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On Global Convergence of Third-Order Chebyshev-Type Method under General Continuity Conditions

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Abstract: There are very few papers that talk about the global convergence of iterative methods with the help of Banach spaces. The main purpose of this paper is to discuss the global convergence of third order iterative method. The convergence analysis of this method is proposed under the assumptions that Fréchet derivative of first order satisfies continuity condition of the Hölder. Finally, we consider some integral equation and boundary value problem (BVP) in order to illustrate the suitability of theoretical results.

Keywords: global convergence; nonlinear equations; Hölder condition; recurrence relation

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1. Introduction

Solutions for nonlinear equations in Banach spaces are widely studied in numerical analysis and computational mathematics. There are several real life problems that can reduce into nonlinear equations, the solutions of which can be obtained by using iterative methods. Generally, we have three types of convergence analysis for iterative methods: local, semilocal, and global convergence. The assumption of semilocal convergence [2,3] is based on initial guess and domain estimates, the local convergence [4,5] assumptions are established on the information surrounding the solution, and the global convergence [1,6] stands on domain assumptions. The difference between these convergence approaches is that the condition at initial point x_0 is imposed in results of semilocal convergence and the condition on solution x^* is imposed in results of local convergence. There is a main difficulty in the case of local convergence result in the solution x^* of the equation, which is usually unknown. On the other hand, semilocal convergence results are a good starting point for the solution. The semilocal convergence of Newton's method is established by Kantorovich [4] under the different assumptions. There is a lot of literature available on higher order iterative methods to discuss the local and semilocal convergence (for reference please see [2,4,7–11]).

Let F' satisfy the condition $\|F'(x) - F'(y)\| \leq K\|x - y\|^q, x, y \in D, q \in (0,1]$; this hypothesis is called Hölder continuity condition. In [4], different types of convergence analyses have been suggested for a third order family under the first order of Fréchet differentiable operator that satisfies different continuity conditions. The convergence analysis of modified Halley method is presented in [9]. Argyros et al. [8] gave the convergence analysis of Chebyshev–Halley-type methods with a parameter under the assumption F' satisfies the Lipschitz condition. In [1,6], the global convergence of the iterative method proposed under the assumption that Fréchet derivative satisfies the Lipschitz continuity condition. We look into some examples where F' satisfies the Hölder continuity condition but not the Lipschitz condition.

Example 1.

$$F(y)(s) = y(s) - 1 - \frac{1}{3}\int_0^1 K(s,t)y(t)^{3/2}dt, \quad s \in [0,1],$$

where, K(s,t) is the Green's function and F is the functions defined on set of all continuous functions $C[0,1]$ *with max norm* $\|y\| = \max_{s\in[0,1]} |y(s)|$,

$$F'(y)(s) = u(s) - \frac{1}{2}\int_0^1 K(s,t)u(s)^{1/2}dt.$$

Therefore,

$$\|F'(x) - F'(y)\| \leq \frac{1}{16}\|x-y\|^{1/2}.$$

Apparently, for $q = 1/2$, F' *satisfies the Hölder condition where the Lipschitz condition fails.*

Example 2.

$$F(x)(s) = x(s) - f(s) - \mu\int_0^1 \frac{s}{s+t}x(t)^{1+q}dt, \quad s \in [0,1], \quad q \in (0,1]$$

where F is the space of continuous functions on $[0,1]$ *with maximum norm* $\|x\| = \max_{s\in[0,1]} |x(s)|$ *and* μ *is a real number. The first order Fréchet derivative is given below:*

$$[F'(x)u](s) = u(s) - \mu(1+q)\int_0^1 \frac{s}{s+t}u(t)^q$$

Therefore,

$$\|F'(x) - F'(y)\| \leq |\mu|(1+q)\log 2\|x-y\|^q.$$

Apparently, F' *satisfies the Hölder condition for* $q \in (0,1)$ *whereas the Lipschitz continuity fails.*

Let us consider

$$F(x) = 0 \tag{1}$$

to be a nonlinear equation defined on a non-empty open convex set Ω from Banach spaces X to Y. Typically, we represent the nonlinear Equation (1) as a system of nonlinear equations, boundary value problems, and integral equations, etc. The third order iterative process for solving such problems (1), is given by

$$\begin{cases} y_n = x_n - [F'(x_n)]^{-1}F(x_n) \\ z_n = x_n + p(y_n - x_n), p \in (0,1] \\ x_{n+1} = y_n - \dfrac{1}{p^2}[F'(x_n)]^{-1}((p-1)F(x_n) + F(z_n)) \end{cases} \tag{2}$$

This method was developed by Ezquerro and Hernendez [12], using another method in [13] to obtain (2). In [12], they proposed the derivative-free method, which reduces the cost of computing, number of function evaluation, and the convergence order maintained. At $p = 1$, the iterative method (2) becomes the Newton frozen two-step method [14].

