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On Starlike Functions of Negative Order Defined by *q*-Fractional Derivative

Sadia Riaz [1], Ubaid Ahmed Nisar [2], Qin Xin [3], Sarfraz Nawaz Malik [4,*] and Abdul Raheem [1]

1 Department of Mathematics, National University of Modern Languages, Rawalpindi 46000, Pakistan; sadia.riaz@numl.edu.pk (S.R.); abdul.raheem@numl.edu.pk (A.R.)
2 Department of Mathematics, NUTECH School of Applied Sciences and Humanities (NUSASH), National University of Technology, Sector I-12, Islamabad 44000, Pakistan; ubaidahmed@nutech.edu.pk
3 Faculty of Science and Technology, University of the Faroe Islands, Vestarabryggja 15, FO 100 Torshavn, Faroe Islands; qinx@setur.fo
4 Department of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt 47040, Pakistan
* Correspondence: snmalik110@ciitwah.edu.pk or snmalik110@yahoo.com

Abstract: In this paper, two new classes of q-starlike functions in an open unit disc are defined and studied by using the q-fractional derivative. The class $\widetilde{S_q^*}(\alpha)$, $\alpha \in (-3, 1]$, $q \in (0, 1)$ generalizes the class S_q^* of q-starlike functions and the class $\widetilde{T_q^*}(\alpha)$, $\alpha \in [-1, 1]$, $q \in (0, 1)$ comprises the q-starlike univalent functions with negative coefficients. Some basic properties and the behavior of the functions in these classes are examined. The order of starlikeness in the class of convex function is investigated. It provides some interesting connections of newly defined classes with known classes. The mapping property of these classes under the family of q-Bernardi integral operator and its radius of univalence are studied. Additionally, certain coefficient inequalities, the radius of q-convexity, growth and distortion theorem, the covering theorem and some applications of fractional q-calculus for these new classes are investigated, and some interesting special cases are also included.

Keywords: analytic functions; starlike functions; q-fractional differential operator; fractional derivative; q-starlike functions; q-Bernardi operator

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1. Introduction

Quantum calculus is basically usual calculus without the notion of limits. It has wide applications in mathematics and physics. The q-derivative and q-integral are the main tools introduced by Jackson [1,2] in a systematic way. The linear q-difference equation, and q-differential equations, are studied in [3,4]. Mansour [5] investigated linear sequential q-differential equation of fractional order. Using the q-derivative, Ismail [6] introduced the class of q-starlike functions. In the recent past, the theory of q-calculus operators has been applied in general fractional calculus. Al-Salam [7] and Agarwal [8] introduced several types of fractional q-integral operators and fractional q-derivatives. Rajkovi'c [9] investigated fractional integral and derivatives in q-calculus. Additionally, q-integral operators for certain analytic functions by using the concept and theory of fractional q-calculus that was studied by Selvakumaran et al. [10]. Recently, researchers proposed q-version of well known operators like Baskakov Durrmeyer operator, Picard integral operator, Bernardi integral operator and the Gauss–Weierstrass integral operator, see [11–15]. Furthermore, Purohit and Raina [16] applied q-operators on subclasses of analytic functions. Ismail [6] introduced the well known class of q-starlike functions related to q-derivative operator [16,17]. Wingsaijai and Sukantmala [18] presented the class $S_q^*(\alpha)$ of q-starlike functions of order α, $(0 \leq \alpha < 1)$, Certain Coefficient Estimate Problems for Three-Leaf-Type Starlike Functions . Sahoo and Sharma [17] defined and studied $q-$ analogue of a close-to-convex function.

The starlikeness of normalized bessel functions with symmetric points is studied in [19]. Recently, certain generalized classes of q-starlike functions have been investigated, see [20,21]. Zainab et al. [22] defined a new class of q-starlike functions by using q-Ruscheweyh differential operator. The recent contributions on fractional derivatives by several researchers are also worth reading, see [23–25]. Sokół [26] introduced a one-parameter family of functions $\widetilde{p_\alpha}(z)$, as shown in (5). Using this family of functions, he defined a classes of starlike functions, and certain properties of these functions were investigated. However, $\widetilde{p_\alpha}(z)$ has not been studied under the q-analogue of analytic and univalent functions of negative order, which has vital applications in different zones of mathematics. This was the main motivation behind Definitions 1 and 3 and their related results, and to keep in mind recent developments on starlike functions and their associated functions, we have categorized our main results into two sections. In the first section, we have investigated some interesting properties for our new class $\widetilde{S_q^*}(\alpha)$, $q \in (0,1)$ of q-starlike functions of order α, $\alpha \in (-3,1]$, which is introduced in Definition 1. We primarily focus on q-integral representation of functions belonging to this class, and its related results. Further, we have investigated distortion bounds and order of starlikeness in class of convex functions. In the second section, we have defined the class $\widetilde{T_q^*}(\alpha)$ of q-starlike functions of order α ($\alpha \in [-1,1]$) with negative coefficients. It is investigated that functions belonging to this class are preserved under q-Bernardi integral operator and its radius of univalence is also determined. Several other properties such as coefficient inequities, radius of q-convexity, growth and distortion theorem, covering result and some applications of fractional q-calculus for the said class are presented. It is noted that obtained results are the advancement of several known results, proved by researchers in their research articles.

Let A consist of the analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E = \{z : |z| < 1\}. \tag{1}$$

Let $S \subset A$ be the class of univalent functions in $E = \{z : |z| < 1\}$. The classes $S^*(\gamma)$ of starlike functions of order γ and $C(\gamma)$ of convex functions of order γ, which are subclasses of S are defined as:

$$S^*(\gamma) = \left\{ f \in A : \Re\left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, \quad 0 \le \gamma < 1, z \in E \right\} \tag{2}$$

$$C(\gamma) = \left\{ f \in A : \Re\left\{ \frac{(zf'(z))'}{f'(z)} \right\} > \gamma, \quad 0 \le \gamma < 1, z \in E \right\} \tag{3}$$

When $\gamma = 0$, we have the well-known class S^* of starlike functions and the class C of convex functions, see [27]. Let $f_i \in A$, $i = 1, 2$. Then, we say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$, if there exists a function w, analytic in E with $w(0) = 0$ and $|w(z)| < 1$, $z \in E$, such that $f_1(z) = f_2(w(z))$. If $f_2 \in S$, it is known that the above subordination is equivalent to $f_1(0) = f_2(0)$ and $f_1(E) \subset f_2(E)$, see [27].

Let T denote the subclass of S consisting of the analytic and univalent functions, whose functions can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, z \in E. \tag{4}$$

Silverman [28] introduced and studied the classes $T^*(\gamma)$ and $K(\gamma)$ of starlike functions of order γ and convex functions of order γ, $(0 \le \gamma < 1)$ in the open unit disc $E = \{z : |z| < 1\}$. He defined these classes as follows:

$$T^*(\gamma) = \left\{ f \in T : \Re\left(\frac{zf'(z)}{f(z)} \right) > \gamma, \, 0 \le \gamma < 1, \, z \in E \right\},$$

$$K(\gamma) = \left\{ f \in T : \Re\left(\frac{(zf'(z))'}{f'(z)} \right) > \gamma,\ 0 \leq \gamma < 1,\ z \in E \right\}.$$

When $\gamma = 0$, the above classes reduce to the classes T^* and K, of starlike and convex functions of negative coefficients, respectively, see [28].

Our work is related to a one-parameter family of functions defined and studied by Sokół [26]. We recall the properties of these functions, which we shall need to derive our results.

Remark 1. *Let*

$$\widetilde{p_\alpha}(z) = \frac{1}{3 + (\alpha - 3)z + \alpha z^2} = \frac{3}{3+\alpha}\left\{ \frac{1}{1-z} + \frac{\alpha}{\alpha z + 3} \right\},\ \alpha \in (-3, 1]. \tag{5}$$

Then, the following assertions are true.

1. $\widetilde{p_\alpha}$ *is univalent in E.*
2. $\frac{9(1+\alpha)}{2(3+\alpha)^2} \leq \Re\{\widetilde{p_\alpha}(z)\} \leq \Re\left\{ \frac{-3}{(z-1)(\alpha z+3)} \right\} > \frac{3}{2(3-\alpha)}$
3. *When $\alpha \in [-1, 1]$, $\widetilde{p_\alpha}$ is convex univalent function in E.*

Now, we include some basic definitions and concepts of q-calculus, which are used in this work.

