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Article

Relevance of Factorization Method to Differential and Integral Equations Associated with Hybrid Class of Polynomials

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Abstract: This article has a motive to derive a new class of differential equations and associated integral equations for some hybrid families of Laguerre–Gould–Hopper-based Sheffer polynomials. We derive recurrence relations, differential equation, integro-differential equation, and integral equation for the Laguerre–Gould–Hopper-based Sheffer polynomials by using the factorization method.

Keywords: Laguerre–Gould–Hopper-based Sheffer polynomials; factorization method; generating function; differential equation; integral equation

MSC: 44A45; 33C45; 05A15; 33E20

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1. Introduction and Preliminaries

The class of Sheffer polynomial sequences is one of the significant classes of the polynomial sequences [1–3]. The Sheffer polynomial sequences emerge in many fields of applied mathematics, mathematical physics, engineering sciences, estimation theory, and various branches of applied sciences. The class of Sheffer sequences is a group of non-abelian type with reference to the umbral calculus.

During recent years, the generalized and multi-variable varieties of the special functions have remained as notable key functions in the development of mathematical physics. The special functions of two variables are the solutions of differential equations which are encountered in many different fields.

By Roman [4], the Sheffer sequence is elaborated by the generating relation

$$\frac{e^{af^{-1}(t)}}{g(f^{-1}(t))} = \sum_{n=0}^{\infty} S_n(a)\frac{t^n}{n!}, \tag{1}$$

$\forall$ a in c, where $f^{-1}(t)$ is the compositional inverse of $f(t)$ and $f(t)$ are uniquely determined by two power series.

Let $f(t)$ be a delta, $g(t)$ be an invertible series given by

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, f_1 \neq 0, f_0 = 0, \tag{2}$$

and

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, g_0 \neq 0. \tag{3}$$

Then, $\exists$ a sequence $s_n(x)$ of polynomials satisfy the orthogonality condition

$$\langle g(t)f(t)^k | s_n(x)\rangle = n!\delta_{n,k}, \forall n,k \geq 0.$$

For $(g(t) = 1)$, it is the associated Sheffer sequence and with condition $(f(t) = t)$, it becomes the Appell sequence [5].

Some important sequences of polynomials, such as Hermite, Laguerre, Bessel, etc., are the special cases of the Sheffer polynomial sequence. These polynomials are very useful in mathematical physics, and various fields of engineering.

In 2012, Khan et al. [6] presented the 3-variables Laguerre–Gould–Hopper polynomials by the following generating relation

$$J_0(-at^p)e^{(bt+ct^s)} = \sum_{n=0}^{\infty} {}_LH_n^{(p,s)}(a,b,c)\frac{t^n}{n!}, \tag{4}$$

where $J_0(x)$ is a Bessel–Tricomi function of 0th order. $J_n(x)$ is nth order Bessel–Tricomi function, which can be defined by the generating function

$$exp(t - \frac{x}{t}) = \sum_{n=0}^{\infty} J_n(x)t^n, \tag{5}$$

for $t \neq 0$ and $\forall$ finite x.

In 2016, Raza et al. [7] presented a new class of Laguerre–Gould–Hopper–Sheffer polynomials (3VLGHSP) ${}_{LH^{(p,s)}}S_n(a,b,c)$ by using the generating function. The development and properties of hybrid special polynomials [8–10] have been most substantial theme in applied mathematics in last decades. The generating function of Laguerre–Gould–Hopper–Sheffer polynomials (3VLGHSP) ${}_{LH_n^{(p,s)}}S_n(a,b,c)$ [7] can be given as follows

$$\frac{1}{g(f^{-1}(t))}exp\Big(b(f^{-1}(t)) + D_a^{-1}(f^{-1}(t))^p + c(f^{-1}(t))^s\Big) = \sum_{n=0}^{\infty} {}_{LH^{(p,s)}}S_n(a,b,c)\frac{t^n}{n!}, \tag{6}$$

$J_0(ax)$ is Bessel–Tricomi function of order zero [11].

Since

$$J_n(xt) = \sum_{n=0}^{\infty} \frac{(-1^r)(xt)^n}{r^2!},$$

where D_a^{-1} denotes the inverse derivative operator $D_a = \frac{\partial}{\partial a}$ and is defined by

$$D_a^{-1}\{f(a)\} = \int_0^a f(\xi)d\xi.$$

The series representation of Laguerre–Gould–Hopper-based Sheffer polynomials of 3-variables (3VLGHSP) ${}_{LH^{(p,s)}}S_n(a,b,c)$ is as follows

$${}_{LH^{(p,s)}}S_n(a,b,c) = \sum_{k=0}^{n} \binom{n}{k} {}_LH_{n-k}^{(p,s)}(a,b,c)S_k. \tag{7}$$

The factorization method [12,13] is used to derive the differential and integro-differential equations for Appell polynomial. Recently, the differential equations are investigated for hybrid forms of Appell polynomials (see, example [14–18]). In 2020, Wani et al. [19] used an idea to derive the differential equations, recurrence relations and integral equations for Laguerre–Gould–Hopper-based Appell and related polynomials by factorization method. We mention here a few preliminaries associated with the factorization technique.

