned}$$

2. Hermite–Jensen–Mercer-Type Inequalities via the Caputo–Fabrizio Fractional Operator

Theorem 4. *Let $\xi : I = [v,\mu] \to \mathbb{R}$ be a h-convex function and $\xi \in L_1[v,\mu]$. If h is a super-additive function and $\theta \in [0,1]$, then*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\xi\left(v+\mu-\frac{x_1+x_2}{2}\right) \le{}& \frac{B(\theta)}{\theta(x_2-x_1)} \\ &\times\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)-\frac{2(1-\theta)}{B(\theta)}\xi(t)\right] \\ &\le \int_0^1 h(1)d\chi\left(M[\xi(v)+\xi(\mu)]-\frac{\xi(x_1)+\xi(x_2)}{2}\right), \end{aligned} \tag{2}$$

holds for all $x_1, x_2 \in [v,\mu]$, $t \in [v,\mu]$, $B(\theta) > 0$ is a normalization function and $M = sup$ $\{h(\chi) : \chi \in (0,1)\}$.

Proof. Since ξ is h-convex function on $[x_1, x_2]$ yields that

$$\begin{aligned}\xi\left(v+\mu-\frac{x_1+x_2}{2}\right) &= \xi\left(\frac{v+\mu-x_1+v+\mu-x_2}{2}\right) \\ &\le h\left(\frac{1}{2}\right)\Big(\xi(v+\mu-x_1)+\xi(v+\mu-x_2)\Big) \\ &= h\left(\frac{1}{2}\right)\Big(\xi(v+\mu-(\chi x_1+(1-\chi)x_2)) \\ &\quad +\xi(v+\mu-((1-\chi)x_1+\chi x_2))\Big),\end{aligned}$$

holds for all $x_1, x_2 \in [v,\mu]$.

The above inequality is integrated with respect to χ over $[0,1]$ and by change of variable technique, we can deduce

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}\xi\left(v+\mu-\frac{x_1+x_2}{2}\right) &\le \frac{2}{x_2-x_1}\int_{v+\mu-x_2}^{v+\mu-x_1}\xi(z)dz \\ &= \frac{2}{x_2-x_1}\left(\int_{v+\mu-x_2}^{\chi}\xi(z)dz+\int_{\chi}^{v+\mu-x_1}\xi(z)dz\right).\end{aligned} \tag{3}$$

Both sides of (3) multipled by $\frac{\theta(x_2-x_1)}{2B(\theta)}$ and adding $\frac{2(1-\theta)}{B(\theta)}\xi(t)$, we have

$$\begin{aligned}&\frac{2(1-\theta)}{B(\theta)}\xi(t)+\frac{\theta(x_2-x_1)}{2h\left(\frac{1}{2}\right)B(\theta)}\xi\left(v+\mu-\frac{x_1+x_2}{2}\right) \\ &\le \frac{2(1-\theta)}{B(\theta)}\xi(t)+\frac{\theta}{B(\theta)}\left(\int_{v+\mu-x_2}^{t}\xi(z)dz+\int_t^{v+\mu-x_1}\xi(z)dz\right) \\ &= \left(\frac{(1-\theta)}{B(\theta)}\xi(t)+\frac{\theta}{B(\theta)}\int_{v+\mu-x_2}^{t}\xi(z)dz\right)+\left(\frac{(1-\theta)}{B(\theta)}\xi(t)+\int_t^{v+\mu-x_1}\xi(z)dz\right) \\ &= \left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t).\end{aligned} \tag{4}$$

Suitable rearrangement of (4) yields the first inequality of (2).

By using h-convexity of ξ, we have

$$\xi(\chi(v+\mu-x_1)+(1-\chi)(v+\mu-x_2)) \le h(\chi)\xi(v+\mu-x_1)+h(1-\chi)\xi(v+\mu-x_2),$$

and

$$\xi((1-\chi)(v+\mu-x_1)+\chi(v+\mu-x_2)) \le h(1-\chi)\xi(v+\mu-x_1)+h(\chi)\xi(v+\mu-x_2).$$

Adding the above two inequalities and then by using the super additivity of function and Jensen–Mercer inequality yields that

$$\begin{aligned}
&\xi(\chi(v+\mu-x_1)+(1-\chi)(v+\mu-x_2))+\xi((1-\chi)(v+\mu-x_1)+\chi(v+\mu-x_2)) \\
&\quad\leq h(1)\Big(\xi(v+\mu-x_1)+\xi(v+\mu-x_2)\Big) \\
&\quad\leq h(1)\Big(2M[\xi(v)+\xi(\mu)]-(\xi(x_1)+\xi(x_2))\Big).
\end{aligned} \tag{5}$$

Integrating the inequality (5) with respect to χ over $[0,1]$ and by the change of variable technique, we can write

$$\frac{2}{x_2-x_1}\int_{v+\mu-x_2}^{v+\mu-x_1}\xi(z)dz \leq \int_0^1 h(1)d\chi\Big(2M[\xi(v)+\xi(\mu)]-(\xi(x_1)+\xi(x_2))\Big). \tag{6}$$

By making use of the same operations with (3) in (6), we have

$$\begin{aligned}
&\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t) \\
&\quad\leq \frac{2(1-\theta)}{B(\theta)}\xi(t)+\frac{\theta(x_2-x_1)}{2B(\theta)}\left[\int_0^1 h(1)d\chi\Big(2M[\xi(v)+\xi(\mu)]-(\xi(x_1)+\xi(x_2))\Big)\right].
\end{aligned} \tag{7}$$

By suitable rearrangement of (7), we obtain inequality (2). □

Remark 1. *By putting $h(\chi)=\chi$, M = sup $\{h(\chi):\chi\in(0,1)\}=1$, $x_1=v$ and $x_2=\mu$ in Theorem 2, then we obtain Theorem 2 of (see [45]).*

Theorem 5. *Assume that $\xi: I=[v,\mu]\to\mathbb{R}$ is a h-convex function and $\xi\in L_1[v,\mu]$. If $\theta\in[0,1]$, then*

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\xi\left(v+\mu-\frac{x_1+x_2}{2}\right)\int_0^1 h(\chi)d\chi \\
&\quad\leq \frac{1}{h\left(\frac{1}{2}\right)}M[\xi(v)+\xi(\mu)]\int_0^1 h(\chi)d\chi \\
&\qquad-\frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{x_1}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{x_2}\xi\right)(t)-\frac{2(1-\theta)}{B(\theta)}\xi(t)\right] \\
&\quad\leq \frac{1}{h\left(\frac{1}{2}\right)}M[\xi(v)+\xi(\mu)]\int_0^1 h(\chi)d\chi-\frac{1}{2h\left(\frac{1}{2}\right)}\xi\left(\frac{x_1+x_2}{2}\right),
\end{aligned} \tag{8}$$

holds $\forall\ x_1,x_2\in[v,\mu]$, $t\in[v,\mu]$, $B(\theta)>0$ is a normalization function and M = sup $\{h(\chi)$: $\chi\in(0,1)\}$.