The q-derivative of a function $f \in A$ is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (z \neq 0), \tag{6}$$

and $D_q f(0) = f'(0)$, where $q \in (0, 1)$, see [2]. For a function $g(z) = z^n$, the q-derivative is

$$D_q g(z) = [n]_q z^{n-1}, \tag{7}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q}. \tag{8}$$

We note that as $q \to 1^-$, $D_q f(z) \to f'(z)$, here $f'(z)$ is ordinary derivative and $[n]_q \to n$ as $q \to 1^-$. From (4), one can deduce that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^n. \tag{9}$$

Jackson [1] introduced the q-integral of a function f, which is given by

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(q^n z), \tag{10}$$

provided that series converges.

In [18], Wongsaijai and Sukantamala introduce the class $S_q^*(\gamma)$ of q-starlike functions of order γ as follows:

$$S_q^*(\gamma) = \left\{ f \in A : \Re\left\{ \frac{zD_q f(z)}{f(z)} \right\} > \gamma, \quad 0 \leq \gamma < 1, z \in E \right\}. \tag{11}$$

The corresponding class $C_q(\gamma)$ of q-convex functions is defined as

$$C_q(\gamma) = \left\{ f \in A : \Re\left\{ \frac{D_q(zD_qf(z))}{D_qf(z)} \right\} > \gamma, \quad 0 \leq \gamma < 1, z \in E \right\}. \tag{12}$$

By seting $\gamma = 0$ in above definitions, we get C_q of q-convex functions and S_q^* of q-starlike functions introduced in [6].

Then, we define a new class $\widetilde{S_q^*}(\alpha) \subset S$, which is the refinement of the above known classes of starlike functions. Results related to this class will be derived in Section 2.

Definition 1. *A function from the class A is said to be in the class $\widetilde{S_q^*}(\alpha)$ if and only if it satisfies the condition*

$$\frac{zD_qf(z)}{f(z)} \prec \widetilde{p_\alpha}(z), \quad \alpha \in (-3, 1], z \in E, \tag{13}$$

where $\widetilde{p_\alpha}$ is given by (5).

From the Remark 1, we have

$$\Re\left\{ \frac{zD_qf(z)}{f(z)} \right\} > \frac{9(1+\alpha)}{2(3+\alpha)^2}, \ (\alpha \in (-3, 1], \ z \in E), \tag{14}$$

when $f \in \widetilde{S_q^*}(\alpha)$.

Our aim is to investigate geometric properties of class $\widetilde{S_q^*}(\alpha)$ of q-starlike functions of order α. It deals with several ideas and techniques used in geometric function theory. The order of starlikeness in the class of convex functions of negative order and distortion bounds is also formulated. It provides an interesting connection of our above-defined class with well known classes in the form of following special cases.

Special Cases

1. When $\alpha = -2$, we have

$$\Re\left\{ \frac{zD_qf(z)}{f(z)} \right\} > \frac{-9}{2}, \ (z \in E).$$

2. For $\alpha = -\frac{3}{2}$, we obtain

$$\Re\left\{ \frac{zD_qf(z)}{f(z)} \right\} > -1, \ (z \in E).$$

3. Let $\alpha \in [-1, 1]$, and taking $q \to 1^-$, we get the class of starlike functions, which is univalent in E, see [27].
4. If $\alpha = -1$, then f belongs to the class $\widetilde{S_q^*}(-1) \subset S_q^*$ of q-starlike functions, which is defined and studied in [6].
5. For $\alpha = 0$, we have the known class $\widetilde{S_q^*}(0) \subset S_q^*(\frac{1}{2})$ of q-starlike functions of order $\frac{1}{2}$, see [27].
6. When $\alpha = 1$, then f belongs to the class $\widetilde{S_q^*}(1) \subset S_q^*(\frac{9}{16})$ of starlike function with order $\frac{9}{16}$.

Now we define another class $S_q^*[M]$, a subclass of S_q^*. This class will be used in derivation of Theorems 1 and 8.

Definition 2. *For $-1 \leq M \leq 1$, $M \neq 0$, the class $S_q^*[M]$ is defined as follows.*

$$S_q^*[M] = \left\{ f \in A : \frac{zD_qf(z)}{f(z)} \prec \frac{1}{1+Mz} \right\}, \tag{15}$$

We note that, for $M = 1$, then the function $p(z) = \frac{1}{1+Mz}$ maps the unit disc E onto half plane $\Re(w) > \frac{1}{2}$ and onto the disc with center $\frac{1}{1-M^2}$ and radius $\frac{|M|}{1-M^2}$, for $M \neq 0$.

Next, we define the class $\widetilde{T_q^*}(\alpha) \subset T$ of q-starlike functions of order α with negative coefficients. Results regarding this class are presented in Section 3.

Definition 3. *A function from the class T is said to be in the class $\widetilde{T_q^*}(\alpha)$ if and only if it satisfies the condition*

$$\Re\left\{\frac{zD_qf(z)}{f(z)}\right\} > \frac{9(1+\alpha)}{2(3+\alpha)^2}, \ (\alpha \in [-1,1], \ z \in E). \tag{16}$$

Special Cases

1. If $\alpha = -1$, we have

$$T_q^* = \left\{f \in A : \Re\left\{\frac{zD_qf(z)}{f(z)}\right\} > 0, \ z \in E\right\},$$

which contains q-starlike functions with negative coefficients, and taking $q \to 1^-$, we obtain the known class T^* introduced in [28].

2. For $\alpha = 0$, we have

$$T_q^*(0) = \left\{f \in A : \Re\left\{\frac{zD_qf(z)}{f(z)}\right\} > \frac{1}{2}, \ z \in E\right\},$$

of q-starlike functions of order $\frac{1}{2}$ with negative coefficients, and taking $q \to 1^-$, we obtain the known class K of convex functions defined and studied in [28].

3. When $\alpha = 1$, we obtain

$$T_q^*(1) = \left\{f \in A : \Re\left\{\frac{zD_qf(z)}{f(z)}\right\} > \frac{9}{16}, \ z \in E\right\},$$

of starlike function of order $\frac{9}{16}$ with negative coefficients.

Next, we define the corresponding class $\widetilde{C_q}(\alpha)$ of q-convex functions of order α and having negative coefficients. An application of this class will be shown in investigating the radius problem as given in Theorem 10.

Definition 4. *A function from the class T is said to be in the class $\widetilde{C_q}(\alpha)$ if and only if it satisfies the condition*

$$\Re\left\{\frac{D_q(zD_qf(z))}{D_qf(z)}\right\} > \frac{9(1+\alpha)}{2(3+\alpha)^2}, \ (\alpha \in [-1,1], \ z \in E). \tag{17}$$

We require following lemma to obtain our main results.

Lemma 1 (q-Jack's Lemma, [29])**.** *Let $\phi(z)$ be analytic in E with $\phi(0) = 0$. Then, if $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_o \in E$, then we have*

$$z_oD_q\phi(z_o) = m\phi(z_o),$$

$m \geq 1$ real number.

2. The Class $\widetilde{S_q^*}(\alpha)$

In this section, we obtain some results related to newly defined class $\widetilde{S_q^*}(\alpha)$ of q-starlike functions of order α. For the following results, we consider $\alpha \in (-3,1]$, $q \in (0,1)$, $z \in E$, unless otherwise stated. To prove our main results, we first prove the following lemma.