Let $\{p_n(x)\}_{k=0}^{\infty}$ be a sequence of polynomials as the degree of $(p_n(x)) = n(p_n \in \mathbb{N}_0 := 0,1,2,3,...)$. The operators satisfying the following relations

$$\Phi_n^- p_n(x) = p_{n-1}(x), \tag{8}$$

$$\Phi_n^+ p_n(x) = p_{n+1}(x), \tag{9}$$

are known as the derivative operator and multiplicative operators, respectively. Dattoli et al. [20–24] used the monomiality principle and operational rules to derive new classes of hybrid polynomials and their properties.

The differential equation of hybrid polynomials

$$(\Phi_{n+1}^- \Phi_n^+) p_n(x) = p_n(x), \tag{10}$$

can be derived using the Φ_n^- and Φ_n^+ operators. For finding the derivative Φ_n^- and multiplicative Φ_n^+ operator, we use the factorization technique [25] as the Equation (10) holds.

2. Recurrence Relation and Differential Equations for Laguerre–Gould–Hopper-Based Sheffer Polynomial (3VLGHSP) ${}_{LH^{(p,s)}}S_n(a,b,c)$

First, we derive the recurrence relation for the Laguerre–Gould–Hopper-based Sheffer polynomial (3VLGHSP) ${}_{LH^{(p,s)}}S_n(a,b,c)$.

Theorem 1. *The Laguerre–Gould–Hopper–Sheffer polynomials* ${}_{LH^{(p,s)}}S_n(a,b,c)$ *satisfy the recurrence relation*

$$\begin{aligned}
{}_{LH^{(p,s)}}S_{n+1}(a,b,c) = \Big(&(b+\alpha_0)\,{}_{LH^{(p,s)}}S_n(a,b,c) \\
&+ \sum_{k=1}^{n} \tbinom{n}{k} \alpha_k\, {}_{LH^{(p,s)}}S_{n-k}(a,b,c) \\
&+ \tfrac{n!}{(n-p+1)!} p D_a^{-1}\, {}_{LH^{(p,s)}}S_{n-p+1}(a,b,c) \\
&+ \tfrac{n!}{(n-s+1)!} sc\, {}_{LH^{(p,s)}}S_{n-s+1}(a,b,c) \Big) \tfrac{1}{f'(t)},
\end{aligned} \tag{11}$$

where the coefficients $\{a_k\}_{k\in\mathbb{N}_0}$ *are given by the expansion*

$$\frac{g'(t)}{g(t)} = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k. \tag{12}$$

Proof. Differentiating the each sides of generating relation (6) of Laguerre–Gould–Hopper-based Sheffer polynomials ${}_{LH^{(p,s)}}S_n(a,b,c)$ (3VLGHSP) with respect to t, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_{LH^{(p,s)}}S_{n+1}(a,b,c)\frac{t^n}{n!} \\
&= \frac{1}{g(f^{-1}(t))} exp\Big(D_b^{-1}(f^{-1}(t))^s + af^{-1}(t)\Big) \\
&\quad \times \left(b + pD_a^{-1}\left(f^{-1}(t))^{p-1} cs(f^{-1}(t))^{s-1} + \frac{g'(f^{-1}(t))}{g(f^{-1}(t))}\right)\right) \frac{1}{f'(f^{-1}(t))}.
\end{aligned}$$

Using the Equations (6) and (12) in the above equation, and then applying the cauchy product rule in the left-hand side of the resultant equation, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{L}H^{(p,s)}S_{n+1}(a,b,c)\tfrac{t^n}{n!} = \sum_{n=0}^{\infty}\Big(&\sum_{k=0}^{n}\tbinom{n}{k}\alpha_k {}_{L}H^{(p,s)}S_{n-k}(a,b,c) \\ &+ b\, {}_{L}H^{(p,s)}S_n(a,b,c) \\ &+ \tfrac{n!}{(n-p+1)!}pD_a^{-1} {}_{L}H^{(p,s)}S_{n-p+1}(a,b,c) \\ &+ \tfrac{n}{(n-s+1)!}cs\, {}_{L}H^{(p,s)}S_n(a,b,c)\Big)\tfrac{t^n}{n!}\tfrac{1}{f'(f^{-1}(t))}.\end{aligned} \tag{13}$$

Now, comparing the coefficients of equal powers of t on both sides of the above equation and solving the resultant equation for $t = f^{-1}(t)$, then we get the Equation (11). □

Furthermore, we derive the shift operators for Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP) ${}_{L}H^{(p,s)}S_n(a,b,c)$.