Proof. By the Jensen–Mercer inequality, we have

$$\xi\left(v+\mu-\frac{x_1+x_2}{2}\right)\leq M[\xi(v)+\xi(\mu)]-h\left(\frac{1}{2}\right)[\xi(x_1)+\xi(x_2)].$$

Both sides of the above inequality are multiplied by $h(\chi)$ and integrated with respect to χ over [0,1], and we obtain

$$\begin{aligned}
&\xi\left(v+\mu-\frac{x_1+x_2}{2}\right)\int_0^1 h(\chi)d\chi \\
&\quad\leq M[\xi(v)+\xi(\mu)]\int_0^1 h(\chi)d\chi-h\left(\frac{1}{2}\right)[\xi(x_1)+\xi(x_2)]\int_0^1 h(\chi)d\chi,
\end{aligned}$$

which implies that

$$\frac{1}{h\left(\frac{1}{2}\right)}\xi\left(v+\mu-\frac{x_1+x_2}{2}\right)\int_0^1 h(\chi)d\chi$$
$$\leq \frac{1}{h\left(\frac{1}{2}\right)}M[\xi(v)+\xi(\mu)]\int_0^1 h(\chi)d\chi-[\xi(x_1)+\xi(x_2)]\int_0^1 h(\chi)d\chi.$$

Now, we will use the right-hand side of the Hermite–Hadamard inequality for the h-convex function, and we obtain

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\xi\left(v+\mu-\frac{x_1+x_2}{2}\right)\int_0^1 h(\chi)d\chi \\
&\leq \frac{1}{h\left(\frac{1}{2}\right)}M[\xi(v)+\xi(\mu)]\int_0^1 h(\chi)d\chi-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\xi(z)dz \\
&= \frac{1}{h\left(\frac{1}{2}\right)}M[\xi(v)+\xi(\mu)]\int_0^1 h(\chi)d\chi-\frac{1}{x_2-x_1}\left(\int_{x_1}^{t}\xi(z)dz+\int_t^{x_2}\xi(z)dz\right). \qquad (9)
\end{aligned}$$

Both sides of (9) multiplying by $\frac{\theta(x_2-x_1)}{B(\theta)}$ and subtracting $\frac{2(1-\theta)}{B(\theta)}\xi(t)$, we have

$$\begin{aligned}
&\frac{\theta(x_2-x_1)}{B(\theta)h\left(\frac{1}{2}\right)}\xi\left(v+\mu-\frac{x_1+x_2}{2}\right)\int_0^1 h(\chi)d\chi-\frac{2(1-\theta)}{B(\theta)}\xi(t) \\
&\leq \frac{\theta(x_2-x_1)}{B(\theta)h\left(\frac{1}{2}\right)}M[\xi(v)+\xi(\mu)]\int_0^1 h(\chi)d\chi \\
&\quad -\frac{\theta}{B(\theta)}\left(\int_{x_1}^{t}\xi(z)dz+\int_t^{x_2}\xi(z)dz\right)-\frac{2(1-\theta)}{B(\theta)}\xi(t) \\
&= \frac{\theta(x_2-x_1)}{B(\theta)h\left(\frac{1}{2}\right)}M[\xi(v)+\xi(\mu)]\int_0^1 h(\chi)d\chi-\left[\left(\frac{\theta}{B(\theta)}\int_{x_1}^{t}\xi(z)dz+\frac{(1-\theta)}{B(\theta)}\xi(t)\right)\right. \\
&\quad \left.+\left(\frac{\theta}{B(\theta)}\int_t^{x_2}\xi(z)dz+\frac{(1-\theta)}{B(\theta)}\xi(t)\right)\right] \\
&= \frac{\theta(x_2-x_1)}{h\left(\frac{1}{2}\right)B(\theta)}M[\xi(v)+\xi(\mu)]\int_0^1 h(\chi)d\chi-\left[\left({}^{CF}_{x_1}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{x_2}\xi\right)(t)\right]. \qquad (10)
\end{aligned}$$

After suitable rearrangement, (10) yields the first inequality of (8).

For the second part of the inequality of (8), we will use the right-hand side of the Hermite–Hadamard integral inequality for the h-convex function, and we can write

$$-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\xi(z)dz\leq-\frac{1}{2h\left(\frac{1}{2}\right)}\xi\left(\frac{x_1+x_2}{2}\right). \qquad (11)$$

By using the same operations with (9) in (11), we have

$$-\frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{x_1}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{x_2}\xi\right)(t)-\frac{2(1-\theta)}{B(\theta)}\xi(t)\right]\leq-\frac{1}{2h\left(\frac{1}{2}\right)}\xi\left(\frac{x_1+x_2}{2}\right). \qquad (12)$$

Adding $\frac{1}{h\left(\frac{1}{2}\right)} M[\xi(v)+\xi(\mu)] \int_0^1 h(\chi)d\chi$ to both sides of (12), we have