In this work, we set up the domain of convergence for an iterative method (2) similar to that obtained from local and semilocal convergence results. The main differences are that there is no need to look at the situation in the solution x^* and the initial guesses x_0. Therefore, we are interested in discussing the global convergence of third order iterative method of the form (2), in this manuscript. The existence of a uniqueness solution x^* of (1) is to be established with the assumption that Fréchet differentiable operator of the first order satisfies Hölder condition. The globally convergent domain also proposed. Some examples are designed to discuss the global convergence of the iterative method.

The paper frame is as follows. The introduction constitutes Section 1. In Section 2, we give some assumptions for the global convergence of iterative method. Additionally, we discuss the development recurrence relations of proposed method in order to establish global convergence. In Section 3, the global convergence analysis of the proposed scheme is discussed. In Section 4, some numerical examples are designed to show the existence and uniqueness of the solution of our scheme. Section 5 contains concluding remarks.

2. Recurrence Relations

In this section, we develop a system of recurrence relations with the assumptions C_1, C_2. We consider the following assumptions to discuss the global convergence of (2). The assumptions are:

C_1. $\|\bar{\Gamma}\| \leq \beta$ and $\|\bar{\Gamma}F(\bar{x})\| \leq \eta$ for some $\bar{x} \in \Omega$ and $\bar{\Gamma} = [F'(\bar{x}]^{-1}$

C_2. $\|F'(x) - F'(y)\| \leq K\|x-y\|^q$, $\forall x, y \in \Omega$, $q \in (0,1]$ and $K \geq 0$

From the condition C_2, we observe that $\|F'(x) - F'(\bar{x})\| \leq \bar{K}\|x - \bar{x}\|^q$, $\forall x \in \Omega$, with $\bar{K} \leq K$.

Here, we introduce some lemmas that used to derive recurrence relation. This also helps us to prove convergence Theorem 1.

Lemma 1 (see [12]). *Let Fréchet differential operator F be defined on a non empty open convex set Ω of a Banach space X with values in Banach space Y. Then*

(a) $F(x_0) = F(\bar{x}) + F'(\bar{x})(x_0 - \bar{x}) + \int_{\bar{x}}^{x_0}(F'(x) - F'(\bar{x}))(x_0 - x)dx$

(b) *For $x_n, z_n \in \Omega$, $p \in (0,1]$, we have*

$$F(z_n) = (1-p)F(x_n) + \int_{x_n}^{z_n}(F'(x) - F'(x_n))(z_n - x)dx.$$

(c) *For $x_n, x_{n+1} \in \Omega$, we have*

$$\begin{aligned} F(x_{n+1}) &= F(x_n) + F'(x_n)(x_{n+1} - x_n) + \int_{x_n}^{x_{n+1}}(F'(x) - F'(x_n))dx \\ &= \frac{1}{p}\int_0^1 [F'(x_n) - F'(x_n + pt(y_n - x_n))](y_n - x_n)dt \\ &+ \int_0^1 [F'(x_n + t(x_{n+1} - x_n)) - F'(x_n)](x_{n+1} - x_n)dt. \end{aligned}$$

(d) *For $x_n \in \Omega$, we have*

$$F(x_n) + F'(x_n)(\bar{x} - x_n) = F(\bar{x}) - \int_{x_n}^{\bar{x}}(F'(x) - F'(x_n))(\bar{x} - x)dx.$$