Lemma 2. *Let h be analytic in E with $h(0) = 1$. A function G is in the class*

$$S_q(h) = \left\{ f \in A : \frac{zD_q f(z)}{f(z)} \prec h(z) \right\} \quad z \in E, \tag{18}$$

if and only if there exists an analytic function p_q, $p_q \prec h$, such that

$$G(z) = z\left(exp \int_0^z \frac{p_q(t) - 1}{t} d_q t \right)^{\frac{lnq}{1-q}}, \quad z \in E. \tag{19}$$

Proof. Consider

$$p_q(z) = \frac{zD_q G(z)}{G(z)}, \tag{20}$$

where p_q is analytic and $p_q(0) = 1$ in E. Using q-integral properties, we get

$$\begin{aligned} \int_o^z \frac{p_q(t) - 1}{t} d_q t &= \int_o^z \frac{tD_q G(t) - G(t)}{tG(t)} d_q t \\ &= \left(\frac{1-q}{lnq} \right) \log(G(z)) - \left(\frac{1-q}{lnq} \right) \log(z) \\ &= \log\left(\frac{G(z)}{z} \right)^{\frac{1-q}{lnq}}. \end{aligned}$$

It follows that

$$z\left(\exp \int_o^z \frac{p_q(t) - 1}{t} d_q t \right)^{\frac{lnq}{1-q}} = G(z),$$

which is (19). Now conversely, let (19) hold, that is

$$\frac{G(z)}{z} = \left(\exp \int_o^z \frac{p_q(t) - 1}{t} d_q t \right)^{\frac{lnq}{1-q}}, \tag{21}$$

The Logarithmic q-differentiation of (21) gives us

$$D_q(ln(G(z)) - D_q(ln(z)) = \frac{lnq}{1-q} D_q \int_o^z \frac{p_q(t) - 1}{t} d_q t.$$

Using the formulation $D_q(lnf(z)) = \left(\frac{lnq}{1-q} \right) \left(\frac{D_q f(z)}{f(z)} \right)$, and the fundamental theorem of q-calculus, see [30], we get

$$\frac{lnq}{1-q} \left(\frac{D_q G(z)}{G(z)} \right) - \frac{lnq}{1-q} \left(\frac{1}{z} \right) = \frac{lnq}{1-q} \left(\frac{p_q(z) - 1}{z} \right),$$

which implies that

$$\frac{zD_q G(z)}{G(z)} = p_q(z);$$

it follows that $p_q \prec h$, and this implies that $G \in S_q(h)$ in E. This completes the proof. □

By taking $q \to 1^-$ in Lemma 2, we obtain the known result proved by Sokół [26].

Then, by using the class $S_q^*[M]$ given by (15), we derive the following theorem for the function $f \in \widetilde{S_q^*}(\alpha)$, $\alpha \in (-3, 1]$.

Theorem 1. *Let $f \in A$, $\alpha \in (-3,1]\backslash\{0\}$. If $f \in \widetilde{S_q^*}(\alpha)$, then there exists a function $F_1 \in S_q^*(\frac{1}{2})$ and the function $F_2 \in S_q^*\left[\frac{\alpha}{3}\right]$, such that*

$$f(z) = [F_1(z)]^{\frac{3}{3+\alpha}}[F_2(z)]^{\frac{\alpha}{3+\alpha}}, \quad z \in E.$$

(We note that, if $\alpha = 0$; then, $\widetilde{S_q^*}(0) = S_q^*(\frac{1}{2})$).

Proof. Let $f \in \widetilde{S_q^*}(\alpha)$. Then, by Lemma 2, there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in E$, such that

$$f(z) = z\left(exp\int_0^z \frac{\widetilde{p_\alpha}(w(t)) - 1}{t} d_q t\right)^{\frac{lnq}{1-q}}, \tag{22}$$

and from (5), we have

$$f(z) = z\left(exp\int_0^z \frac{\frac{3}{3+\alpha}\left\{\frac{1}{1-w(t)} + \frac{\alpha}{\alpha w(t)+3}\right\} - 1}{t} d_q t\right)^{\frac{lnq}{1-q}},$$

which implies that

$$f(z) = z\left(exp\int_0^z \frac{\frac{3}{3+\alpha}\left(\frac{1}{1-w(t)}\right) + \frac{\alpha}{3+\alpha}\left(\frac{1}{1+\frac{\alpha}{3}w(t)}\right) - 1}{t} d_q t\right)^{\frac{lnq}{1-q}}.$$

It follows that

$$f(z) = z\left(exp\int_0^z \frac{\frac{3}{3+\alpha}\left(\frac{1}{1-w(t)}\right) + \frac{\alpha}{3+\alpha}\left(\frac{1}{1+\frac{\alpha}{3}w(t)}\right) - 1}{t} d_q t\right)^{\frac{lnq}{1-q}}.$$

This implies that

$$f(z) = \left(z\, exp\int_0^z \frac{\left[\frac{1}{1-w(t)} - 1\right]}{t} d_q t\right)^{\left(\frac{3}{3+\alpha}\right)\left(\frac{lnq}{1-q}\right)} \times \left(z\, exp\int_0^z \frac{\left[\frac{1}{1+\frac{\alpha}{3}w(t)} - 1\right]}{t} d_q t\right)^{\left(\frac{\alpha}{3+\alpha}\right)\left(\frac{lnq}{1-q}\right)}.$$

Using the Lemma 2, we have

$$f(z) = [F_1(z)]^{\frac{3}{3+\alpha}}[F_2(z)]^{\frac{\alpha}{3+\alpha}},$$

which shows that the functions F_1 and F_2 satisfy $F_1 \in S_q^*\left(\frac{1}{2}\right)$ and $F_2 \in S_q^*\left[\frac{\alpha}{3}\right]$, and this completes the proof. □

Theorem 2. *Let $f \in A$, $\alpha \in (-3,1]$. If there exists a function $F_1 \in S_q^*\left(\frac{1}{2}\right)$ and $F_2 \in S_q^*\left[\frac{\alpha}{3}\right]$, such that*

$$\frac{zD_qF_1(z)}{F_1(z)} = \frac{1}{1-w(z)}, \tag{23}$$

and

$$\frac{zD_qF_2(z)}{F_2(z)} = \frac{1}{1+\frac{\alpha}{3}w(z)}, \quad z \in E, \tag{24}$$

for analytic function $w(0) = 0$, *and* $|w(z)| < 1, z \in E$, *then the function*

$$f(z) = [F_1(z)]^{\frac{3}{3+\alpha}}[F_2(z)]^{\frac{\alpha}{3+\alpha}} \in \widetilde{S_q^*}(\alpha), \quad z \in E. \tag{25}$$

Proof. From (23) and (24), we find that $F_1 \in S_q^*\left(\frac{1}{2}\right)$ and $F_2 \in S_q^*\left[\frac{\alpha}{3}\right]$ are generated by (19) wih the same function w, so by using Theorem 1 and Lemma 2, we have

$$[F_1(z)]^{\frac{3}{3+\alpha}}[F_2(z)]^{\frac{\alpha}{3+\alpha}} = z\left(exp\int_0^z \frac{\widetilde{p_\alpha}(w(t))-1}{t}d_qt\right)^{\frac{lnq}{1-q}}.$$

Hence, we have

$$f(z) = [F_1(z)]^{\frac{3}{3+\alpha}}[F_2(z)]^{\frac{\alpha}{3+\alpha}} \in \widetilde{S_q^*}(\alpha),$$

which is the required result. □

Next, we obtain distortion result for our class $\widetilde{S_q^*}(\alpha)$, $\alpha \in (-3,1]$, by using Theorems 1 and 2.

Theorem 3. *If* $f \in \widetilde{S_q^*}(\alpha)$, $\alpha \in (-3,1]$, *and* $|z| < r, 0 \le r < 1$, *then*

$$\left[\left(\frac{r}{1+r}\right)^{\frac{3}{1+\alpha}}\left(\frac{r}{1+\frac{\alpha r}{3}}\right)^{\frac{\alpha}{3+\alpha}}\right]^{\frac{1-q}{\log q^{-1}}} \le |f(z)| \le \left[\left(\frac{r}{1-r}\right)^{\frac{3}{3+\alpha}}\left(\frac{r}{1-\frac{\alpha r}{3}}\right)^{\frac{\alpha}{3+\alpha}}\right]^{\frac{1-q}{\log q^{-1}}}$$

Proof. Let $f \in \widetilde{S_q^*}(\alpha)$. Then, by Theorem 1, there exist $F_1 \in S_q^*\left(\frac{1}{2}\right)$ and $F_2 \in S^*\left[\frac{\alpha}{3}\right]$, such that (25) holds.
Let $F_1 \in {S^*}_q\left[\frac{\alpha}{3}\right]$. Then, we have

$$\frac{zD_qF_1(z)}{F_1(z)} \prec \frac{1}{1-z},$$

which implies that

$$\left|\frac{zD_qF_1(z)}{F_1(z)} - \frac{1}{1-r^2}\right| < \frac{r}{1-r^2}.$$

Using q-differential properties and partial q-derivatives, we get

$$\frac{1}{r(1+r)} \le \frac{\partial_q}{\partial_r}\log|F_1(re^{i\theta})| \le \frac{1}{r(1-r)}.$$

The q-integration gives us

$$\left(\frac{r}{1+r}\right)^{\frac{1-q}{\log q^{-1}}} \le |F_1(z)| \le \left(\frac{r}{1-r}\right)^{\frac{1-q}{\log q^{-1}}}. \tag{26}$$