Theorem 2. *The Shift operators for the Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP)* ${}_{L}H^{(p,s)}S_n(a,b,c)$ *are given by*

$${}_{a}\pounds_n^{-} = \frac{1}{n}D_a D_b^{-(p-1)}, \tag{14}$$

$${}_{b}\pounds_n^{-} = \frac{1}{n}D_b, \tag{15}$$

$${}_{c}\pounds_n^{-} = \frac{1}{n}D_c D_b^{-(p-1)}, \tag{16}$$

$$\begin{aligned}{}_{a}\pounds_n^{+} = \Big(&(b+\alpha_0) + \textstyle\sum_{k=1}^{n}\tfrac{\alpha_k}{k!}D_a^k D_b^{-k(p-1)} + pD_a^{p-2}D_b^{-(p-1)^2} \\ &+ scD_a^{s-1}D_b^{(1-r)(p-1)}\Big)\tfrac{1}{f'(t)},\end{aligned} \tag{17}$$

$${}_{b}\pounds_n^{+} = \left((b+\alpha_0) + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_b^k + pD_a^{-1}D_b^{p-1} + scD_a^{(s-1)}\right)\frac{1}{f'(t)}, \tag{18}$$

$$\begin{aligned}{}_{c}\pounds_n^{+} = \Big(&(b+\alpha_0) + \sum_{k=1}^{n}\tfrac{\alpha_k}{k!}D_c^k D_b^{k(s-1)} + pD_a^{-1}D_b^{(p-1)(1-s)}D_c^{p-1} \\ &+ scD_c^{(s-1)}D^{-(s-1)^2}\Big)\tfrac{1}{f'(t)},\end{aligned} \tag{19}$$

where $D_a = \frac{\partial}{\partial a}$, $D_b = \frac{\partial}{\partial b}$, $D_c = \frac{\partial}{\partial c}$, *and* $D_x^{-1} = \int_0^x f(\xi)d\xi$.

Proof. Differentiating the generating function of (3VLGHSP) with respect to b, and then equating the coefficients of same powers of t from both sides, then we get

$$\frac{\partial}{\partial b}\Big({}_{L}H^{(p,s)}S_n(a,b,c)\Big) = n\, {}_{L}H^{(p,s)}S_{n-1}(a,b,c), \tag{20}$$

so that

$$\frac{1}{n}\frac{\partial}{\partial b}\Big({}_{L}H^{(p,s)}S_n(a,b,c)\Big) = {}_{L}H^{(p,s)}S_{n-1}(a,b,c). \tag{21}$$

Consequently,

$${}_{b}\pounds_n^{-}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = \frac{1}{n}D_b\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = {}_{L}H^{(p,s)}S_{n-1}(a,b,c), \tag{22}$$

which proves (15).

Next, differentiating the generating relation (6) with respect to a, and then equating the same powers of t from both sides, we obtain

$$\frac{\partial}{\partial a}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = \frac{n!}{(n-p)!}\{{}_{L}H^{(p,s)}S_{n-p}(a,b,c)\}. \tag{23}$$

From the Equation (20), it may be written as

$$\frac{\partial}{\partial a}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = n\frac{\partial^{p-1}}{\partial b^{p-1}} = n\{{}_{L}H^{p,s}S_{n-1}(a,b,c)\}, \tag{24}$$

$$\begin{aligned} {}_{a}\pounds_n^{-}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} &= \tfrac{1}{n}D_a D_b^{-(p-1)}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} \\ &= {}_{L}H^{(p,s)}S_{n-1}(a,b,c). \end{aligned} \tag{25}$$

Consequently,

$${}_{a}\pounds_n^{-} = \frac{1}{n!}D_a D_b^{-(p-1)},$$

which proves (14).

Again differentiating the generating relation (6) with respect to c and comparing the coefficients of equal powers of t, we obtain

$$\frac{\partial}{\partial c}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = \frac{n!}{(n-s)!}\{{}_{L}H^{(p,s)}S_{n-s}(a,b,c)\}, \tag{26}$$

which, in view of Equation (20), may be written as

$$\frac{\partial}{\partial c}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = n\frac{\partial^{s-1}}{\partial b^{s-1}} = n\{{}_{L}H^{(p,s)}S_{n-1}(a,b,c)\}. \tag{27}$$

Consequently,

$${}_{c}\pounds_n^{-}[{}_{L}H^{(p,s)}S_n(a,b,c)] = \frac{1}{n}D_c D_b^{-(s-1)}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = {}_{L}H^{(p,s)}S_{n-1}(a,b,c), \tag{28}$$

which proves assertion (16).

Next, to find the raising operator for ${}_{b}\pounds_n^{+}$

$${}_{L}H^{(p,s)}S_{n-k}(a,b,c) = \left({}_{b}\pounds_{n-k+1}^{-}\,{}_{b}\pounds_{n-k+2}^{-}\cdots\cdots{}_{b}\pounds_n^{-}\right){}_{L}H^{(p,s)}S_n(a,b,c),$$

which, on using Equation (22), may be written as

$${}_{L}H^{(p,s)}S_{n-k}(a,b,c) = \frac{(n-k)!}{n!}D_b^k\{{}_{L}H^{(p,s)}S_n(a,b,c)\}, \tag{29}$$

by the use of this equation in recurrence relation in view of the fact that

$${}_{b}\pounds_n^{+}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = {}_{L}H^{(p,s)}S_{n+1}(a,b,c),$$

$${}_{b}\pounds_n^{+} = \left((b+\alpha_0) + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_b^k + pD_a^{-1}D_b^{(p-1)} + scD_b^{s-1}\right)\frac{1}{f'(t)},$$

which proves (18) equation.