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)} M[\xi(v)+\xi(\mu)] \int_0^1 h(\chi)d\chi \\
&\quad - \frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{x_1}I^{\theta}\xi\right)(t) + \left({}^{CF}I^{\theta}_{x_2}\xi\right)(t) - \frac{2(1-\theta)}{B(\theta)}\xi(t)\right] \\
&\qquad \le \frac{1}{h\left(\frac{1}{2}\right)} M[\xi(v)+\xi(\mu)] \int_0^1 h(\chi)d\chi - \frac{1}{2h\left(\frac{1}{2}\right)}\xi\left(\frac{x_1+x_2}{2}\right),
\end{aligned}$$

which completes the proof. □

Theorem 6. *Let $\xi_1, \xi_2 : I \subseteq \mathbb{R} \to \mathbb{R}$ be an h-convex function on I. If $\xi_1\xi_2 \in L[v,\mu]$, then*

$$\begin{aligned}
&\frac{2B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi_1\xi_2\right)(t) + \left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi_1\xi_2\right)(t) - \frac{2(1-\theta)}{B(\theta)}\xi_1(t)\xi_2(t)\right] \\
&\le 2M^2 B_1(v,\mu) - 2MB_2(v,\mu,x_1)\int_0^1 h(1-\chi)d\chi \\
&\quad - 2MB_3(v,\mu,x_2)]\int_0^1 h(\chi)d\chi + 2B_4(x_1,x_2)\int_0^1 h(\chi)h(1-\chi)d\chi \\
&\quad + 2K_1(x_1)\int_0^1 (h(1-\chi))^2 d\chi + 2K_2(x_2)\int_0^1 (h(\chi))^2 d\chi,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
&B_1(v,\mu) = \xi_1(v)\xi_2(v) + \xi_1(v)\xi_2(\mu) + \xi_1(\mu)\xi_2(v) + \xi_1(\mu)\xi_2(\mu), \\
&B_2(v,\mu,x_1) = \xi_1(v)\xi_2(x_1) + \xi_1(\mu)\xi_2(x_1) + \xi_1(x_1)\xi_2(v) + \xi_1(x_1)\xi_2(\mu), \\
&B_3(v,\mu,x_2) = \xi_1(v)\xi_2(x_2) + \xi_1(\mu)\xi_2(x_2) + \xi_1(x_2)\xi_2(v) + \xi_1(x_2)\xi_2(\mu), \\
&B_4(x_1,x_2) = \xi_1(x_1)\xi_2(x_2) + \xi_1(x_2)\xi_2(x_1), \\
&K_1(x_1) = \xi_1(x_1)\xi_2(x_1),
\end{aligned}$$

and

$$K_2(x_2) = \xi_1(x_2)\xi_2(x_2),$$

holds $\forall\ x_1, x_2 \in [v,\mu]$, M = sup $\{h(\chi) : \chi \in (0,1)\}$, $t \in [v,\mu]$ and $B(\theta) > 0$ is a normalization function.

Proof. Since ξ_1 and ξ_2 are h-convex functions on $[x_1, x_2]$ and making use of the Jensen–Mercer inequality, we have

$$\begin{aligned}
&\xi_1(v+\mu-((1-\chi)x_1+\chi x_2)) \\
&\quad \le M[\xi_1(v)+\xi_1(\mu)] - (h(1-\chi)\xi_1(x_1) + h(\chi)\xi_1(x_2)), \ \ \forall \chi \in [0,1], x_1, x_2 \in I,
\end{aligned}$$

and

$$\begin{aligned}
&\xi_2(v+\mu-((1-\chi)x_1+\chi x_2)) \\
&\quad \le M[\xi_2(v)+\xi_2(\mu)] - (h(1-\chi)\xi_2(x_1) + h(\chi)\xi_2(x_2)), \ \ \forall \chi \in [0,1], x_1, x_2 \in I.
\end{aligned}$$

Multiplying both sides of the above inequalities, we can write

$$
\begin{aligned}
&\xi_1(v+\mu-((1-\chi)x_1+\chi x_2))\xi_2(v+\mu-((1-\chi)x_1+\chi x_2)) \\
&\quad \le M^2[\xi_1(v)\xi_2(v)+\xi_1(v)\xi_2(\mu)+\xi_1(\mu)\xi_2(v)+\xi_1(\mu)\xi_2(\mu)] \\
&\qquad - Mh(1-\chi)[\xi_1(v)\xi_2(x_1)+\xi_1(\mu)\xi_2(x_1)+\xi_1(x_1)\xi_2(v)+\xi_1(x_1)\xi_2(\mu)] \\
&\qquad - Mh(\chi)[\xi_1(v)\xi_2(x_2)+\xi_1(\mu)\xi_2(x_2)+\xi_1(x_2)\xi_2(v)+\xi_1(x_2)\xi_2(\mu)] \\
&\qquad + h(\chi)h(1-\chi)[\xi_1(x_1)\xi_2(x_2)+\xi_1(x_2)\xi_2(x_1)]+(h(1-\chi))^2[\xi_1(x_1)\xi_2(x_1)] \\
&\qquad + (h(\chi))^2[\xi_1(x_2)\xi_2(x_2)].
\end{aligned}
$$

Integrating the above inequality with respect to χ over [0,1] and then by the change of variable technique, we obtain

$$
\begin{aligned}
&\frac{1}{x_2-x_1}\int_{v+\mu-x_2}^{v+\mu-x_1}\xi_1(z)\xi_2(z)dz \\
&\quad \le M^2[\xi_1(v)\xi_2(v)+\xi_1(v)\xi_2(\mu)+\xi_1(\mu)\xi_2(v)+\xi_1(\mu)\xi_2(\mu)] \\
&\qquad - M[\xi_1(v)\xi_2(x_1)+\xi_1(\mu)\xi_2(x_1)+\xi_1(x_1)\xi_2(v)+\xi_1(x_1)\xi_2(\mu)]\int_0^1 h(1-\chi)d\chi \\
&\qquad - M[\xi_1(v)\xi_2(x_2)+\xi_1(\mu)\xi_2(x_2)+\xi_1(x_2)\xi_2(v)+\xi_1(x_2)\xi_2(\mu)]\int_0^1 h(\chi)d\chi \\
&\qquad + [\xi_1(x_1)\xi_2(x_2)+\xi_1(x_2)\xi_2(x_1)]\int_0^1 h(\chi)h(1-\chi)d\chi \\
&\qquad + [\xi_1(x_1)\xi_2(x_1)]\int_0^1(h(1-\chi))^2d\chi+[\xi_1(x_2)\xi_2(x_2)]\int_0^1(h(\chi))^2d\chi,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{2}{x_2-x_1}\left[\int_{v+\mu-x_2}^{\chi}\xi_1(z)\xi_2(z)dz+\int_{\chi}^{v+\mu-x_1}\xi_1(z)\xi_2(z)dz\right] \\
&\quad \le 2M^2B_1(v,\mu)-2MB_2(v,\mu,x_1)\int_0^1 h(1-\chi)d\chi \\
&\qquad - 2MB_3(v,\mu,x_2)]\int_0^1 h(\chi)d\chi+2B_4(x_1,x_2)\int_0^1 h(\chi)h(1-\chi)d\chi \\
&\qquad + 2K_1(x_1)\int_0^1(h(1-\chi))^2d\chi+2K_2(x_2)\int_0^1(h(\chi))^2d\chi.
\end{aligned}
$$