Raising (26) to power $\frac{3}{1+\alpha}$, we get

$$\left(\frac{r}{1+r}\right)^{\left(\frac{1-q}{\log q^{-1}}\right)\left(\frac{3}{1+\alpha}\right)} \leq |F_1(z)|^{\frac{3}{1+\alpha}} \leq \left(\frac{r}{1-r}\right)^{\left(\frac{1-q}{\log q^{-1}}\right)\left(\frac{3}{1+\alpha}\right)}. \tag{27}$$

Now, we suppose that $F_2 \in {S^*}_q\left[\frac{\alpha}{3}\right]$, so we can write

$$\frac{zD_qF_2(z)}{F_2(z)} \prec \frac{1}{1+\left[\frac{\alpha}{3}\right]z}.$$

It follows that

$$\left|\frac{zD_qF_2(z)}{F_2(z)} - \frac{1}{1-\left(\frac{\alpha}{3}\right)^2 r^2}\right| \leq \frac{\left|\frac{\alpha}{3}\right| r}{1-\left(\frac{\alpha}{3}\right)^2 r^2},$$

as the linear transformation $\frac{1}{1+\left[\frac{\alpha}{3}\right]z}$ maps $|z| = r$ onto disc of center $\frac{1}{1-\left(\frac{\alpha}{3}\right)^2 r^2}$ and radius $\frac{\left|\frac{\alpha}{3}\right| r}{1-\left(\frac{\alpha}{3}\right)^2 r^2}$. Additionally, we know that

$$\Re\left[\frac{zD_qF_2(z)}{F_2(z)}\right] = r\frac{\partial_q}{\partial_r}\log\left|F_2(re^{i\theta})\right|,$$

so, we have

$$\frac{1}{r\left(1-\frac{\alpha}{3}r\right)} \leq \frac{\partial_q}{\partial_r}\log\left|F_2(re^{i\theta})\right| \leq \frac{1}{r\left(1+\frac{\alpha}{3}r\right)}.$$

The q-Integration on both sides gives us

$$\left(\frac{r}{1-\frac{\alpha r}{3}}\right)^{\frac{1-q}{\log q^{-1}}} \leq |F_2(z)| \leq \left(\frac{r}{1+\frac{\alpha r}{3}}\right)^{\frac{1-q}{\log q^{-1}}}. \tag{28}$$

Raising (28) to the power $\frac{\alpha}{3+\alpha}$, we get

$$\left(\frac{r}{1+\frac{\alpha r}{3}}\right)^{\left(\frac{\alpha}{3+\alpha}\right)\left(\frac{1-q}{\log q^{-1}}\right)} \leq |F_2(z)|^{\frac{\alpha}{3+\alpha}} \leq \left(\frac{r}{1-\frac{\alpha r}{3}}\right)^{\left(\frac{\alpha}{3+\alpha}\right)\left(\frac{1-q}{\log q^{-1}}\right)}, \tag{29}$$

because $\left(\frac{\alpha}{3+\alpha}\right) < 0$, when $\alpha \in (-3, 0]$. Multiplying (29) and (27), we obtain our required result. □

Next, we will obtain the order of starlikeness in the class of convex functions.

Theorem 4. *Let $f \in A$, and let $\beta = \frac{9(1+\alpha)}{2(3+\alpha)^2}$, for $\alpha \in [-1, 1]$,*

$$\Re\left\{\frac{D_q(zD_qf(z))}{D_qf(z)}\right\} > \beta - \frac{1}{2} - \frac{\beta(1-2\beta)}{2(1-\beta)^2}. \tag{30}$$

Then,

$$\Re\left\{\frac{zD_qf(z)}{f(z)}\right\} > \beta, \quad z \in E.$$

Proof. Consider

$$\frac{zD_qf(z)}{f(z)} = \frac{1+(1-2\beta)\phi(z)}{(1-\phi(z))}, \tag{31}$$

where ϕ is analytic with $\phi(0) = 1$ in E. The q-logarithmic differentiation of (31) gives us

$$\left(\frac{lnq}{1-q}\right)\left\{\frac{D_q(zD_qf(z))}{D_qf(z)} - \frac{zD_qf(z)}{f(z)}\right\} = \left\{\frac{(1-2\beta)zD_q\phi(z)}{1+(1-2\beta)\phi(z)} + \frac{zD_q\phi(z)}{1-\phi(z)}\right\}\left(\frac{lnq}{1-q}\right).$$

On contrary, we assume $z_o \in E$, such that $m \geq 1$, $|\phi(z_o)| = 1$ and $\phi(z_o) = e^{i\theta}$, $z_oD_q\phi(z_o) = m\phi(z_o)$; thus, we have

$$\Re\left\{\frac{D_q(z_oD_qf(z_o))}{D_qf(z_o)}\right\} = \Re\left\{\frac{z_oD_qf(z_o)}{f(z_o)} + \frac{(1-2\beta)z_oD_q\phi(z_o)}{1+(1-2\beta)\phi(z_o)} + \frac{zD_q\phi(z_o)}{1-\phi(z_o)}\right\}.$$

Using (31), we have

$$\begin{aligned}
\Re\left\{\frac{D_q(z_oD_qf(z_o)}{D_qf(z_o)}\right\} &= \Re\left\{\frac{1+(1-2\beta)\phi(z_o)}{1-\phi(z_o)}\right\} + \\
&\quad \Re\left\{\frac{(1-2\beta)z_oD_q\phi(z_o)}{1+(1-2\beta)\phi(z_o)} + \frac{z_oD_q\phi(z_o)}{1-\phi(z_o)}\right\} \\
&= \Re\left\{\frac{1+(1-2\beta)e^{i\theta}}{1-e^{i\theta}}\right\} + \\
&\quad \Re\left\{\frac{(1-2\beta)me^{i\theta}}{1+(1-2\beta)e^{i\theta}} + \frac{me^{i\theta}}{1-e^{i\theta})}\right\} \\
&= \frac{1-\beta_1-cos\theta(1-\beta_1)}{2(1-2cos\theta)} + \frac{m\beta_1cos\theta+m\beta_1^2}{1-2\beta_1\cos\theta+\beta_1{}^2} - \frac{m}{2},
\end{aligned}$$

where $\beta_1 = 1-2\beta$. If $\theta = \pi$, we have

$$\Re\left\{\frac{D_q(z_oD_qf(z_o))}{D_qf(z_o)}\right\} = \frac{1-\beta_1}{2} + \frac{m\beta_1(\beta_1-1)}{(1+\beta_1)^2} - \frac{m}{2}.$$

Re-substituting $\beta_1 = 1-2\beta$, and since $m \geq 1$, so we have

$$\Re\left\{\frac{D_q(z_oD_q(f(z_o))}{D_qf(z_o)}\right\} \leq \beta - \frac{\beta(1-2\beta)}{2(1-\beta)^2} - \frac{1}{2}.$$

where $\beta = \frac{9(1+\alpha)}{2(3+\alpha)^2}$, $\quad \alpha \in [-1,1]$, which is contraction to our given hypothesis. Thus, the required result follows. □

We note that by substituting various values to the parameters involved in above result, we get known and new results, as shown in the following corollaries.

Corollary 1. *Let $f \in A$, and $\alpha = 0$. Then, if*

$$\Re\left\{\frac{D_q(zD_qf(z)}{D_qf(z))}\right\} > 0,$$

then

$$\Re\left\{\frac{zD_qf(z)}{f(z)}\right\} > \frac{1}{2}.$$

By further taking $q \to 1^-$, we obtain the well known result that a convex function of order zero is starlike of order one-half, see [27].

Corollary 2. *Let $f \in A$, and $\alpha = -1$. Then,*

$$\Re\left\{\frac{D_q(zD_qf(z)}{D_qf(z))}\right\} > \frac{-1}{2},$$

implies that

$$\Re\left\{\frac{zD_qf(z)}{f(z)}\right\} > 0, z \in E.$$

Corollary 3. *Let $f \in A$ and $\alpha = 1$. Then, if*

$$\Re\left\{\frac{D_q(zD_qf(z)}{D_qf(z))}\right\} > 0.0692,$$

then

$$\Re\left\{\frac{zD_qf(z)}{f(z)}\right\} > \frac{9}{16}.$$

The following theorem shows the coefficients inequality for the functions of the class $\widetilde{S_q^*}(\alpha)$.