Next, to find the raising operator for ${}_{a}\pounds_n^{+}$

$${}_{L}H^{(p,s)}S_{n-k}(a,b,c) = \left({}_{a}\pounds_{n-k+1}^{-}\,{}_{a}\pounds_{n-k+2}^{-}\cdots\cdots{}_{a}\pounds_n^{-}\right){}_{L}H^{(p,s)}S_n(a,b,c),$$

which, on using Equation (25), can be presented as

$$ {}_{L}H^{(p,s)}S_{n-k}(a,b,c) = \frac{(n-k)!}{n!} D_b^{k-(p-1)} D_a \{ {}_{L}H^{(p,s)}S_n(a,b,c) \}, \tag{30} $$

by using this equation in recurrence relation in view of the fact that

$$ {}_{a}\pounds_n^{+} \{ {}_{L}H^{(p,s)}S_n(a,b,c) \} = {}_{L}H^{(p,s)}S_{n+1}(a,b,c). $$

We have

$$ {}_{a}\pounds_n^{+} = \left((b+\alpha_0) + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_a^k D_b^{-k(p-1)} \right. $$
$$ \left. + pD_a^{p-2} D_b^{-(p-1)^2} + scD_a^{s-1} D_b^{(1-s)(p-1)} \right) \frac{1}{f'(t)}, $$

which proves (17) equation.

Now, finally, the raising operator for ${}_{c}\pounds_n^{+}$

$$ {}_{L}H^{(p,s)}S_{n-k}(a,b,c) = \left({}_{c}\pounds_{n-k+1}^{-} {}_{c}\pounds_{n-k+2}^{-} \cdots\cdots {}_{c}\pounds_{n-k+1}^{-} \right) {}_{L}H^{(p,s)}S_{n-k}(a,b,c), $$

which, on using the Equation (28), can be written as

$$ {}_{L}H^{p,s}S_{n-k}(a,b,c) = \frac{(n-k)!}{n!} D_c^k D_b^{k(1-s)} \{ {}_{L}H^{(}{}_{p,s)}S_n(a,b,c) \}. \tag{31} $$

By using the Equation (31) in recurrence relation in view of the fact that

$$ {}_{c}\pounds_n^{+} \{ {}_{L_H^{p,s}} S_n(a,b,c) \} = {}_{L}H^{(p,s)}S_{n+1}(a,b,c). $$

We find

$$ {}_{c}\pounds_n^{+} = \left((b+\alpha_0) + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_c^k D_b^{-k(s-1)} + pD_a^{(-1)} D_b^{-(1-s)(p-1)} D_c^{p-1} \right. $$
$$ \left. + scD_c^{s-1} D_b^{-(1-s)^2} \right) \frac{1}{f'(t)}, $$

which proves assertion (19). □

Now, we derive the differential equation for Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP) ${}_{L}H^{(p,s)}S_n(a,b,c)$.

Theorem 3. *The differential equation for Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP)* ${}_{L}H^{(p,s)}S_n(a,b,c)$ *can be given as follows*

$$ \left(\{ (b+\alpha_0) D_b + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_b^{k+1} + pD_a^{-1} D_b^p + scD_b^s \} \frac{1}{f'(t)} - n \right) {}_{L}H^{(p,s)}S_n(a,b,c) = 0. \tag{32} $$

Proof. Now, we use the factorization method to derive the differential equation

$$ {}_{b}\pounds_{n+1}^{-} {}_{b}\pounds_n^{+} \{ {}_{L}H^{(p,s)}S_n(a,b,c) \} = {}_{L}H^{(p,s)}S_n(a,b,c). $$

Putting the values of the shift operators from Equations (15) and (18) in the left-hand side of the above equation

$$\left(\{(b+\alpha_0)D_b + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_a^{k+1} + pD_a^{-1}D_b^p + scD_b^s\}\frac{1}{f'(t)} - n\right){}_LH^{(p,s)}S_n(a,b,c) = 0,$$

which proves (32). Using the relation $D_b^p = D_a a D_a$ [6], we get

$$\left(\{(b+\alpha_0)D_b + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_b^{k+1} + paD_a + scD_b^s\}\frac{1}{f'(t)} - n\right){}_LH^{(p,s)}S_n(a,b,c) = 0.$$

□

3. Integro-Differential Equation for Laguerre–Gould–Hopper-Based Sheffer Polynomial (3VLGHSP)

Theorem 4. *The integro-differential equations of Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP)* ${}_{L_H^{(p,s)}}S_n(a,b,c)$ *are given as the follows equations*

$$(i)\quad \left(\{(b+\alpha_0)D_a + \sum_{k=1}^{n}\tfrac{\alpha_k}{k!}D_b^{-k(p-1)}D_a^{k+1} + pD_b^{((p-1)^2)}D_a^{(m-1)} + scD_b^{-(s-1)(1-p)}D_a^s\}\tfrac{1}{f'(t)} - (n+1)D_b^{p-1}\right){}_LH^{(p,s)}S_n(a,b,c) = 0. \tag{33}$$

$$(ii)\quad \left(\{(b+\alpha_0)D_c + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_b^{-k(s-1)}D_c^{k+1} + pD_b^{-(1-s)(1-p))}D_a^{-1}D_c^p + scD_b^{-(1-s)^2}D_c^s + sD_b^{-(1-s)^2}D_c^{s-1}\}\frac{1}{f'(t)} - (n+1)D_b^{s-1}\right){}_LH^{(p,s)}S_n(a,b,c) = 0. \tag{34}$$