The above inequality is multipled by $\frac{\theta(x_2-x_1)}{2B(\theta)}$, and adding $\frac{2(1-\theta)}{B(\theta)}\xi_1(t)\xi_2(t)$, we have

$$
\begin{aligned}
&\frac{\theta}{B(\theta)}\left[\int_{v+\mu-x_2}^{t}\xi_1(z)\xi_2(z)dz+\int_{t}^{v+\mu-x_1}\xi_1(z)\xi_2(z)dz\right]+\frac{2(1-\theta)}{B(\theta)}\xi_1(t)\xi_2(t) \\
&\quad \le \frac{\theta(x_2-x_1)}{2B(\theta)}\left[2M^2B_1(v,\mu)-2MB_2(v,\mu,x_1)\int_0^1 h(1-\chi)d\chi\right. \\
&\qquad - 2MB_3(v,\mu,x_2)]\int_0^1 h(\chi)d\chi+2B_4(x_1,x_2)\int_0^1 h(\chi)h(1-\chi)d\chi \\
&\qquad \left.+ 2K_1(x_1)\int_0^1(h(1-\chi))^2d\chi+2K_2(x_2)\int_0^1(h(\chi))^2d\chi\right]+\frac{2(1-\theta)}{B(\theta)}\xi_1(t)\xi_2(t).
\end{aligned}
$$

Therefore,

$$\left[\frac{(1-\theta)}{B(\theta)}\xi_1(t)\xi_2(t)+\frac{\theta}{B(\theta)}\int_{v+\mu-x_2}^{t}\xi_1(w)\xi_2(w)dw\right]+\left[\frac{(1-\theta)}{B(\theta)}\xi_1(t)\xi_2(t)\right.$$
$$\left.+\frac{\theta}{B(\theta)}\int_{t}^{v+\mu-x_1}\xi_1(w)\xi_2(w)dw\right]$$
$$\leq\frac{\theta(x_2-x_1)}{2B(\theta)}\left[2M^2B_1(v,\mu)-2MB_2(v,\mu,x_1)\int_0^1 h(1-\chi)d\chi\right.$$
$$-2MB_3(v,\mu,x_2)]\int_0^1 h(\chi)d\chi+2B_4(x_1,x_2)\int_0^1 h(\chi)h(1-\chi)d\chi$$
$$\left.+2K_1(x_1)\int_0^1 (h(1-\chi))^2d\chi+2K_2(x_2)\int_0^1 (h(\chi))^2d\chi\right]+\frac{2(1-\theta)}{B(\theta)}\xi_1(t)\xi_2(t).$$

Thus,

$$\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi_1\xi_2\right)(t)+\left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi_1\xi_2\right)(t)\right]$$
$$\leq\frac{\theta(x_2-x_1)}{2B(\theta)}\left[2M^2B_1(v,\mu)-2MB_2(v,\mu,x_1)\int_0^1 h(1-\chi)d\chi\right.$$
$$-2MB_3(v,\mu,x_2)]\int_0^1 h(\chi)d\chi+2B_4(x_1,x_2)\int_0^1 h(\chi(1-\chi))$$
$$\left.+2K_1(x_1)\int_0^1 h((1-\chi)^2)d\chi+2K_2(x_2)\int_0^1 h(\chi^2)d\chi\right]+\frac{2(1-\theta)}{B(\theta)}\xi_1(t)\xi_2(t). \tag{14}$$

By suitable rearrangement, (14) yields required inequality (13). □

Remark 2. *By putting $h(\chi)=\chi$, M = sup $\{h(\chi):\chi\in(0,1)\}=1$, $x_1=v$ and $x_2=\mu$ in Theorem 2, then we obtain Theorem 3 of [45].*

3. Some Novel Results Related to the Caputo–Fabrizio Fractional Operator

In this section, we will present some new Lemmas, and then we develop some novel results for an h-convex function with the help of the Caputo–Fabrizio fractional integral operator.

Lemma 1. *Let $\xi:I=[v,\mu]\to\mathbb{R}$ be a differentiable mapping on I°, where $v,\mu\in I$ with $v<\mu$. If $\xi'\in L_1[v,\mu]$, then*

$$\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2}-\frac{1}{x_2-x_1}\int_{v+\mu-x_2}^{v+\mu-x_1}\xi(z)dz$$
$$=\frac{x_2-x_1}{2}\int_0^1(1-2\chi)\xi'(v+\mu-((1-\chi)x_1+\chi x_2))d\chi, \tag{15}$$

holds for all $x_1,x_2\in[v,\mu]$.