Lemma 2. *If there exists $R > 0$, for a real valued function $f(t) = t\left(\frac{p^{q-1}}{(q+1)(q+2)} + \frac{(1+\frac{t}{(q+1)(q+2)})^{q+1}}{(q+1)}\right)$, $a_0 = Kde^q$, $\Gamma = F'(x)^{-1}$, and $a_0 \in (0.38748, 0.4953]$, then the following inequalities are true for $n \geq 1$,*

$$
\begin{aligned}
&(I)\ \|\Gamma_n\| \le f(a_{n-1})\|\Gamma_{n-1}\|,\\
&(II)\ Kd\|\Gamma_n\| \le a_n,\\
&(III)\|y_n - \bar{x}\| \le \frac{(q+1)(q+2)\eta + K\beta R^{q+2}}{(q+1)(q+2)(1-\bar{K}\beta R)},\\
&(IV)\ \|z_n - x_n\| \le pf(a_{n-1})\|y_{n-1} - x_{n-1}\|,\\
&(V)\ \|x_{n+1} - x_n\| \le \left(1 + \frac{a_n p^{q-1}}{(q+1)(q+2)}\right)\|y_n - x_n\|,\\
&(VI)\ \|x_{n+1} - \bar{x}\| \le \frac{(q+1)(q+2)\eta + K\beta R^{q+2}}{(q+1)(q+2)(1-\bar{K}\beta R)} + \frac{a_n e}{(q+1)(q+2)} \le R
\end{aligned} \tag{3}
$$

Proof. By Banach Lemma and from $\|I - \bar{\Gamma}F'(x)\| \le \|\bar{\Gamma}(F'(\bar{x}) - F'(x))\|$, we have

$$\|\Gamma\| = \|[F'(x)]^{-1}\| \le \frac{\beta}{1-\bar{K}\beta R} = d$$

and

$$\|\Gamma F'(\bar{x})\| \le \frac{1}{1-\bar{K}\beta R}, \tag{4}$$

where $R > 0$ as $x \in B(\bar{x}, R) \subset \Omega$ and $\bar{K}\beta R < 1$.

From the condition (a) of Lemma 1 and (4), we have

$$
\begin{aligned}
\|y_0 - x_0\| &\le \|\Gamma_0 F(x_0)\|\\
&\le \|\Gamma_0 F'(\bar{x})\|\|\bar{\Gamma}F(x_0)\|\\
&\le \|\Gamma_0 F'(\bar{x}\|\left(\|\bar{\Gamma}F(\bar{x}) + \bar{\Gamma}F'(\bar{x})(x_0 - \bar{x}) + \bar{\Gamma}\int_{\bar{x}}^{x_0}(F'(x) - F'(\bar{x}))(x_0 - x)dx\|\right)\\
&\le \|\Gamma_0 F'(\bar{x}\|\left(\|\bar{\Gamma}F(\bar{x})\| + \|x_0 - \bar{x}\| + \|\bar{\Gamma}\|\bar{K}\int_{\bar{x}}^{x_0}\|x - \bar{x}\|^q\|x_0 - x\|dx\right)\\
&\le \|\Gamma_0 F'(\bar{x}\|\left(\|\bar{\Gamma}F(\bar{x})\| + \|x_0 - \bar{x}\| + \|\bar{\Gamma}\|\bar{K}\int_0^1 t^q(1-t)\|x_0 - \bar{x}\|^{q+2}\right)\\
&\le \frac{\eta + R + \frac{\bar{K}\beta R^{q+1}}{(q+1)(q+2)}}{(1-\bar{K}\beta R)}\\
&\le \frac{(q+1)(q+2)\eta + (q+1)(q+2)R + \bar{K}\beta R^{q+1}}{(q+1)(q+2)(1-\bar{K}\beta R)} = e,
\end{aligned}
$$

from (2) and the hypothesis (d) of Lemma 1 we find

$$
\begin{aligned}
y_0 - \bar{x} &= x_0 - \bar{x} - F'(x_0)^{-1}F(x_0)\\
&= -\Gamma_0(F(x_0) + F'(x_0)(\bar{x} - x_0))\\
&= -\Gamma_0(F(\bar{x}) - \int_{x_0}^{\bar{x}}(F'(x) - F'(x_0))(\bar{x} - x_))dx
\end{aligned}
$$