$$(iii)\quad \left(\{(b+\alpha_0)D_a + \sum_{k=1}^{n}\tfrac{\alpha_k}{k!}D_aD_c^kD_b^{-k(s-1)} + pD_b^{-(1-s)(1-p))}D_c^{p-1} + scD_b^{-(1-s)^2}D_c^{s-1}D_a\}\tfrac{1}{f'(t)} - (n+1)D_b^{p-1}\right){}_LH^{(p,s)}S_n(a,b,c) = 0. \tag{35}$$

$$(iv)\quad \left(\{(b+\alpha_0)D_c + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_b^{-k(p-1)}D_a^kD_c + pD_b^{-(1-p)^2}D_cD_x^{p-2} + scD_b^{-(1-s)(1-p)}D_a^{s-1} + sD_b^{-(1-s)(1-p)}D_a^{s-1}D_c\}\frac{1}{f'(t)} - (n+1)D_b^{s-1}\right){}_LH^{(p,s)}S_n(a,b,c) = 0. \tag{36}$$

Proof. By using the factorization method

$$\pounds_{n+1}^{-}\pounds_n^{+}\{{}_LH^{(p,s)}S_n(a,b,c)\} = {}_LH^{(p,s)}S_n(a,b,c). \tag{37}$$

By using the shift operators expressions (14); (16) and (17); (19) in Equation (37), we find the integro-differential Equations (33) and (34), respectively. By the same process, taking the pair of shift operators (14); (16) and (19), (17) in the relation (37) and we find the integro-differential Equations (35) and (36) respectively. □

Remark 1. *The partial differential equations for the Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP)* ${}_LH^{(p,s)}S_n(a,b,c)$ *are calculated as the following consequence of Theorem 4.*

Corollary 1. *By differentiating the Equation (33) integro-differential equation $n(p-1)$-times with respect to b then we get a partial differential equation of Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP)*

$$(i) \quad \Big(\{(b+\alpha_0)D_a D_b^{n(p-1)} + n(p-1)D_b^{np-n-1}D_a + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_b^{n-k(p-1)}D_a^{k+1} + pD_b^{(p-1)(n-p+1)}D_a^{(p-1)} + scD_b^{(p-1)(n-s+1)}D_a^{s}\}\frac{1}{f'(t)} - (n+1)D_b^{p-1}\Big)\,{}_LH^{(p,s)}S_n(a,b,c) = 0. \tag{38}$$

By differentiating Equation (34) integro-differential equation $n(s-1)$-times with respect to b, then we get a partial differential equation of Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP)

$$(ii) \quad \Big(\{(b+\alpha_0)D_c D_b^{n(s-1)} + n(s-1)D_b^{ns-n-1}D_c + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_b^{(n-k)(s-1)}D_c^{k+1} + pD_a^{-1}D_b^{(s-1)(n-p+1)}D_c^{p} + scD_b^{(r-1)(n-s+1)}D_c^{s} + sD_b^{-(1-s)^2+n(s-1)}D_c^{s-1}\}\frac{1}{f'(t)} - (n+1)D_b^{(s-1)+n(s-1)}\Big)\,{}_LH^{(p,s)}S_n(a,b,c) = 0. \tag{39}$$

By differentiating Equation (35) integro-differential equation $n(s-1)$-times with respect to b, then we get a partial differential equation of Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP)

$$(iii) \quad \Big(\{(b+\alpha_0)D_a D_b^{n(s-1)} + n(s-1)D_b^{ns-n-1}D_a + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_a D_b^{(n-k)(s-1)}D_c^{k} + pD_b^{(s-1)(n-p+1)}D_c^{p-1} + scD_b^{(s-1)(n-s+1)}D_c^{s-1}D_a\}\frac{1}{f'(t)} - (n+1)D_b^{(p-1)+n(s-1)}\Big)\,{}_LH^{(p,s)}S_n(a,b,c) = 0. \tag{40}$$

By differentiating equation of (36) integro-differential equation $n(p-1)$-times with respect to b, we get a partial differential equation of Laguerre–Gould–Hopper-based Sheffer polynomials (3VLGHSP)

$$(iv) \quad \Big(\{(b+\alpha_0)D_c D_b^{n(p-1)} + n(p-1)D_b^{np-n-1}D_c + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_b^{(n-k)(p-1)}D_a^{k}D_c + pD_b^{(p-1)(n-p+1)}D_c D_a^{p-2} + scD_b^{(p-1)(n-s+1)}D_a^{s-1} + sD_b^{(p-1)(n-s+1)}D_a^{s-1}D_c\}\frac{1}{f'(t)} - (n+1)D_b^{n(s-1)+(p-1)}\Big)\,{}_LH^{(p,s)}S_n(a,b,c) = 0. \tag{41}$$

Proof. Differentiating the integro-differential Equation (33) by $n(p-1)$-times with respect to b, then we get a partial differential equation of Laguerre–Gould–Hopper-based Sheffer polynomials and differentiating the Equation (34) by $n(s-1)$-times, with respect to b; then, we get a partial differential equation of Laguerre–Gould–Hopper-based Sheffer polynomials. By the same process, the partial differential Equation (40) can be obtained by taking the derivatives of the integro-differential Equation (35) $n(s-1)$-times with respect to b and Equation (41) can be obtained by taking the derivatives of the integro-differential Equation (36) $n(p-1)$-times with respect to b. □

4. Integral Equation of the Laguerre–Gould–Hopper-Based Sheffer Polynomial (3VLGHSP)

Here, we derive the integral equations for the Laguerre–Gould–Hopper-based Sheffer polynomial. The importance of these integral equations of the Laguerre–Gould–Hopper-based Sheffer polynomials may be observed in different engineering sciences.

