fractal and fractional

Article

The Grüss-Type and Some Other Related Inequalities via Fractional Integral with Respect to Multivariate Mittag-Leffler Function

Yabin Shao [1], Gauhar Rahman [2], Yasser Elmasry [3,4], Muhammad Samraiz [5], Artion Kashuri [6] and Kamsing Nonlaopon [7,*]

1 School of Science, Chongqing University of Posts and Telecommunications, Nan'an, Chongqing 400065, China
2 Department of Mathematics and Statistics, Hazara University, Mansehra 21300, Pakistan
3 Department of Mathematics, Faculty of Science, King Khalid University, P.O. Box 9004, Abha 61466, Saudi Arabia
4 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
5 Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan
6 Department of Mathematics, Faculty of Technical and Natural Sciences, University "Ismail Qemali", 9400 Vlora, Albania
7 Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
* Correspondence: nkamsi@kku.ac.th

Abstract: In the recent era of research, the field of integral inequalities has earned more recognition due to its wide applications in diverse domains. The researchers have widely studied the integral inequalities by utilizing different approaches. In this present article, we aim to develop a variety of certain new inequalities using the generalized fractional integral in the sense of multivariate Mittag-Leffler (M-L) functions, including Grüss-type and some other related inequalities. Also, we use the relationship between the Riemann-Liouville integral, the Prabhakar integral, and the generalized fractional integral to deduce specific findings. Moreover, we support our findings by presenting examples and corollaries.

Keywords: fractional integrals; generalized fractional integrals; Prabhakar integral; Mittag-Leffler function; inequalities

Citation: Shao, Y.; Rahman, G.; Elmasry, Y.; Samraiz, M.; Kashuri, A.; Nonlaopon, K. The Grüss-Type and Some Other Related Inequalities via Fractional Integral with Respect to Multivariate Mittag-Leffler Function. *Fractal Fract.* **2022**, *6*, 546. https://doi.org/10.3390/fractalfract6100546

Academic Editor: Mahamadi Warma

Received: 28 August 2022
Accepted: 21 September 2022
Published: 27 September 2022

1. Introduction

The field of fractional calculus is the branch of mathematical analysis which deals with the study of arbitrary order integrals and derivatives. In the last few years, this field has gained more recognition and significance due to its wide applications in diverse domains. The researchers have considered that this field is the most powerful tool in determining the anomalous kinetics and its wide applications in diverse domains. Several problems such as statistical, mathematical, engineering, chemical, and biological can be easily modelled by employing ordinary differential equations containing fractional derivatives. The researchers have extensively studied a variety of types of fractional integrals and derivatives operators such as Riemann-Liouville, Caputo, Riesz, Hilfer, Hadamard, Erdélyi-Kober, Saigo, Marichev-Saigo-Maeda and so on. We suggest the readers to see [1–4].

Khalil et al. [5] proposed the notion of fractional conformable derivatives operators. Abdeljawad [6] gave the properties of the fractional conformable derivative operators. Jarad et al. [7] proposed the fractional conformable integral and derivative operators. Anderson and Unless [8] propose the idea of the conformable derivative by considering local proportional derivatives. Abdeljawad and Baleanu [9] investigated certain monotonicity results for fractional difference operators with discrete exponential kernels. In [10], Abdeljawad and Baleanu proposed the fractional derivative operator involving an exponential kernel and their discrete version. Atangana and Baleanu [11] proposed a new fractional derivative operator with the non-local and non-singular kernel. Fractional derivative without a singular kernel can be found in the work of Caputo and Fabrizio [12].

Recently, the researchers have studied the field of fractional calculus extensively and developed certain new and interesting fractional integral and derivative operators. These new operators have gained more attention from researchers due to their wide applications in the field of both applied and pure. Inequalities are well recognised to have potential uses in technology, scientific research, and analysis as well as in a wide range of mathematical topics including approximation theory, statistical analysis, and the social sciences; see for example [13–15]. Regarding wider uses, these versions have received a lot of attention. Authors have now presented a new version of these inequalities, which may be useful in the research of various integro-differential and difference equation forms. Sousa et al. [16] presented Grüss-type and some other integral inequalities by employing the Katugampola fractional operator. In particular, many remarkable inequalities, properties and applications for the fractional conformable integrals and generalized proportional integrals can be found in the literature [17–19].