Proof. Note that

$$I=\int_0^1(1-2\chi)\xi'(v+\mu-((1-\chi)x_1+\chi x_2))d\chi$$
$$=\frac{\xi(v+\mu-((1-\chi)x_1+\chi x_2))}{x_1-x_2}(1-2\chi)\Bigg|_0^1+2\int_0^1\frac{\xi(v+\mu-((1-\chi)x_1+\chi x_2))}{x_1-x_2}d\chi$$
$$=\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{x_2-x_1}-\frac{2}{x_2-x_1}.\frac{1}{x_2-x_1}\int_{v+\mu-x_2}^{v+\mu-x_1}\xi(z)dz.$$

After suitable rearrangements, we obtain the required inequality (15). □

Remark 3. *For $x_1 = \upsilon$ and $x_2 = \mu$ in Lemma 3, we obtain Lemma 2.1 of (see [46]).*

Lemma 2. *Suppose that $\xi : I = [\upsilon, \mu] \to \mathbb{R}$ is a differentiable mapping on I°, $\upsilon, \mu \in I$ with $\upsilon < \mu$. If $\xi' \in L_1[\upsilon, \mu]$ and take $\theta \in [0,1]$, then*

$$\begin{aligned}
&\frac{x_2 - x_1}{2} \int_0^1 (1-2\chi)\xi'(\upsilon + \mu - ((1-\chi)x_1 + \chi x_2))d\chi - \frac{2(1-\theta)}{\theta(x_2 - x_1)}\xi(t) \\
&= \frac{\xi(\upsilon + \mu - x_1) + \xi(\upsilon + \mu - x_2)}{2} - \frac{B(\theta)}{\theta(x_2 - x_1)}\Big[\Big({}^{CF}_{\upsilon+\mu-x_2}I^\theta \xi\Big)(t) + \Big({}^{CF}I^\theta_{\upsilon+\mu-x_1}\xi\Big)(t)\Big],
\end{aligned}$$

holds for all $x_1, x_2 \in [\upsilon, \mu]$, where $t \in [\upsilon, \mu]$ and $B(\theta) > 0$ is a normalization function.

Proof. It is easy to see that

$$\begin{aligned}
&\int_0^1 (1-2\chi)\xi'(\upsilon + \mu - ((1-\chi)x_1 + \chi x_2))d\chi \\
&= \frac{\xi(\upsilon + \mu - x_1) + \xi(\upsilon + \mu - x_2)}{x_2 - x_1} - \frac{2}{(x_2 - x_1)^2}\left(\int_{\upsilon+\mu-d}^t \xi(z)dz + \int_t^{\upsilon+\mu-x_1} \xi(z)dz\right).
\end{aligned}$$

With both sides of the above inequality multiplied by $\frac{\theta(x_2-x_1)^2}{2B(\theta)}$ and subtracting $\frac{2(1-\theta)}{B(\theta)}\xi(t)$, we have

$$\begin{aligned}
&\frac{\theta(x_2 - x_1)^2}{2B(\theta)} \int_0^1 (1-2\chi)\xi'(\upsilon + \mu - ((1-\chi)x_1 + \chi x_2))d\chi - \frac{2(1-\theta)}{B(\theta)}\xi(t) \\
&= \frac{\theta(x_2 - x_1)(\xi(\upsilon + \mu - x_1) + \xi(\upsilon + \mu - x_2))}{2B(\theta)} - \frac{2(1-\theta)}{B(\theta)}\xi(t) \\
&\quad - \frac{\theta}{B(\theta)}\left(\int_{\upsilon+\mu-x_2}^t \xi(z)dz + \int_t^{\upsilon+\mu-x_1} \xi(z)dz\right) \\
&= \frac{\theta(x_2 - x_1)(\xi(\upsilon + \mu - x_1) + \xi(\upsilon + \mu - x_2))}{2B(\theta)} - \left(\frac{(1-\theta)}{B(\theta)}\xi(t) + \frac{\theta}{B(\theta)}\int_{\upsilon+\mu-x_2}^t \xi(z)dz\right) \\
&\quad - \left(\frac{(1-\theta)}{B(\theta)}\xi(t) + \frac{\theta}{B(\theta)}\int_t^{\upsilon+\mu-x_1} \xi(z)dz\right) \\
&= \frac{\theta(x_2 - x_1)(\xi(\upsilon + \mu - x_1) + \xi(\upsilon + \mu - x_2))}{2B(\theta)} - \Big[\Big({}^{CF}_{\upsilon+\mu-x_2}I^\theta \xi\Big)(t) + \Big({}^{CF}I^\theta_{\upsilon+\mu-x_1}\xi\Big)(t)\Big].
\end{aligned}$$

After suitable rearrangements, we obtain the desired result. □

Remark 4. *For $x_1 = \upsilon$ and $x_2 = \mu$ in Lemma 3, then we obtain Lemma 2 of (see [45]).*

Theorem 7. *Let $\xi : I \to \mathbb{R}$ be a positive differentiable function on I°. If $|\xi'|$ is a h-convex function on $[\upsilon, \mu]$ where $x_1, x_2 \in I$ with $\upsilon < \mu$, $\xi' \in L_1[\upsilon, \mu]$ and $\theta \in [0,1]$, then*

$$\begin{aligned}
&\Bigg|\frac{\xi(\upsilon + \mu - x_1) + \xi(\upsilon + \mu - x_2)}{2} - \frac{B(\theta)}{\theta(x_2 - x_1)}\Big[\Big({}^{CF}_{\upsilon+\mu-x_2}I^\theta \xi\Big)(t) + \Big({}^{CF}I^\theta_{\upsilon+\mu-x_1}\xi\Big)(t)\Big] \\
&\quad + \frac{2(1-\theta)}{\theta(x_2 - x_1)}\xi(t)\Bigg| \\
&\le \frac{x_2 - x_1}{2}\left[\frac{1}{2}M\Big(\big|\xi'(\upsilon)\big| + \big|\xi'(\mu)\big|\Big) - \Big\{B_h(1-\chi)\big|\xi'(x_1)\big| + B_h(\chi)\big|\xi'(x_2)\big|\Big\}\right],
\end{aligned} \tag{16}$$

where

$$B_h(1-\chi) = \int_0^{\frac{1}{2}} (1-2\chi)h(1-\chi)d\chi + \int_{\frac{1}{2}}^1 (2\chi-1)h(1-\chi)d\chi,$$
$$B_h(\chi) = \int_0^{\frac{1}{2}} (1-2\chi)h(\chi)d\chi + \int_{\frac{1}{2}}^1 (2\chi-1)h(\chi)d\chi,$$

holds $\forall\, x_1, x_2 \in [v,\mu]$, $t \in [v,\mu]$, $B(\theta) > 0$ *is a normalization function and* $M = sup\ \{h(\chi)\colon \chi \in (0,1)\}$.