Theorem 5. *Following homogeneous volterra integral equation for the Lagurre–Gould–Hopper-based Sheffer polynomials (3VLGHSP)* ${}_{_LH^{(p,s)}}S_n(a,b,c)$ *holds true*

$$\begin{aligned}\phi(b,c) = &-\tfrac{\alpha_1}{sc}\Big(\mathbb{P}_{s-2}\tfrac{b^{s-3}}{(s-3)!}+\mathbb{P}_{s-3}\tfrac{b^{s-4}}{(s-4)!}+\ldots+\mathbb{P}_2(b)+\mathbb{P}_1\Big)\\ &-\tfrac{\alpha_0}{sc}\Big(\mathbb{P}_{s-2}\tfrac{b^{s-2}}{(s-2)!}+\mathbb{P}_{s-3}\tfrac{b^{s-3}}{(s-3)!}\\ &\quad+\ldots+\mathbb{P}_2\tfrac{a^2}{2!}+\mathbb{P}b+n\mathbb{R}_{n-1}\Big)-\tfrac{1}{sc}\Big(\mathbb{P}_{s-2}\tfrac{b^{s-1}}{(s-1)!}+\mathbb{P}_{s-3}\tfrac{b^{s-2}}{(s-2)!}\\ &\quad+\ldots+\mathbb{P}_2\tfrac{b^3}{2!}\mathbb{P}_1b^2+n\mathbb{R}n-1b\Big)\\ &+\tfrac{nf'(t)-paD_a}{sc}\Big(\mathbb{P}_{s-2}\tfrac{b^{s-1}}{(s-1)!}+\mathbb{P}_{s-3}\tfrac{b^{s-2}}{(s-2)!}\\ &\quad+\ldots+\mathbb{P}_2\tfrac{b^3}{3!}+\mathbb{P}_1\tfrac{b^2}{2!}+n\mathbb{R}_{n-1}b+\mathbb{R}_n\Big)\\ &-\tfrac{1}{sc}\int_0^b\Big(\alpha_1\tfrac{(b-\xi)^{s-3}}{(s-3)!}+(b+\alpha_0)\tfrac{(b-\xi)^{s-2}}{(s-2)!}\\ &\quad+(paD_a-nf'(t))\tfrac{(b-\xi)^{s-1}}{(s-1)!}\Big)\phi(\xi,c)d\xi,\end{aligned} \tag{42}$$

where

$${}_{_LH^{(p,s)}}S_n(0,b,c)=S_n(b,c)={}_{H^s}S_n(b,c)=\mathbb{R}_n,$$

$$\mathbb{P}_{s-q}=\prod_{k=0}^{s-q}(n-k)\mathbb{R}_{n-s+(q-1)},\qquad q=s-1,s-2,\ldots 3,2.$$

Proof. Now, we take the differential equation with $k=1$,

$$\left([D_b^s+\{b+\alpha_0+\alpha_1D_b^2+paD_a\}\frac{1}{sc}]\frac{1}{f'(t)}-n\frac{1}{sc}\right){}_{_LH^{(p,s)}}S_n(a,b,c)=0, \tag{43}$$

from the generating function (6) with $a=0$

$$\begin{aligned}&\frac{1}{g(f^{-1}(t))}exp(bf^{-1}(t)+D_a^{-1}(f'(t))^p+c(f'(t))^s)\\ &\qquad=\sum_{n=0}^{\infty}{}_{_LH^{(p,s)}}S_n(0,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}{}_{H^s}S_n(b,c)\frac{t^n}{n!}.\end{aligned}$$

Since, by $A(t)=\frac{1}{g(f^{-1}(t))}$ ([26], (p. 923)) and expanding the exponential in the left-hand side and then using the cauchy product rule in the left-hand side in the resultant equation, we have the following presentation of ${}_{H^s}S_n(b,c)$

$${}_{H^s}S_n(b,c)=n!\sum_{k=0}^{n}\sum_{l=0}^{\frac{k}{s}}\frac{S_{n-k}c^l\Upsilon^{k-sl}}{(n-k)!l!(k-sl)!}. \tag{44}$$