Theorem 5. *Let $f \in A$, $\alpha \in [-1,1]$. If*

$$\sum_{n=2}^{\infty}\left([n]_q - \frac{9(1+\alpha)}{2(3+\alpha)^2}\right)|a_n| \leq 1 - \frac{9(1+\alpha)}{2(3+\alpha)^2}, \tag{32}$$

where $[n]_q$ is given by (8), then $f \in \widetilde{S_q^*}(\alpha)$.

Proof. It is sufficient to prove that the values for $\frac{zD_qf}{f}$ lie in a circle centered at 1 and radius $1 - \frac{9(1+\alpha)}{2(3+\alpha)^2}$. For this, consider

$$\begin{aligned}\left|\frac{zD_qf(z)}{f(z)} - 1\right| &= \left|\frac{zD_qf(z) - f(z)}{f(z)}\right| \qquad (33)\\ &= \left|\frac{\sum_{n=2}^{\infty}\left([n]_q - 1\right)a_n z^n}{z + \sum_{n=2}^{\infty} a_n z^n}\right| \\ &\leq \frac{\sum_{n=2}^{\infty}\left([n]_q - 1\right)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n||z|^{n-1}} \\ &\leq \frac{\sum_{n=2}^{\infty}\left([n]_q - 1\right)|a_n|}{1 - \sum_{n=2}^{\infty}|a_n|}. \qquad (34)\end{aligned}$$

As (34) is bounded by $\left(1 - \frac{9(1+\alpha)}{2(3+\alpha)^2}\right)$ if

$$\sum_{n=2}^{\infty}\left([n]_q - 1\right)|a_n| \leq \left(1 - \frac{9(1+\alpha)}{2(3+\alpha)^2}\right)\left(1 - \sum_{n=2}^{\infty}|a_n|\right),$$

which is equivalent to

$$\sum_{n=2}^{\infty}\left([n]_q - \frac{9(1+\alpha)}{2(3+\alpha)^2}\right)|a_n| \leq 1 - \frac{9(1+\alpha)}{2(3+\alpha)^2}. \tag{35}$$

However, (35) is true by hypothesis. Thus, we have $\left|\frac{zD_q f(z)}{f(z)} - 1\right| \leq 1 - \frac{9(1+\alpha)}{2(3+\alpha)^2}$, and this gives us the required result. □

Taking $q \to 1^-$ and $\alpha = -1$ in Theorem 5, we get the following known result.

Corollary 4 ([31]). *Let $f \in A$, $\alpha = -1$. Then, if*

$$\sum_{n=2}^{\infty} n|a_n| \leq 1, \tag{36}$$

then $f \in S^*$.

Taking $q \to 1^-$, and $\alpha = 0$ in Theorem 5, we obtain a result proved by Schild [32], as shown in the following corollary.

Corollary 5 ([32]). *Let $f \in A$. Then, if*

$$\sum_{n=2}^{\infty} \left(n - \frac{1}{2}\right)|a_n| \leq \frac{1}{2}, \tag{37}$$

then $f \in S^*(\frac{1}{2}) \subset C$.

Set $\alpha = -1$ in Theorem 5, this gives us the following result.

Corollary 6. *Let $f \in A$. Then, if*

$$\sum_{n=2}^{\infty} n|a_n| \leq 1, \tag{38}$$

then $f \in S_q^*$.

3. The Class $\widetilde{T_q^*}(\alpha)$

In this section, we shall study the properties of the class $\widetilde{T_q^*}(\alpha)$ shown in Definition 3. For the following results, we have $\alpha \in [-1, 1]$, $q \in (0, 1)$, $z \in E$ unless otherwise stated.

3.1. Coefficient Inequalities

Coefficient inequalities for functions belong to the class $\widetilde{T_q^*}(\alpha)$ are derived in following theorem.

Theorem 6. *Let f be given by (4). Then, $f \in \widetilde{T_q^*}(\alpha)$, if and only if*

$$\sum_{n=2}^{\infty} \left(\frac{1-q^n}{1-q} - \frac{9(1+\alpha)}{2(3+\alpha)^2}\right)|a_n| < 1 - \frac{9(1+\alpha)}{2(3+\alpha)^2}, \quad z \in E. \tag{39}$$

Proof. In view of Theorem 5, it is sufficient to prove the only if part. Let $f \in \widetilde{T_q^*}(\alpha)$, that is,

$$\Re\left\{\frac{zD_q f(z)}{f(z)}\right\} = \Re\left\{\frac{z - \sum_{n=2}^{\infty}[n]_q|a_n|z^n}{z - \sum_{n=2}^{\infty}|a_n|z^n}\right\} > \frac{9(1+\alpha)}{2(3+\alpha)^2}. \tag{40}$$

Choose values of z on real axis so that $\frac{zD_q f(z)}{f(z)}$ is real. Upon clearing the denominator in (40), and letting $z \to 1$ through real values, we have

$$1 - \sum_{n=2}^{\infty} [n]_q |a_n| \geq \frac{9(1+\alpha)}{2(3+\alpha)^2} \left(1 - \sum_{n=2}^{\infty} |a_n| \right). \tag{41}$$

By (8), we have

$$\sum_{n=2}^{\infty} \left(\frac{1-q^n}{1-q} - \frac{9(1+\alpha)}{2(3+\alpha)^2} \right) |a_n| \leq 1 - \frac{9(1+\alpha)}{2(3+\alpha)^2},$$

and this completes the proof. □

Corollary 7. *Let $f \in T$. Then, if $f \in \widetilde{T_q^*}(\alpha)$, then*

$$|a_n| \leq \frac{(2\alpha^2 + 3\alpha + 9)(1-q)}{2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q)}. \tag{42}$$

This result is sharp for the extremal function of the form

$$f_n(z) = z - \frac{(2\alpha^2 + 3\alpha + 9)(1-q)}{2(1-q^n)(3+\alpha)^2 - 9(1-q)(1+\alpha)} z^n. \tag{43}$$

3.2. Distortion Theorems

The growth and distortion theorems for the functions in the class $\widetilde{T_q^*}(\alpha)$, for $\alpha \in [-1, 1]$.

Theorem 7. *Let $f \in T$. If $f \in \widetilde{T_q^*}(\alpha)$, then*

$$r - \frac{2\alpha^2 + 3\alpha + 9}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} r^2 \leq |f(z)| \leq r + \frac{2\alpha^2 + 3\alpha + 9}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} r^2 \; (|z| = \pm r).$$

Equality holds for the extremal function $g_0(z)$, given as

$$g_0(z) = z - \frac{2\alpha^2 + 3\alpha + 9}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} z^2 \quad (|z| = \pm r). \tag{44}$$

Proof. From Theorem 6, we have

$$\left(\frac{1-q^2}{1-q} - \frac{9(1+\alpha)}{2(3+\alpha)^2} \right) \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \left(\frac{1-q^n}{1-q} - \frac{9(1+\alpha)}{2(3+\alpha)^2} \right) |a_n| \leq 1 - \frac{9(1+\alpha)}{2(3+\alpha)^2}.$$

That is,

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{2\alpha^2 + 3\alpha + 9}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)}. \tag{45}$$

Consider

$$\begin{aligned} |f(z)| &\leq r + \sum_{n=2}^{\infty} |a_n| r^n \\ &\leq r + r^2 \sum_{n=2}^{\infty} |a_n| \\ &\leq r + \frac{2\alpha^2 + 3\alpha + 9}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} r^2, \end{aligned} \tag{46}$$

by (45). Similarly, we have

$$
\begin{aligned}
|f(z)| &\geq r - \sum_{n=2}^{\infty} |a_n| r^n \\
&\geq r - r^2 \sum_{n=2}^{\infty} |a_n| \\
&\geq r - \frac{2\alpha^2 + 3\alpha + 9}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} r^2.
\end{aligned} \tag{47}
$$

From (46) and (47), we get our desired result. □

Theorem 8. *Let $f \in T$. If $f \in \widetilde{T_q^*}(\alpha)$, then*

$$
1 - \frac{2(2\alpha^2 + 3\alpha + 9)r}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} \leq |D_q f(z)| \leq 1 + \frac{2(2\alpha^2 + 3\alpha + 9)r}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)}, (|z| = \pm r).
$$

Equality holds for the function $g_0(z)$ given by (44).