Applying norm on both sides, we find

$$\begin{aligned}
\|y_0 - \bar{x}\| &\leq \| - \Gamma_0(F(x_0) + F'(x_0)(\bar{x} - x_0))\| \\
&\leq \| - \Gamma_0\Big(F(\bar{x}) - \int_{x_0}^{\bar{x}} (F'(x) - F'(x_0))(\bar{x} - x)dx\Big)\| \\
&\leq \|\Gamma_0 F'(\bar{x})\|\|F'(\bar{x})^{-1}F(\bar{x})\| \qquad (5) \\
&\quad + \|\Gamma_0 F'(\bar{x})\|\|F'(\bar{x})^{-1}\|\Big(\int_{x_0}^{\bar{x}} \|F'(x) - F'(x_0)\|\|\bar{x} - x\|dx\Big) \\
&\leq \frac{(q+1)(q+2)\eta + K\beta R^{q+2}}{(q+1)(q+2)(1 - \bar{K}\beta R)}. \qquad (6)
\end{aligned}$$

From (2), we have

$$z_0 - x_0 = p(y_0 - x_0), \qquad (7)$$

and

$$\begin{aligned}
z_0 - \bar{x} &= z_0 - x_0 + x_0 - \bar{x} \\
&= p(y_0 - x_0) + x_0 - \bar{x} \\
&= p(y_0 - \bar{x} + \bar{x} - x_0) + x_0 - \bar{x} \\
&= p(y_0 - \bar{x}) + (1 - p)(x_0 - \bar{x}). \qquad (8)
\end{aligned}$$

Applying norm on both sides to (7) and (8), we find

$$\|z_0 - x_0\| = p\|y_0 - x_0\| \leq pe,$$

$$\|z_0 - \bar{x}\| \leq (1 - p)\|x_0 - \bar{x}\| + p\|y_0 - \bar{x}\| \leq R. \qquad (9)$$

From (5) and (9), we conclude that $y_0, z_0 \in B(\bar{x}, R)$, with the condition that

$$\frac{(q+1)(q+2)\eta + K\beta R^{q+2}}{(q+1)(q+2)(1 - \bar{K}\beta R)} \leq R.$$

Furthermore,

$$\begin{aligned}
\|x_1 - x_0\| &= \|y_0 - x_0 - \frac{1}{p^2}F'(x_0)^{-1}((p-1)F(x_0) + F(z_0))\| \\
&= \| - F'(x_0)^{-1}F(x_0) - \frac{1}{p^2}F'(x_0)^{-1}((p-1)F(x_0) + F(z_0))\| \\
&= \| - F'(x_0)^{-1}\Big(F(x_0) + \frac{1}{p^2}((p-1)F(x_0) + F(z_0))\Big)\| \\
&= \| - \frac{1}{p^2}\Gamma_0\Big((p^2 + p - 1)F(x_0) + F(z_0)\Big)\| \\
&= \| - \frac{1}{p^2}\Gamma_0\Big((p^2F(x_0) + (p-1)F(x_0) + (1-p)F(x_0) \\
&\quad + \int_{x_0}^{z_0}(F'(x) - F'(x_0))(z_0 - x)dx\Big)\| \\
&\leq \|\Gamma_0 F(x_0)\| + \frac{1}{p^2}\Gamma_0\| \int_{x_0}^{z_0} \|(F'(x) - F'(x_0))\|\|(z_0 - x)\|dx \\
&\leq \|\Gamma_0 F(x_0)\| + \frac{Kd\|z_0 - x_0\|^{q+1}}{(q+1)(q+2)p^2} \\
&\leq e + \frac{Kdp^{q+1}e^{q+1}}{(q+1)(q+2)p^2} \\
&\leq (1 + \frac{a_0 p^{q-1}}{(q+1)(q+2)})\|y_0 - x_0\|,
\end{aligned}$$

and

$$\begin{aligned}
\|x_1 - \bar{x}\| &= \|y_0 - \bar{x} - \frac{1}{p^2}F'(x_0)^{-1}((p-1)F(x_0) + F(z_0))\| \\
&\leq \|y_0 - \bar{x}\| + \|\frac{1}{p^2}F'(x_0)^{-1}((p-1)F(x_0) + F(z_0))\| \\
&\leq \frac{(q+1)(q+2)\eta + K\beta R^{q+2}}{(q+1)(q+2)(1 - \bar{K}\beta R)} + \frac{a_0}{(q+1)(q+2)})e \leq R,
\end{aligned}$$

where $a_0 = Kde^q$ and $x_1 \in B(\bar{x}, R)$, provided that

$$\frac{(q+1)(q+2)\eta + K\beta R^{q+2}}{(q+1)(q+2)(1 - \bar{K}\beta R)} + \frac{a_0}{(q+1)(q+2)}e \leq R \tag{10}$$