The initial condition obtained

$$ {}_{L}H^{(p,s)}S_n(0,b,c) = S_n(b,c) = {}_{H^s}S_n(b,c) = \mathbb{R}_n, $$

and letting ${}_{H^s}S_n(b,c) = \mathbb{R}_n$

$$ \frac{d}{db}{}_{L}H^{(p,s)}S_n(0,b,c) = D_b\{{}_{H^s}S_n(b,c)\} = n\mathbb{R}_{n-1}, $$

$$ \begin{aligned} \frac{d^2}{d^2b}{}_{L}H^{(p,s)}S_n(0,b,c) &= D_b^2\{{}_{H^s}S_n(b,c)\} \\ &= n(n-1)\mathbb{R}_{n-2} = \prod_{k=0}^{1}(n-k)\mathbb{R}_{n-2} = \mathbb{P}_1. \end{aligned} $$

$$ \begin{aligned} D_b^{s-2}\{{}_{L}H^{(p,s)}S_n(0,b,c)\} &= n(n-1)(n-2)...(n-s+3)\mathbb{R}_{n-s+2} \\ &= \mathbb{P}_{s-3} = \prod_{k=0}^{s-3}(n-k)\mathbb{R}_{n-s+2}, \end{aligned} $$

$$ \begin{aligned} D_b^{s-1}\{{}_{L_H^{p,s}}S_n(0,b,c)\} &= n(n-1)(n-2)...(n-s+1)\mathbb{R}_{n-s+1} \\ &= \mathbb{P}_{s-2} = \textstyle\prod_{k=0}^{s-2}(n-k)\mathbb{R}_{n-s+1}. \end{aligned} \tag{45} $$

Consider the equation

$$ D_b^s\{{}_{L_H^{p,s}}S_n(a,b,c)\} = \phi(b,c). \tag{46} $$

Now, integrate the Equation (46) with the initial conditions

$$ D_b^{s-1}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = \int_0^b \phi(\xi,c)d\xi + \mathbb{P}_{s-2} $$

$$ D_b^{s-2}\{{}_{L}H^{(p,s)}S_n(a,b,c)\} = \int_0^a \phi(\xi,c)d^2\xi + \mathbb{P}_{s-3} $$

$$ \begin{aligned} D_b^2\{{}_{L}H^{(p,s)}S_n(a,b,c)\} &= \int_0^b \phi(\xi,c)d\xi^{s-2} + \mathbb{P}_{s-2}\frac{b^{s-3}}{(s-3)!} + \mathbb{P}_{s-3}\frac{b^{s-4}}{(s-4)!} \\ &+ ... + \mathbb{P}_2 b + \mathbb{P}_1 \end{aligned} $$

$$ \begin{aligned} D_b\{{}_{L}H^{(p,s)}S_n(a,b,c)\} &= \int_0^b \phi(\xi,c)d\xi^{s-1} + \mathbb{P}_{s-2}\frac{b^{s-2}}{(s-2)!} + \mathbb{P}_{s-3}\frac{b^{s-3}}{(s-3)!} \\ &+ ... + \mathbb{P}_1 b + ... + n\mathbb{R}n-1, \end{aligned} $$

$$ \begin{aligned} {}_{L}H^{(p,s)}S_n(a,b,c) &= \textstyle\int_0^b \phi(\xi,c)d\xi^s + \mathbb{P}_{s-2}\frac{b^{s-1}}{(s-1)!} + \mathbb{P}_{s-3}\frac{b^{s-2}}{(s-2)!} \\ &+ ... + \mathbb{P}_1\frac{b^2}{2!} + n\mathbb{R}_{n-1}x + \mathbb{R}_n, \end{aligned} \tag{47} $$

where

$$ \mathbb{P}_{s-q} = \prod_{k=0}^{s-q}(n-k)\mathbb{R}_{n-s+(q-1)}, \qquad q = s-1, s-2, ...3, 2. \tag{48} $$

Using the Equation (4) in Equation (43), we get

$$\begin{aligned}\phi(b,c) = &-\frac{(b+\alpha_0)}{sc}\bigg(\int_0^b \phi(\xi,c)d\xi^{s-1} + \mathbb{P}_{s-2}\frac{b^{s-2}}{(s-2)!} + \mathbb{P}_{s-3}\frac{b^{s-3}}{(s-3)!} \\ &+ ... + \mathbb{P}_2\frac{b^2}{2!} + \mathbb{P}_1\frac{b}{1!} + n\mathbb{R}_{n-1}\bigg) \\ &-\frac{\alpha_1}{sc}\bigg(\int_0^b \phi(\xi,c)d\xi^{s-2} + \mathbb{P}_{s-2}\frac{b^{s-3}}{(s-3)!} \\ &+\mathbb{P}_{s-3}\frac{b^{s-4}}{(s-4)!} + ... + \mathbb{P}_2 b + \mathbb{P}_1\bigg) \\ &+\frac{(paD_a - nf'(t))}{sc}\bigg(\int_0^b \phi(\xi,c)d\xi + \mathbb{P}_{s-2}\frac{b^{s-1}}{(s-1)!} + \mathbb{P}_{s-3}\frac{b^{s-2}}{(s-2)!} \\ &+ ... + \mathbb{P}_1\frac{b^2}{2!} + n\mathbb{R}_{n-1}b + \mathbb{R}_n\bigg),\end{aligned}$$

which proves assertion (42). □

5. Conclusions

We derived differential equations, shift operators, and integral equations for the Laguerre–Gould–Hopper-based Sheffer polynomial by the factorization method in the present investigation. The development of these types of techniques may be useful in different scientific areas. The Laguerre–Gould–Hopper-based Sheffer polynomial and for their relatives can be taken in further investigations of mathematical and engineering sciences. Specifically, the integral, differential and integro-differential equations are of particular motivation of this paper for the applications purpose in engineering sciences.