Proof. Consider

$$
\begin{aligned}
|D_q f(z)| &\leq 1 + \sum_{n=2}^{\infty} [n]_q |a_n| |z|^{n-1} \\
&\leq 1 + r \sum_{n=2}^{\infty} [n]_q |a_n|.
\end{aligned} \tag{48}
$$

From Theorem 6, we have

$$
\begin{aligned}
\left| \sum_{n=2}^{\infty} [n]_q |a_n| \right| &\leq 1 - \frac{9(1+\alpha)}{2(3+\alpha)^2} + \frac{9(1+\alpha)}{2(3+\alpha)^2} \left(\frac{(2\alpha^2 + 3\alpha + 9)}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} \right) \\
&\leq \frac{2(2\alpha^2 + 3\alpha + 9)}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)}.
\end{aligned} \tag{49}
$$

By substituting (49) in (48), we get the right hand side of required inequality.

Similarly, we have

$$
\begin{aligned}
|D_q f(z)| &\geq 1 - \sum_{n=2}^{\infty} [n]_q |a_n| |z|^{n-1} \\
&\geq 1 - r \sum_{n=2}^{\infty} [n]_q |a_n| \\
&\geq 1 - \frac{2(2\alpha^2 + 3\alpha + 9)}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} r.
\end{aligned} \tag{50}
$$

This completes the proof. □

Setting $q \to 1^-$ and $\alpha \in [-1, 1]$ in Theorems 7 and 8, we get the results, derived by Silverman [28].

3.3. Covering Results

Following is the covering result deduced by letting $r \to 1$ in Theorem 7.

Theorem 9. *Let Let $f \in T$, and let $f \in \widetilde{T_q^*}(\alpha)$. Then, $f(E)$ contains an open unit disc of radius*

$$\frac{2q(3+\alpha)^2}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)}.$$

Equality holds for the function $g_0(z)$ given by (44).

3.4. Radius of q-Convexity for $\widetilde{T_q}^*(\alpha)$

Now, we investigate the radius of q-convexity for functions in class $\widetilde{T_q^*}(\alpha)$.

Theorem 10. *Let $f \in T$ and let $f \in \widetilde{T_q^*}(\alpha)$. Then, f is q-convex in the disk*

$$|z| < r = r(\alpha) =^{\inf}_{n} \left(\frac{2(1-q^n)(1-q)(3+\alpha)^2 - 9(1+\alpha)(1-q)^2}{(1-q^n)^2(2\alpha^2+3\alpha+9)} \right)^{\frac{1}{n-1}} \tag{51}$$

This result is sharp. Extremal function $g_0(z)$ is given by (44).

Proof. We are required to show that $|\frac{D_q(zD_qf(z))}{D_qf(z)}| \leq 1$ for $|z| < r(\alpha)$, we have

$$\begin{aligned} \left| \frac{D_q(zD_qf(z)}{D_qf(z)} \right| &= \left| \frac{-\sum_{n=2}^{\infty} a_n[n]_q([n]_q-1)z^{n-1}}{1-\sum_{n=2}^{\infty} a_n[n]_q z^{n-1}} \right| \\ &= \frac{\sum_{n=2}^{\infty}[n]_q([n]_q-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}|a_n|[n]_q|z|^{n-1}}. \end{aligned} \tag{52}$$

The left side of expression given in (52) is bounded above by 1 if

$$\sum_{n=2}^{\infty}[n]_q([n]_q-1)|a_n||z|^{n-1} \leq 1 - \sum_{n=2}^{\infty}[n]_q|a_n||z|^{n-1},$$

or

$$\sum_{n=2}^{\infty}([n]_q)^2|a_n||z|^{n-1} \leq 1,$$

which will be true, if by Theorem 6,

$$([n]_q)^2|z|^{n-1} \leq \frac{2[n]_q(3+\alpha)^2 - 9(1+\alpha)}{(2\alpha^2+3\alpha+9)}, \quad (n = 2, 3, \ldots).$$

It follows that

$$|z| \leq \left(\frac{2(1-q^n)(1-q)(3+\alpha)^2 - 9(1+\alpha)(1-q)^2}{(1-q^n)^2(2\alpha^2+3\alpha+9)} \right)^{\frac{1}{n-1}} \quad (n = 2, 3, \ldots).$$

Set $|z| = r(\alpha)$, we have

$$r(\alpha) \leq \left(\frac{2(1-q^n)(1-q)(3+\alpha)^2 - 9(1+\alpha)(1-q)^2}{(1-q^n)^2(2\alpha^2+3\alpha+9)} \right)^{\frac{1}{n-1}} \quad (n = 2, 3, \ldots), \tag{53}$$

which is the required result. □

Taking $q \to 1^-$ and $\alpha \in [-1, 1]$ in Theorem 10, we obtain the known result, see [28].

3.5. Integral Operators

In [15], q-Bernardi operator is defined as:
Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and let $c > -1$, $q \in (0,1)$, $z \in E$. Then,

$$F(z) = \frac{[c+1]_q}{z^c} \int_0^z t^{c-1} f(t) d_q t = \sum_{n=2}^{\infty} \frac{1-q^{1+c}}{1-q^{c+n}} a_n. \tag{54}$$

We note that $F(z)$ is well defined. Next, we prove that the class $\widetilde{T_q^*}(\alpha)$ is closed under q-Bernardi operator $F(z)$ given by (54) for $f \in T$, and f is defined by (4), and we also discuss the converse case by investigating the radius of univalence.

Theorem 11. *Let $f \in T$ be the function defined by (4), and let $f \in \widetilde{T_q^*}(\alpha)$ and c be real such that $c > -1$. Then, the function $F(z)$ defined by (54) belongs to the class $\widetilde{T_q^*}(\alpha)$.*

Proof. From (54), we have

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n,$$

where

$$b_n = \frac{1-q^{1+c}}{1-q^{c+n}} a_n, \quad (a_n \geq 0). \tag{55}$$

Consider

$$\begin{aligned} \sum_{n=2}^{\infty} \left([n]_q - \frac{9(1+\alpha)}{2(3+\alpha)^2} \right) b_n &= \sum_{n=2}^{\infty} \left([n]_q - \frac{9(1+\alpha)}{2(3+\alpha)^2} \right) \left(\frac{1-q^{1+c}}{1-q^{c+n}} \right) a_n \\ &\leq \sum_{n=2}^{\infty} \left([n]_q - \frac{9(1+\alpha)}{2(3+\alpha)^2} \right) a_n \\ &\leq 1 - \frac{9(1+\alpha)}{2(3+\alpha)^2}, \end{aligned} \tag{56}$$

as $f \in \widetilde{T_q^*}(\alpha)$. Therefore, an application of Theorem 6 leads us to the fact that $F \in \widetilde{T_q^*}(\alpha)$, which is the required result. □

Next, we will investigate the radius problem for the function $F(z)$ given in (54). For this purpose, we prove the following lemma.

Lemma 3. *(q-Noshiro–Warchowsky theorem) Let $f \in T$. If for all z belongs to a convex domain D and some real α, $\Re\{D_q f(z)\} > 0$, , then f is said to be a univalent function in E.*

Proof. Let $z_1 \neq z_2 \in D$. Since D is convex domain, so

$$L = \{z : z = (1-t)z_1 + tz_2,\ 0 < t < 1\},$$

L lies in D, and

$$d_q z = (z_2 - z_1) d_q t, \tag{57}$$

consider

$$\begin{aligned} f(z_2) - f(z_1) &= \int_L D_q f(z) d_q z \\ &= \int_0^1 D_q(f(z(t))(z_2 - z_1) d_q t, \end{aligned} \tag{58}$$

by using (57). We have

$$f(z_2) - f(z_1) = (z_2 - z_1)\int_0^1 D_q f(z(t))d_q t.$$

Since $\Re\{D_q f(z)\} > 0$, so

$$\int_0^1 D_q f(z(t))d_q t \neq 0,$$

and also we have $z_2 - z_1 \neq 0$; therefore, by Fundamental Theorem of q-Calculus, see [30], we have $f(z_2) - f(z_1) \neq 0$. Thus, $f(z)$ is univalent in D. □

Taking $q \to 1^-$ in above lemma, we get the well known Noshiro–Warchowsky theorem, see [27].