Proof. By making use of Lemma 3, the properties of the absolute value, the h-convexity of $|\xi'|$ and the Jensen–Mercer inequality yields

$$\begin{aligned}
&\Bigg|\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2} - \frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t) + \left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)\right] \\
&\quad + \frac{2(1-\theta)}{\theta(x_2-x_1)}\xi(t)\Bigg| \\
&\leq \frac{x_2-x_1}{2}\int_0^1 |1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|d\chi \\
&\leq \frac{x_2-x_1}{2}\int_0^1 |1-2\chi|\left(M\left[\left|\xi'(v)\right|+\left|\xi'(\mu)\right|\right] - \left(h(1-\chi)\left|\xi'(x_1)\right| + h(\chi)\left|\xi'(x_2)\right|\right)\right)d\chi \\
&\leq \frac{x_2-x_1}{2}\Bigg(\int_0^{\frac{1}{2}} (1-2\chi)\left(M\left[\left|\xi'(v)\right|+\left|\xi'(\mu)\right|\right] - \left(h(1-\chi)\left|\xi'(x_1)\right| + h(\chi)\left|\xi'(x_2)\right|\right)\right)d\chi \\
&\quad + \int_{\frac{1}{2}}^1 (2\chi-1)\left(M\left[\left|\xi'(v)\right|+\left|\xi'(\mu)\right|\right] - \left(h(1-\chi)\left|\xi'(x_1)\right| + h(\chi)\left|\xi'(x_2)\right|\right)\right)d\chi\Bigg) \\
&\leq \frac{x_2-x_1}{2}\Bigg[\frac{1}{2}M\left(\left|\xi'(v)\right|+\left|\xi'(\mu)\right|\right) \\
&\quad - \Bigg\{\left|\xi'(x_1)\right|\left(\int_0^{\frac{1}{2}}(1-2\chi)h(1-\chi)d\chi + \int_{\frac{1}{2}}^1 (2\chi-1)h(1-\chi)d\chi\right) \\
&\quad + \left|\xi'(x_2)\right|\left(\int_0^{\frac{1}{2}}(1-2\chi)h(\chi)d\chi + \int_{\frac{1}{2}}^1 (2\chi-1)h(\chi)d\chi\right)\Bigg\}\Bigg] \\
&\leq \frac{x_2-x_1}{2}\left[\frac{1}{2}M\left(\left|\xi'(v)\right|+\left|\xi'(\mu)\right|\right) - \left\{B_h(1-\chi)\left|\xi'(x_1)\right| + B_h(\chi)\left|\xi'(x_2)\right|\right\}\right].
\end{aligned}$$

This completes the proof. □

Remark 5. *By putting* $h(\chi) = \chi$, $M = sup\ \{h(\chi) : \chi \in (0,1)\} = 1$, $x_1 = v$ *and* $x_2 = \mu$ *in Theorem 3, we obtain Theorem 5 of [45].*

Theorem 8. *Suppose that* $\xi : I \to \mathbb{R}$ *is a positive differentiable function on* I° *and* $|\xi'|^q$ *is a h-convex function on* $[v,\mu]$, $v,\mu \in I^\circ$ *with* $v < \mu$ *for* $p,q > 1$ *with* $\frac{1}{p}+\frac{1}{q} = 1$, *where* $v,\mu \in I$ *with* $v < \mu$. *If* $\xi' \in L_1[v,\mu]$ *and* $\theta \in [0,1]$, *then*

$$\begin{aligned}
&\Bigg|\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2} - \frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t) + \left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)\right] \\
&\quad + \frac{2(1-\theta)}{\theta(x_2-x_1)}\xi(t)\Bigg| \\
&\leq \frac{x_2-x_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg(M\left[\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right] \\
&\quad - \left(\left|\xi'(x_1)\right|^q\int_0^1 h(1-\chi)d\chi + \left|\xi'(x_2)\right|^q\int_0^1 h(\chi)d\chi\right)\Bigg)^{\frac{1}{q}}, \qquad (17)
\end{aligned}$$

holds $\forall\, x_1, x_2 \in [v,\mu]$, $t \in [v,\mu]$, $B(\theta) > 0$ *is a normalization function and* $M = sup\ \{h(\chi)\colon \chi \in (0,1)\}$.

Proof. From Lemma 3, Hölder's integral inequality, the h-convexity of $|\xi'|^q$ and the Jensen–Mercer inequality yields that

$$\begin{aligned}
&\left|\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2}-\frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)\right]\right.\\
&\left.+\frac{2(1-\theta)}{\theta(x_2-x_1)}\xi(t)\right|\\
&\leq \frac{x_2-x_1}{2}\int_0^1|1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|d\chi\\
&\leq \frac{x_2-x_1}{2}\left(\int_0^1|1-2\chi|^p d\chi\right)^{\frac{1}{p}}\left(\int_0^1\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\\
&\leq \frac{x_2-x_1}{2}\left(\int_0^1|1-2\chi|^p d\chi\right)^{\frac{1}{p}}\left(\int_0^1\left(M\left[\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right]\right.\right.\\
&\left.\left.-\left(h(1-\chi)\left|\xi'(x_1)\right|^q+h(\chi)\left|\xi'(x_2)\right|^q\right)\right)d\chi\right)^{\frac{1}{q}}\\
&\leq \frac{x_2-x_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(M\left[\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right]\right.\\
&\left.-\left(\left|\xi'(x_1)\right|^q\int_0^1 h(1-\chi)d\chi+\left|\xi'(x_2)\right|^q\int_0^1 h(\chi)d\chi\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Remark 6. *By putting $h(\chi)=\chi$, $M = sup\ \{h(\chi):\chi\in(0,1)\}=1$, $x_1=v$ and $x_2=\mu$ in Theorem 3, we obtain Theorem 6 of [45].*

Next, we will prove the following theorems using the Hölder–*İscan* integral inequality and for improved power mean integral inequality, respectively.