Next, from $\|y_1 - x_1\| \leq \|\Gamma_1 F(x_1)\|$ and hypothesis (c) of Lemma 1, we see

$$\begin{aligned}
\|y_1 - x_1\| &= \|\Gamma_1\|\|\frac{1}{p}\int_0^1 [F'(x_0) - F'(x_0 + pt(y_0 - x_0))](y_0 - x_0)dt \\
&\quad \int_0^1 [F'(x_0 + t(x_1 - x_0)) - F'(x_0)](x_1 - x_0)dt\| \\
&\leq K\|\Gamma_1\|\|\frac{1}{p}\int_0^1 \|pt(y_0 - x_0)\|^q \|y_0 - x_0\| \\
&\quad + \int_0^1 \|x_0 + t(x_1 - x_0) - x_0\|^q \|x_1 - x_0\| \\
&\leq K\|\Gamma_1\|\Big(\frac{p^{q-1}}{q+1}\|(y_0 - x_0)\|^{q+1} + \frac{\|x_1 - x_0\|^{q+1}}{q+1}\Big) \\
&\leq K\|\Gamma_1\|\Big(\frac{p^{q-1}}{q+1}\|(y_0 - x_0)\|^{q+1} + \frac{(1 + \frac{a_0}{(q+1)(q+2)})^{q+1}}{q+1}\|y_0 - x_0\|^{q+1}\Big) \quad (11) \\
&\leq \Big(Kd\frac{p^{q-1}}{q+1}e^q + Kd\frac{(1 + \frac{a_0}{(q+1)(q+2)})^{q+1}}{q+1}e^q\Big)\|y_0 - x_0\| \\
&\leq f(a_0)\|y_0 - x_0\|, \quad (12)
\end{aligned}$$

where

$$f(t) = t\Big(\frac{p^{q-1}}{(q+1)(q+2)} + \frac{(1 + \frac{t}{(q+1)(q+2)})^{q+1}}{(q+1)}\Big). \quad (13)$$

For $t > 0$, the function f is increasing, therefore it follows that

$$kd\|y_1 - x_1\| \leq a_1,$$

$$\begin{gathered}
\|y_1 - \bar{x}\| \leq \frac{(q+1)(q+2)\eta + K\beta R^{q+1}}{(q+1)(q+2)(1 - \bar{K}\beta R)}, \\
\|z_1 - x_1\| \leq pf(a_0)\|y_0 - x_0\|, \\
\|z_1 - \bar{x}\| \leq R, \\
\|x_2 - x_1\| \leq (1 + a_1 p^{q-1}/(q+1)(q+2))\|y_1 - x_1\|, \\
\|x_2 - \bar{x}\| \leq \Big(\frac{(q+1)(q+2)\eta + K\beta R^{q+1}}{(q+1)(q+2)(1 - \bar{K}\beta R)} + \frac{e}{(q+1)(q+2)}a_0 f(a_0)^2\Big) \leq R.
\end{gathered}$$

We note that last condition holds for $f(a_0) < 1$. By using mathematical induction, we can prove the above inequalities. The first step is already proven for $n = 1$, so we continue same procedure to demonstrate the inequalities.

Now, we define as $a_1 = a_0 f(a_0)$ and define the real sequence

$$a_n = a_{n-1} f(a_{n-1}), n \geq 1.$$

We notice that for $f(a_0) < 1$, for $a_0 \in (0.3774, 0.4953]$, and for $q \in (0, 1]$, the sequence a_n is decreasing. □

3. Global Convergence

For the global analysis of the iterative method, we followed the same procedure as in the case of semilocal convergence for different values of the radius R condition (10) satisfied. We can choose the most appropriate value from these values. We can say that global convergence gives the largest value and the best location of solution.

Theorem 1. *Let F be a Fréchet differentiable function in an open convex set Ω. Let hypotheses (C_1) and (C_2) be satisfied. We denote $a_0 = K\beta e^q$ and $0.3774 < a_0 \leq 0.4953$. Then, the sequence of $\{x_n\}$ is defined in (9) and start point x_0 converge to the solution x^* of (1) in $\overline{B}(\bar{x}, R)$ from every point x_0 which is in $B(\bar{x}, R)$.*

Proof. To establish $\{x_n\}$ convergence, it is enough to show that the sequences from the method (2) lies in $\overline{B}(\bar{x}, R)$ and the Cauchy sequence.