Theorem 12. *Let c be real number such that $c > -1$. If $F(z) \in \widetilde{T_q^*}(\alpha)$; then, f is defined by (4) is univalent in $|z| < R$, where*

$$R = \inf\left(\frac{(1-q^{c+1})(2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q))}{(1-q^n)(1-q^{n+c})(2\alpha^2+3\alpha+9)}\right)^{\frac{1}{n-1}},\ (n \geq 2). \tag{59}$$

This result is sharp.

Proof. Let

$$F(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{60}$$

Then, from (54), we have

$$\begin{aligned} f(z) &= \frac{z^{1-c}D_q(z^c F(z))}{[c+1]_q} \\ &= z - \sum_{n=2}^{\infty} \frac{1-q^{c+n}}{1-q^{c+1}} a_n z^n, \quad (c > -1). \end{aligned}$$

Consider

$$|D_q f(z) - 1| = \sum_{n-2}^{\infty} \frac{1-q^{c+n}}{1-q^{c+1}}\left(\frac{1-q^n}{1-q}\right)|a_n||z|^{n-1} < 1. \tag{61}$$

From Theorem 6, we have

$$\frac{2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q)}{(2\alpha^2+3\alpha+9)(1-q)}|a_n| \leq 1.$$

The expression (61) will be satisfied if

$$\frac{(1-q^{c+n})(1-q^n)}{(1-q^{c+1})(1-q)}|z|^{n-1} \leq \frac{2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q)}{(2\alpha^2+3\alpha+9)(1-q)}, \quad (n \geq 2).$$

Solving for $|z|$, we have

$$|z| \leq \left(\frac{(1-q^{c+1})(2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q))}{(1-q^n)(1-q^{n+c})(2\alpha^2+3\alpha+9)}\right)^{\frac{1}{n-1}}, \quad (n \geq 2).$$

It follows that

$$|D_q f(z) - 1| < 1 \quad for \quad |z| < R,$$

that is,

$$\Re\{D_q f(z)\} > 0, \quad for \quad |z| < R.$$

Thus, as an application of Lemma 3 (q-Noshiro–Warchawski theorem), f is univalent for $|z| < R$, where R is given by (59). □

3.6. Extreme Points for $\widetilde{T_q^*}(\alpha)$

To investigate the extreme points of $\widetilde{T_q^*}(\alpha)$, we have the following theorem.

Theorem 13. *Let $f_1(z) = z$, and $n = (2, 3, \dots)$*

$$f_n(z) = z - \frac{(2\alpha^2 + 3\alpha + 9)(1-q)z^n}{2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q)}. \tag{62}$$

Then, $f \in \widetilde{T_q^*}(\alpha)$ if and only if it has the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \tag{63}$$

where $\mu_n > 0$ and $\sum_{n=1}^{\infty} \mu_n = 1$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=2}^{\infty} \mu_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \mu_n \frac{2\alpha^2 + 3\alpha + 9)(1-q)z^n}{2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q)}. \end{aligned}$$

Then, from Theorem 6, we have

$$\sum_{n=2}^{\infty} \mu_n \frac{(2\alpha^2 + 3\alpha + 9)(1-q)}{2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q)} \left(\frac{2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q)}{(2\alpha^2 + 3\alpha + 9)(1-q)} \right)$$
$$= \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leq 1.$$

Thus $f \in \widetilde{T_q^*}(\alpha)$. Conversely, let $f \in \widetilde{T_q^*}(\alpha)$; from Corollary 7, we have

$$|a_n| \leq \frac{(2\alpha^2 + 3\alpha + 9)(1-q)}{2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q)}, \; (n = 2, 3, \dots).$$

We may set

$$\mu_n = \frac{2(1-q^n)(3+\alpha)^2 - 9(1+\alpha)(1-q)}{(2\alpha^2 + 3\alpha + 9)(1-q)},$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

It follows that

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),$$

and this completes the proof. □

Corollary 8. *The extreme points of* $\widetilde{T_q^*}(\alpha)$ *are functions* $f_n(z)$ *given by (62),* $(n = 1, 2, \dots)$*. We take* $q \to 1^-$ *and* $\alpha \in [-1, 1]$*, and this leads us the result derived in [28].*

Remark 2. *We note that our Theorems 6 and 7 can be derived alternatively by analysis of extreme points shown in above theorem.*

3.7. Application of the Fractional Calculus

In recent past years, the theory of q-calculus operators has been applied in the areas of ordinary fractional calculus, see [3,5,33,34]. Using the concepts of q-theory, Al-Salam [7] and Agarwal [8] introduced fractional q-integral operator and fractional q-derivatives operator as follows.

Definition 5. *The fractional q-integral operator* $\hat{I}^{\hat{\delta}}_{q,z}$ *of a function* $f(z)$ *of order* $\hat{\delta}$*,* $\hat{\delta} > 0$ *is given by*

$$\hat{I}^{\hat{\delta}}_{q,z} f(z) = \tilde{D}^{-\hat{\delta}}_{q,z} f(z) = \frac{1}{\Gamma_q(\hat{\delta})} \int_0^z (z - tq)_{1-\hat{\delta}} f(t) d_q t, \tag{64}$$

where $f(z)$ *is analytic function in simply connected region of z-plane containing the origin. Here, the term* $(z - tq)_{\hat{\delta}-1}$ *is q-binomial function defined by*

$$(z - tq)_{\hat{\delta}-1} = z^{\hat{\delta}-1} \prod \left[\frac{1 - \left(\frac{tq}{z}\right) q^k}{1 - \left(\frac{tq}{z}\right) q^{\hat{\delta}+k-1}} \right]. \tag{65}$$

Definition 6. *The fractional q-derivative operator* $\tilde{D}^{\hat{\delta}}_{q,z}$ *of function* $f(z)$ *of order* $\hat{\delta}$*,* $(0 \le \hat{\delta} < 1)$ *is defined by*

$$\tilde{D}^{\hat{\delta}}_{q,z} f(z) = D_q \hat{I}^{1-\hat{\delta}} f(z) = \frac{1}{\Gamma_q(1 - \hat{\delta})} D_q \int_0^z (z - tq)_{-\hat{\delta}} f(t) d_q t, \tag{66}$$

where $f(z)$ *is suitably contained and the multiplicity of* $(z - tq)_{-\hat{\delta}}$ *is removed as in Definition 5.*

Definition 7. *Fractional q-derivative of order* $(n + \hat{\delta})$ *is defined by*

$$\tilde{D}^{n+\hat{\delta}}_q f(z) = D^n{}_q (\tilde{D}^{\hat{\delta}}_q f(z)), \tag{67}$$

where $0 \le \hat{\delta} < 1$ *and* $n \in N_0 = N \cup \{0\}$*.*

It is noted that, from (64) and (66), and some simple computation, we have

$$\tilde{D}^{-\hat{\delta}}_{q,z} f(z) = \frac{1}{\Gamma_q(2 + \hat{\delta})} z^{\hat{\delta}+1} - \sum_{n=2}^{\infty} \frac{\Gamma_q(n+1)}{\Gamma_q(n + 1 - \hat{\delta})} a_n z^{n+\hat{\delta}} \tag{68}$$

$$\tilde{D}^{\hat{\delta}}_{q,z} f(z) = \frac{1}{\Gamma_q(2 - \hat{\delta})} z^{1-\hat{\delta}} - \sum_{n=2}^{\infty} \frac{\Gamma_q(n+1)}{\Gamma_q(n + 1 - \hat{\delta})} a_n z^{n+\hat{\delta}}. \tag{69}$$

For details see [7,8].

Next, we use the fractional q-derivative operator and fractional q-integral operator to prove the following results for the class $\widetilde{T_q^*}(\alpha)$.