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References

1. Sheffer, I.M. Some properties of polynomials sets of type zero. *Duke Math. J.* **1939**, *5*, 590–622. [CrossRef]
2. Kim, D.S.; Kim, T. Degenerate Sheffer sequences and λ-Sheffer sequences. *J. Math. Anal. Appl.* **2021**, *493*, 124521. [CrossRef]
3. Kim, T.; Kim, D.S. On λ-Bell polynomials associated with umbral calculus. *Russ. J. Math. Phys.* **2017**, *24*, 69–78. [CrossRef]
4. Roman, S. *The Umbral Calculus*; Springer: New York, NY, USA, 2005.
5. Appell, P. Sur une classe de polynômes. *Ann. Sci. Éc. Norm. Supér Appl.* **1880**, *9*, 119–144. [CrossRef]
6. Khan, S.; Al-Gonah, A.A. Operational methods and Laguerre-Gould-Hopper polynomials. *Comput. Appl. Math.* **2012**, *218*, 9930–9942. [CrossRef]
7. Raza, N.; Khan, S.; Ali, M. Properties of certain new special polynomials associated with Sheffer sequences. *Tbilisi Math. J.* **2016**, *9*, 245–270. [CrossRef]
8. Choi, J.; Sabee, S.; Shadab, M. Some identities associated with 2-variable truncated exponential based Sheffer polynomial sequences. *Commun. Korean Math. Soc.* **2020**, *35*, 533–546.
9. Khan, N.U.; Ghayasuddin, M.; Shadab, M. Some generating relations of extended Mittag-Leffler function. *Kyungpook Math. J.* **2019**, *59*, 325–333.

10. Marcellan, F.; Jabee, S.; Sahdab, M. Analytic properties of Touchard based hybrid polynomials via operational techniques. *Bull. Malaysian Math. Sci. Soc.* **2021**, *44*, 223–242. [CrossRef]
11. Andrews, L.C. *Special Functions for Engineers and Applied Mathematicians*; Macmillan: New York, NY, USA, 1985.
12. He, M.X.; Ricci, P.E. Differential equation of the Appell polynomials via factorization method. *J. Comput. Appl. Math.* **2002**, *139*, 231–237. [CrossRef]
13. Ozarslan, M.A.; Yilmaz, B. A set of finite order differential equations for the Appell polynomials. *J. Comput. Appl. Math.* 2000, 118, 111–123.
14. Khan, S.; Riyasat, M. Differential and integral equation for 2-iterated Appell polynomials. *J. Comput. Appl. Math. Anal. Appl.* **2012**, *387*, 116–132. [CrossRef]
15. Khan, S.; Yasmin, G.; Ahmad, N. On a new family related to truncated Exponential and Sheffer polynomials. *J. Math. Anal. Appl.* **2014**, *418*, 921–937. [CrossRef]
16. Srivastava, H.M.; Ozarslan, M.A.; Yilmaz, B. Some families of differential equation associated with the hermite based Appell polynomials and other classes of the hermite-based polynomials. *Filomat* **2014**, *28*, 695–708. [CrossRef]
17. Yılmaz, B.; Özarslan, M.A. Differential equations for the extended 2D Bernoulli and Euler polynomials. *Adv. Differ. Equ.* **2013**, *2013*, 107. [CrossRef]
18. Khan, S.; Riyasat, M. Differential and integral equations for the 2-iterated Bernoulli, 2-iterated Euler and Bernoulli–Euler polynomials. *Georgian Math. J.* **2020**, *27*, 375–389. [CrossRef]
19. Wani, S.A.; Khan, S.; Naikoo, S.A. Differential and integral equations for the Laguerre–Gould–Hopper-based Appell and related polynomials. *Bol. Soc. Mat. Mex.* **2020**, *26*, 617–646. [CrossRef]
20. Dattoli, G. Hermite–Bessel and Laguerre–Bessel functions: A by-product of the monomiality principle. In *Advanced Special Functions and Applications, Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics, Melfi, Italy, 9–12 May 1999*; Aracne: Rome, Italy, 2000; pp. 147–164.
21. Cesarano, G.D.C.; Sacchetti, D. A note on the monomiality principle and generalized polynomials. *J. Math. Anal. Appl.* **1997**, *227*, 98–111.
22. Dattoli, G.; Cesarano, C.; Sacchetti, D. A note on truncated polynomials. *Comput. Appl. Math.* **2003**, *132*, 595–605. [CrossRef]
23. Dattoli, G.; Migliorati, M.; Srivastava, H.M. A class of Bessel summation formulas and associated operational method. *Frac. Calcu. Appl. Anal.* **2004**, *7*, 169–176.
24. Steffensen, J.F. The poweroid an extension of the mathematical notion of power *Acta Math.* **1941**, *73*, 333–366. [CrossRef]
25. Infeld, L.; Hull, T.E. The factorization method. *Rev. Mod. Phys.* **1951**, *23*, 21–68. [CrossRef]
26. Khan, S.; Yasmin, G.; Wani, S.A. Diffential and integral equations for Legendre-Laguerre based hybrid polynomials. *Ukrainian Math. J.* **2021**, *73*, 408–424. [CrossRef]