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \Big(1 + \frac{a_n p^{q-1}}{(q+1)(q+2)}\Big)\|y_n - x_n\| && (14) \\ &\leq \Big(1 + \frac{a_0 p^{q-1}}{(q+1)(q+2)}\Big) f(a_0)^n \|y_0 - x_0\| && (15) \end{aligned}$$

From this, we write,

$$\begin{aligned} \|x_{m+n} - x_m\| &\leq \|x_{m+n} - x_{m+n-1}\| + \ldots + \|x_{m+1} - x_m\| \\ &\leq \sum_{j=n}^{n+m-1} (1 + a_j p^{q-1}/(q+1)(q+2))\|y_i - x_i\| \\ &\leq (1 + a_0 p^{q-1}/(q+1)(q+2)) f(a_0)^n \|y_0 - x_0\| \sum_{j=0}^{m-1} f(a_0)^j \\ &\leq (1 + a_0 p^{q-1}/(q+1)(q+2)) f(a_0)^n \|y_0 - x_0\| \frac{1 - f(a_0)^m}{1 - f(a_0)}. && (16) \end{aligned}$$

From this, $\{x_n\}$ is a Cauchy sequence when we take the limit as n tends to ∞ in inequality (V). Then, we obtain $x^* \in \overline{B}(\bar{x}, R)$. Demonstrating x^* is a $F(x) = 0$ solution, we have that $\|F(x_n)\| \leq \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and sequence $\{\|F'(x_n)\|\}$ bounded as

$$\|F'(x_n)\| \leq \|F'(\bar{x})\| + KR,$$

by the continuity of F and taking limit as n tends ∞, we find x^* is the solution of $F(x) = 0$.

This is the way of analyzing global convergence results of method (2), which also allows us to find the solution x^* in the ball $B(\bar{x}, R)$. Additionally, we define the ball for the global convergence as $\overline{B}(\bar{x}, R)$. □

Theorem 2. *Let F satisfy the assumptions (C_1) and (C_2), then we have unique solution x^* in $B(\bar{x}, R) \cap \Omega$.*

We may prove the uniqueness if z^* is another solution for (1) in $B(x_0, r) \cap \Omega$ and we have

$$0 = \bar{\Gamma}(F(z^*) - F(x^*)) = \int_0^1 \bar{\Gamma} F'(x^* + t(z^* - x^*))dt(z^* - x^*).$$

Obviously, $z^* = x^*$, if $S = \int_0^1 F'(x^* + t(z^* - x^*))dt$ is invertible. This follows from

$$\begin{aligned} \|I - S\| &\leq \bar{\Gamma} \int_0^1 [F'(x^* + t(z^* - x^*)) - F'(\bar{x})]dt\| \\ &\leq \beta \int_0^1 \bar{K}(\|x^* + t(z^* - x^*) - \bar{x}\|^q)dt \\ &\leq \bar{K}\beta \int_0^1 ((1-t)\|x^* - \bar{x}\| + t\|z^* - \bar{x}\|)^q dt \\ &< \bar{K}\beta R^q = 1, && (17) \end{aligned}$$

and by Banach Lemma. Therefore, $z^* = x^*$.

4. Numerical Examples

In this section, we discuss the existence and uniqueness of solution for the numerical problems. Therefore, we choose two real life problems, namely nonlinear integrals of second kind and a boundary value problem (BVP).

Example 3. *Let*

$$x(s) = 1 + \frac{1}{3}\int_0^1 G(s,t)x(t)^{3/2}dt, \quad s \in [0,1], \tag{18}$$

be defined on the set of all continuity functions on $[0,1]$ *with maximum norm* $\|x\| = \max_{s\in[0,1]}|x(s)|$ *and the kernel defined as* $G(s,t)$*,*

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases} \tag{19}$$

Solving (18) is same as solve $F(x) = 0$, where $F : \Omega \subseteq C[a,b] \to C[a,b]$ and

$$[F(x)](s) = x(s) - 1 - \frac{1}{3}\int_0^1 G(s,t)x(t)^{3/2}dt \quad s \in [0,1]. \tag{20}$$