Theorem 9. *Assume that $\xi: I\to\mathbb{R}$ is a positive differentiable mapping on I° and $|\xi'|^q$ is a h-convex function on $[v,\mu]$, $v,\mu\in I^\circ$ with $v<\mu$ for $q\geq 1$, where $v,\mu\in I$ with $v<\mu$. If $\xi'\in L_1[v,\mu]$ and $\theta\in[0,1]$, then*

$$\begin{aligned}
&\left|\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2}-\frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)\right]\right.\\
&\left.+\frac{2(1-\theta)}{\theta(x_2-x_1)}\xi(t)\right|\\
&\leq \frac{x_2-x_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}M\left[\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right]\right.\\
&\left.-\left(\left|\xi'(x_1)\right|^q\int_0^1|1-2\chi|h(1-\chi)d\chi+\left|\xi'(x_2)\right|^q\int_0^1|1-2\chi|h(\chi)d\chi\right)\right)^{\frac{1}{q}}, \qquad (18)
\end{aligned}$$

holds $\forall\ x_1,x_2\in[v,\mu]$, $t\in[v,\mu]$, $B(\theta)>0$ is a normalization function and $M = sup\ \{h(\chi)$: $\chi\in(0,1)\}$.

Proof. Take $q > 1$, by using Lemma 3, the power mean inequality, the h-convexity of $|\xi'|^q$ and the Jensen–Mercer inequality, and we have

$$\begin{aligned}
&\left|\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2} - \frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t) + \left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)\right]\right.\\
&\quad\left. + \frac{2(1-\theta)}{\theta(x_2-x_1)}\xi(t)\right|\\
&\leq \frac{x_2-x_1}{2}\int_0^1 |1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|d\chi\\
&\leq \frac{x_2-x_1}{2}\left(\int_0^1 |1-2\chi|d\chi\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1 |1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\\
&\leq \frac{x_2-x_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2\chi|\right.\\
&\quad\left.\times\left(M\left[\left|\xi'(v)\right|^q + \left|\xi'(\mu)\right|^q\right] - \left(h(1-\chi)\left|\xi'(x_1)\right|^q + h(\chi)\left|\xi'(x_2)\right|^q\right)\right)d\chi\right)^{\frac{1}{q}}\\
&\leq \frac{x_2-x_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}M\left[\left|\xi'(v)\right|^q + \left|\xi'(\mu)\right|^q\right]\right.\\
&\quad\left. - \left(\left|\xi'(x_1)\right|^q\int_0^1 |1-2\chi|h(1-\chi)d\chi + \left|\xi'(x_2)\right|^q\int_0^1 |1-2\chi|h(\chi)d\chi\right)\right)^{\frac{1}{q}}.
\end{aligned} \tag{19}$$

This completes the proof. □

4. Some Results in Improved Hölder Setting

In this section, we will present some results for the h-convex function in the setting of the Hölder–İşcan integral inequality and improved power mean integral inequality via the Caputo–Fabrizio fractional integral operator.

Theorem 10. *Let $\xi : I \to \mathbb{R}$ be a positive differentiable mapping on I° and $|\xi'|^q$ be a h-convex function on $[v,\mu]$, $v,\mu \in I^\circ$ with $v < \mu$ for $p,q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, where $v,\mu \in I$ with $v < \mu$. If $\xi' \in L_1[v,\mu]$ and $\theta \in [0,1]$, then*

$$\begin{aligned}
&\left|\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2} - \frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t) + \left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)\right]\right.\\
&\quad\left. + \frac{2(1-\theta)}{\theta(x_2-x_1)}\xi(t)\right|\\
&\leq \frac{x_2-x_1}{2}\left[\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}M\left(\left|\xi'(v)\right|^q + \left|\xi'(\mu)\right|^q\right)\right.\right.\\
&\quad\left. - \left(\left|\xi'(x_1)\right|^q\int_0^1 (1-\chi)h(1-\chi)d\chi + \left|\xi'(x_2)\right|^q\int_0^1 (1-\chi)h(\chi)d\chi\right)\right)^{\frac{1}{q}}\\
&\quad + \left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}M\left(\left|\xi'(v)\right|^q + \left|\xi'(\mu)\right|^q\right)\right.\\
&\quad\left.\left. - \left(\left|\xi'(x_1)\right|^q\int_0^1 \chi h(1-\chi)d\chi + \left|\xi'(x_2)\right|^q\int_0^1 \chi h(\chi)d\chi\right)\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{20}$$

holds $\forall\ x_1,x_2 \in [v,\mu]$, $t \in [v,\mu]$, $B(\theta) > 0$ is a normalization function and M = sup $\{h(\chi)\colon \chi \in (0,1)\}$.

Proof. From Lemma 3, using the Hölder–İscan integral inequality, the h-convexity of $|\xi'|^q$ and the Jensen–Mercer inequality yields

$$\begin{aligned}
&\left|\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2}-\frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)\right]\right.\\
&\left.+\frac{2(1-\theta)}{\theta(x_2-x_1)}\xi(t)\right|\\
&\leq\frac{x_2-x_1}{2}\int_0^1|1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|d\chi\\
&\leq\frac{x_2-x_1}{2}\left[\left(\int_0^1(1-\chi)|1-2\chi|^p d\chi\right)^{\frac{1}{p}}\right.\\
&\times\left(\int_0^1(1-\chi)\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1\chi|1-2\chi|^p d\chi\right)^{\frac{1}{p}}\left(\int_0^1\chi\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\right]\\
&\leq\frac{x_2-x_1}{2}\left[\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\int_0^1(1-\chi)\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\int_0^1\chi\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\right]\\
&\leq\frac{x_2-x_1}{2}\left[\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\int_0^1(1-\chi)\right.\right.\\
&\times\left.\left(M\left[\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right]-\left(h(1-\chi)\left|\xi'(x_1)\right|^q+h(\chi)\left|\xi'(x_2)\right|^q\right)\right)d\chi\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\int_0^1\chi\right.\\
&\times\left.\left.\left(M\left[\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right]-\left(h(1-\chi)\left|\xi'(x_1)\right|^q+h(\chi)\left|\xi'(x_2)\right|^q\right)\right)d\chi\right)^{\frac{1}{q}}\right]\\
&\leq\frac{x_2-x_1}{2}\left[\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}M\left(\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right)\right.\right.\\
&\left.-\left(\left|\xi'(x_1)\right|^q\int_0^1(1-\chi)h(1-\chi)d\chi+\left|\xi'(x_2)\right|^q\int_0^1(1-\chi)h(\chi)d\chi\right)\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}M\left(\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right)\right.\\
&\left.\left.-\left(\left|\xi'(x_1)\right|^q\int_0^1\chi h(1-\chi)d\chi+\left|\xi'(x_2)\right|^q\int_0^1\chi h(\chi)d\chi\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Theorem 11. *Let $\xi : I \to \mathbb{R}$ be a positive differentiable mapping on I° and $|\xi'|^q$ be a h-convex function on $[v,\mu]$, $v,\mu \in I^{\circ}$ with $v<\mu$ for $q \geq 1$, where $v,\mu \in I$ with $v<\mu$. If $\xi' \in L_1[v,\mu]$ and $\theta \in [0,1]$, then*