Theorem 14. *Let $f \in T$, $0 \leq \hat{\delta} < 0$, and the function f defined by (4) be in the class $\widetilde{T_q^*}(\alpha)$. Then, we have*

$$|\tilde{D}_{q,z}^{\hat{\delta}} f(z)| \geq \frac{|z|^{1-\hat{\delta}}}{\Gamma_q(2-\hat{\delta})}\left\{1 - \frac{(1-q^2)(2\alpha^2+3\alpha+9)(1-q)|z|}{(1-q^{2-\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha))}\right\} \tag{70}$$

and

$$|\tilde{D}_{q,z}^{\hat{\delta}} f(z)| \leq \frac{|z|^{1-\hat{\delta}}}{\Gamma_q(2-\hat{\delta})}\left\{1 + \frac{(1-q^2)(2\alpha^2+3\alpha+9)(1-q)|z|}{(1-q^{2-\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha))}\right\}. \tag{71}$$

Equality holds for the function

$$\tilde{D}_{q,z}^{\hat{\delta}} f(z) = \frac{z^{1-\hat{\delta}}}{\Gamma_q(2-\hat{\delta})}\left\{1 - \frac{(1-q^2)(2\alpha^2+3\alpha+9)(1-q)|z|}{(1-q^{2-\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha))}\right\}. \tag{72}$$

Proof. From (69), we have

$$\begin{aligned} \Gamma_q(2-\hat{\delta}) z^{\hat{\delta}} \tilde{D}_{q,z}^{\hat{\delta}} f(z) &= z - \sum_{n=2}^{\infty} \frac{\Gamma_q(n+1)\Gamma_q(2-\hat{\delta})}{\Gamma_q(n+1-\hat{\delta})} a_n z^n \\ &= z - \sum_{n=2}^{\infty} \phi(n,\hat{\delta}) a_n z^n, \end{aligned} \tag{73}$$

where

$$\phi(n,\hat{\delta}) = \frac{\Gamma_q(n+1)\Gamma_q(2-\hat{\delta})}{\Gamma_q(n+1-\hat{\delta})}, \quad n \geq 2 \tag{74}$$

is decreasing in n, so by using the properties of q-gamma function, we have

$$0 < \phi(n,\hat{\delta}) \leq \phi(2,\hat{\delta}) = \frac{1-q^2}{1-q^{2-\hat{\delta}}},$$

and from Theorem 6, we have

$$\begin{aligned} \Gamma_q(2-\hat{\delta})|z^{\hat{\delta}}||\tilde{D}_{q,z}^{\hat{\delta}} f(z)| &\geq |z| - \phi(n,\hat{\delta})|z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{(1-q^2)(2\alpha^2+3\alpha+9)|z|^2}{(1-q^{2-\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha))}. \end{aligned} \tag{75}$$

Additionally, we have

$$\begin{aligned} \Gamma_q(2-\hat{\delta})|z^{\hat{\delta}}||\tilde{D}_{q,z}^{\hat{\delta}} f(z)| &\leq |z| + \phi(n,\hat{\delta})|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{(1-q^2)(2\alpha^2+3\alpha+9)|z|^2}{(-q^{2-\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha))}. \end{aligned} \tag{76}$$

From (75) and (76), we have the inequalities (70) and (71).
Equality holds for the function

$$\tilde{D}_{q,z}^{\hat{\delta}} f(z) = \frac{z^{1-\hat{\delta}}}{\Gamma_q(2-\hat{\delta})}\left\{1 - \frac{(1-q^2)(2\alpha^2+3\alpha+9)(1-q)}{(1-q^{2-\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha))}\right\}.$$

This completes the proof. □

Theorem 15. *Let $f \in T$, $0 \le \hat{\delta} < 1$, and the function f defined by (4) be in the class $\widetilde{T_q^*}(\alpha)$. Then, we have*

$$|\tilde{D}_{q,z}^{-\hat{\delta}} f(z)| \ge \frac{|z|^{\hat{\delta}+1}}{\Gamma_q(2+\hat{\delta})}\left\{1 - \frac{(2\alpha^2+3\alpha+9)(1-q)|z|}{(1-q^{2+\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha))}\right\} \tag{77}$$

and

$$|\tilde{D}_{q,z}^{-\hat{\delta}} f(z)| \le \frac{|z|^{\hat{\delta}+1}}{\Gamma_q(2+\hat{\delta})}\left\{1 + \frac{(2\alpha^2+3\alpha+9)(1-q)|z|}{(1-q^{2+\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha))}\right\}. \tag{78}$$

Equality holds for the function

$$\tilde{D}_{q,z}^{-\hat{\delta}} f(z) = \frac{z^{\hat{\delta}+1}}{\Gamma_q(2+\hat{\delta})}\left\{1 - \frac{(2\alpha^2+3\alpha+9)(1-q)}{(1-q^{2+\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha))}\right\}. \tag{79}$$

Proof. From (69), we have

$$\begin{aligned}\Gamma_q(2+\hat{\delta}) z^{\hat{\delta}} \tilde{D}_{q,z}^{-\hat{\delta}} f(z) &= z - \sum_{n=2}^{\infty} \frac{\Gamma_q(n+1)\Gamma_q(2+\hat{\delta})}{\Gamma_q(n+1-\hat{\delta})} a_n z^n \\ &= z - \sum_{n=2}^{\infty} \phi_1(n,\hat{\delta}) a_n z^n, \end{aligned} \tag{80}$$

where

$$\phi_1(n,\hat{\delta}) = \frac{\Gamma_q(n+1)\Gamma_q(2+\hat{\delta})}{\Gamma_q(n+1-\hat{\delta})}, \quad n \ge 2 \tag{81}$$

is decreasing in n, so we have

$$0 < \phi_1(n,\hat{\delta}) \le \phi_1(2,\hat{\delta}) = \frac{1-q^2}{1-q^{2+\hat{\delta}}},$$

and an application of Theorem 6, we have

$$\begin{aligned}\Gamma_q(2+\hat{\delta})|z^{\hat{\delta}}||\tilde{D}_{q,z}^{-\hat{\delta}} f(z)| &\ge |z| - \phi_1(n,\hat{\delta})|z|^2 \sum_{n=2}^{\infty} a_n \\ &\ge |z| - \frac{(1-q^2)(2\alpha^2+3\alpha+9)|z|^2}{(1-q^{2+\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha)}. \end{aligned} \tag{82}$$

Additionally, we have

$$\begin{aligned}\Gamma_q(2+\hat{\delta})|z^{\hat{\delta}}||\tilde{D}_{q,z}^{-\hat{\delta}} f(z)| &\le |z| + \phi_1(n,\hat{\delta})|z|^2 \sum_{n=2}^{\infty} a_n \\ &\le |z| + \frac{(1-q^2)(2\alpha^2+3\alpha+9)|z|^2}{(1-q^{2+\hat{\delta}})(2(1+q)(3+\alpha)^2-9(1+\alpha)}. \end{aligned} \tag{83}$$

From (82) and (83), we have the inequalities (77) and (78).

Equality holds for the function

$$\tilde{D}_{q,z}^{-\hat{\delta}} f(z) = \frac{z^{\hat{\delta}+1}}{\Gamma_q(2+\hat{\delta})} \left\{ 1 - \frac{(1-q^2)(2\alpha^2+3\alpha+9)|z|}{(1-q^{2+\hat{\delta}})(2(1+q)(3+\alpha)^2 - 9(1+\alpha))} \right\}.$$

This completes the proof. □

Corollary 9. *Let $f \in T$ and the function f defined by (4) be in the class $\widetilde{T_q^*}(\alpha)$. Then, we have*

$$\frac{|z|^2}{(1+q)!} \left\{ 1 - \frac{(1-q^2)(2\alpha^2+3\alpha+9)|z|}{(1+q+q^2)(2(1+q)(3+\alpha)^2 - 9(1+\alpha))} \right\} \leq \left| \int_0^z f(t) d_q t \right|$$

$$\leq \frac{|z|^2}{(1+q)!} \left\{ 1 + \frac{(1-q^2)(2\alpha^2+3\alpha+9)|z|}{(1+q+q^2)(2(1+q)(3+\alpha)^2 - 9(1+\alpha))} \right\}.$$

and

$$|z| \left\{ 1 - \frac{(2\alpha^2+3\alpha+9)|z|}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} \right\} \leq |f(z)|$$

$$\leq |z| \left\{ 1 + \frac{(2\alpha^2+3\alpha+9)|z|}{2(1+q)(3+\alpha)^2 - 9(1+\alpha)} \right\}.$$

Proof. By Definition 5 and Theorem 15, if we take $\hat{\delta} = 1$, this gives us $\tilde{D}_{q,z}^{-1} = \int_0^z f(t) d_q t$ and the result is true. Additionally, from Definition 6 and Theorem 14, for $\hat{\delta} = 0$, we have $\tilde{D}_{q,z}^{0} = D_q \int_0^z f(t) d_q t = f(z)$, and hence we get the required result. □

4. Conclusions

We introduced and studied two classes of starlike functions defined by q-fractional derivative; one contains negative coefficients and one is of negative order. Both these classes were discussed in detail, and certain geometrical properties were investigated that generalized the already known results. We note that results proved in this article are the q-extension and advancements of several results investigated in [26,28,31,32].

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