Now, we find the First order Fréchet derivative of (18),

$$F'(x)u(s) = u(s) - \frac{1}{2}\int_0^1 G(s,t)x(t)^{1/2}u(t)dt.$$

Then, we have

$$\|F'(x) - F'(y)\| \leq \frac{1}{16}\|x-y\|^{1/2}.$$

Here, we see that F' satisfies the continuity condition of Hölder but fails to satisfy the Lipschitz continuity condition for all $x, y \in \Omega$. Therefore, we see $K = \bar{K} = 1/16$ and $q = 1/2$. For $\bar{x}(s) = 1$, we have $\beta = 16/15$, $\eta = 4/37$. Now, we see that the condition (10) is guaranteed by $R \in (0.110557, 12.5122)$. For all these values, $K\beta R^q < 1$. However, the condition $a_0 < 1$ holding $R \in (0.110557, 10.418)$. Therefore, both conditions are satisfactory, and the iterative method is obviously converge to x^* of $F(x) = 0$ in the $\overline{\mathcal{B}}(\bar{x}, R)$ domain for $R \in (0.110557, 10.418)$. Therefore, the best ball of existence solution is $\overline{\mathcal{B}}(1, 0.110557)$ and the best ball of global convergence of the iterative process is $\overline{\mathcal{B}}(1, 10.418)$. The uniqueness of solution is in the ball $\mathcal{B}(1, 10.418) \cap \Omega$.

Example 4. *Consider the following BVP*

$$u'' + u^{q+1} = 0, q \in (0,1], u(0) = u(1) = 0 \tag{21}$$

First, the interval $[0,1]$ divided into N subintervals by points $t_j = jk$, $j = 0,1,2,\ldots N$ and $k = 1/N$. By using central divided difference formula we approximate the second derivative in $u_j'' \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{k^2}$ for $j = 0,1,2,\ldots N-1$. From (21), we find

$$-u_{j-1} + 2u_j - u_{j+1} - k^2 u_j^{q+1} = 0. \tag{22}$$

This can be written as $F(x) = Hx - h^2 l(x) = 0$, where $F : R^{N-1} \to R^{N-1}$, $x = (x_1, x_2, \dots x_{N-1})$, $l(x) = (x_1^{q+1}, x_2^{q+1}, \dots x_{N-1}^{q+1})^t$ and the matrix given by

$$\begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & 2 \end{bmatrix}$$

Here, $F'(x) = H - (1+q)h^2 M(x)$, where $M(x) = diag(x_1^q, x_2^q, \dots x_{N-1}^q)$ and $\|F'(x) - F'(y)\| \le (1+q)k^2 \|x-y\|^q$. We choose $k = 1/10$ and $q = 1/4$ and

$$\bar{x} = (33 : 5739; 65 : 2025; 91 : 566; 109 : 168; 115 : 363; 109 : 168; 91 : 566; 65 : 2025; 33 : 5739)^t.$$

Here, we see that F' satisfies the Hölder continuity condition but not the Lipschitz continuity condition for all $x, y \in \Omega$. From the assumptions C_1 and C_2, we find $K = \bar{K} = 0.015$, $\beta = 26.5888$, and $\eta = 3.7570 \times 10^{-4}$. Now, we observe that the condition (10) is verified for $R \in (0.408242, 0.804083)$. For all these values $K\beta R^q < 1$. However, the condition $a_0 < 1$ holds for $R \in (0.408242, 0.804083)$. Hence, both the conditions are satisfied, and the iterative process is well defined and converges to x^* of $F(x) = 0$ in $\overline{\mathcal{B}}(\bar{x}, R)$ for $R \in (0.408242, 0.804083)$. Therefore, the best ball of existence for the location of the solution is $\overline{\mathcal{B}}(1, 0.408242)$ and the best ball for the global convergence for the iterative process is $\overline{\mathcal{B}}(1, 0.804083)$. The uniqueness of solution in the ball $\mathcal{B}(1, 0.804083) \cap \Omega$.

5. Conclusions

In this manuscript, we have used the global convergence of method (2) under the assumption that the Fréchet differentiable operator of first order satisfies the Hölder condition. We also demonstrated the existence and uniqueness of the required solution. Two real life examples have been discussed for the convergence regions.

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