$$\begin{aligned}
&\left|\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2}-\frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)\right]\right.\\
&\left.+\frac{2(1-\theta)}{\theta(x_2-x_1)}\xi(t)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{x_2-x_1}{2}\left[\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{4}M\left(\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right)\right.\right.\\
&\quad\left.-\left(\left|\xi'(x_1)\right|^q\int_0^1(1-\chi)|1-2\chi|h(1-\chi)d\chi+\left|\xi'(x_2)\right|^q\int_0^1(1-\chi)|1-2\chi|h(\chi)d\chi\right)\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{4}M\left(\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right)\right.\\
&\quad\left.\left.-\left(\left|\xi'(x_1)\right|^q\int_0^1\chi|1-2\chi|h(1-\chi)d\chi+\left|\xi'(x_2)\right|^q\int_0^1\chi|1-2\chi|h(\chi)d\chi\right)\right)^{\frac{1}{q}}\right],
\end{aligned}
\tag{21}
$$

holds $\forall\ x_1,x_2\in[v,\mu]$, $t\in[v,\mu]$, $B(\theta)>0$ *is a normalization function and* $M=sup\ \{h(\chi)\colon \chi\in(0,1)\}$.

Proof. Take $q>1$, from Lemma 3, and using the improved power-mean integral inequality, the definition of the h-convexity of $|\xi'|^q$, and the Jensen–Mercer inequality, we have

$$
\begin{aligned}
&\left|\frac{\xi(v+\mu-x_1)+\xi(v+\mu-x_2)}{2}-\frac{B(\theta)}{\theta(x_2-x_1)}\left[\left({}^{CF}_{v+\mu-x_2}I^{\theta}\xi\right)(t)+\left({}^{CF}I^{\theta}_{v+\mu-x_1}\xi\right)(t)\right]\right|\\
&\leq\frac{x_2-x_1}{2}\int_0^1|1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|d\chi\\
&\leq\frac{x_2-x_1}{2}\left[\left(\int_0^1(1-\chi)|1-2\chi|d\chi\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left(\int_0^1(1-\chi)|1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\chi|1-2\chi|d\chi\right)^{1-\frac{1}{q}}\\
&\quad\left.\times\left(\int_0^1\chi|1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\right]\\
&\leq\frac{x_2-x_1}{2}\left[\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\chi)|1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\int_0^1\chi|1-2\chi|\left|\xi'(v+\mu-((1-\chi)x_1+\chi x_2))\right|^q d\chi\right)^{\frac{1}{q}}\right]\\
&\leq\frac{x_2-x_1}{2}\left[\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\chi)|1-2\chi|\right.\right.\\
&\quad\left.\times\left(M\left[\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right]-\left(h(1-\chi)\left|\xi'(x_1)\right|^q+h(\chi)\left|\xi'(x_2)\right|^q\right)\right)d\chi\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\int_0^1\chi|1-2\chi|\right.\\
&\quad\left.\left.\times\left(M\left[\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right]-\left(h(1-\chi)\left|\xi'(x_1)\right|^q+h(\chi)\left|\xi'(x_2)\right|^q\right)\right)d\chi\right)^{\frac{1}{q}}\right]\\
&\leq\frac{x_2-x_1}{2}\left[\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{4}M\left(\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right)\right.\right.\\
&\quad\left.-\left(\left|\xi'(x_1)\right|^q\int_0^1(1-\chi)|1-2\chi|h(1-\chi)d\chi+\left|\xi'(x_2)\right|^q\int_0^1(1-\chi)|1-2\chi|h(\chi)d\chi\right)\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{4}M\left(\left|\xi'(v)\right|^q+\left|\xi'(\mu)\right|^q\right)\right.\\
&\quad\left.\left.-\left(\left|\xi'(x_1)\right|^q\int_0^1\chi|1-2\chi|h(1-\chi)d\chi+\left|\xi'(x_2)\right|^q\int_0^1\chi|1-2\chi|h(\chi)d\chi\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

This completes the proof. □

5. Conclusions

In this note, we established the Hermite–Jensen–Mercer-type inequalities for an h-convex function in the Caputo–Fabrizio setting, and various Caputo–Fabrizio fractional integral inequalities are provided as well. We expect that this work will lead to the novel fractional integral research for Hermite–Hadamard inequalities. The remarks at the end of the results verify the generalization of the results. These results are new and set various interesting directions. In the future, we will prove the inequalities (2) and (8) by using any other method.

Author Contributions: Conceptualization, M.V.-C., M.S.S., S.S., M.S.Z. and A.K.; Funding acquisition, M.V.-C.; Investigation, M.V.-C., M.S.S., S.S., M.S.Z. and A.K.; Methodology, M.V.-C. and M.S.Z.; Writing—original draft, M.V.-C., M.S.S., S.S., M.S.Z. and A.K.; Writing—review and editing, M.V.-C., M.S.S., S.S., M.S.Z. and A.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received external funding from Dirección de Investigción in Ponticial Catholic University of Ecuador.

Acknowledgments: The authors thank Ponticial Catholic University of Ecuador for the technical support given for this project.

Conflicts of Interest: The authors declare no conflict of interest.

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