Alzabut et al. and Rahman et al. [20–23] explored the modified proportional derivative and integral operators recently, and they produced certain Gronwall inequality and the Minkowski inequalities that include the above proportional fractional operators.

2. Preliminaries

In this section, recalling the following well-known results:

Theorem 1. [24] *Let $\hbar_1, \hbar_2 : [r,s] \to \mathbb{R}$ be two positive functions with $\mathfrak{m} \leq \hbar_1(\xi) \leq \mathcal{M}$ and $\mathfrak{n} \leq \hbar_2(\xi) \leq \mathcal{N}$ for all $\xi \in [r,s]$, then the following inequality holds:*

$$\left| \frac{1}{s-r} \int_r^s \hbar_1(\xi)\hbar_2(\xi)d\xi - \frac{1}{s-r} \int_r^s \hbar_1(\xi)d\xi \frac{1}{s-r} \int_r^s \hbar_2(\xi)d\xi \right| \leq \frac{1}{4}(\mathcal{M}-\mathfrak{m})(\mathcal{N}-\mathfrak{n}), \quad (1)$$

where $\mathcal{M}, \mathfrak{m}, \mathcal{N}, \mathfrak{n} \in \mathbb{R}$ and $\frac{1}{4}$ is the best constant such that the inequality (1) is sharp.

Definition 1 ([25,26])**.** *A function $\hbar_1(\xi)$ is said to be in the $L_{p,r}[0,\infty)$ space if*

$$L_{p,r}[0,\infty) = \left\{ \hbar_1 : ||\hbar_1||_{L_{p,r}[0,\infty)} = \left(\int_r^s |\hbar_1(\xi)|^p \xi^r d\xi \right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty, r \geq 0 \right\}. \quad (2)$$

If we apply (2) for $r = 0$, then it follows

$$L_p[0,\infty) = \left\{ \hbar_1 : ||\hbar_1||_{L_p[0,\infty)} = \left(\int_r^s |\hbar_1(\xi)|^p d\xi \right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty \right\}.$$

Definition 2 ([27])**.** *Let $n \in N$, $\zeta_1, \sigma_1, \gamma \in \mathbb{C}$, $\Re(\zeta_1) > 0$, $\Re(\sigma_1) > 0$ and $\Re(\gamma_1) > 0$, then the three parameter M-L function is given by*

$$\mathcal{E}^{\gamma_1}_{\sigma_1,\zeta_1}(z_1) = \sum_{l_1=0}^{\infty} \frac{(\gamma_1)_n (z_1)^{l_1}}{\Gamma(\sigma_1 l_1 + \zeta_1) l_1!}.$$

Definition 3 ([28])**.** *The multivariate M-L function is defined as*

$$\begin{aligned} \mathcal{E}^{(\gamma_j)}_{(\sigma_j),\zeta}(z_1, z_2, \ldots, z_j) &= \mathcal{E}^{(\gamma_1,\gamma_2,\ldots,\gamma_j)}_{(\sigma_1,\sigma_2,\ldots,\sigma_j),\zeta}(z_1, z_2, \ldots z_j) \\ &= \sum_{m_1,m_2\ldots m_j=0}^{\infty} \frac{(\gamma_1)_{m_1}(\gamma_2)_{m_2} \ldots (\gamma_j)_{m_j} (z_1)^{m_1} \ldots (z)^{m_j}}{\Gamma(\sigma_1 m_1 + \sigma_2 m_2 + \ldots \sigma_j m_j + \zeta) m_1! \ldots m_j!}, \end{aligned} \quad (3)$$

where $z_i, \sigma_i, \zeta, \gamma_i \in \mathbb{C}$; $i = 1, 2, \ldots, j$, $\Re(\sigma_i) > 0$, $\Re(\zeta) > 0$ *and* $\Re(\gamma_i) > 0$.

The M-L functions with different parameters that have been extensively studied by [29–31] and the references cited therein.

Definition 4 ([1,2])**.** *The Riemann–Liouville (R-L) fractional integral (left and right sided)* ${}_{x_1}\mathcal{I}^{\zeta}$ *and* ${}_{x_2}\mathcal{I}^{\zeta}$ *of order* $\zeta > 0$*, is defined by*

$$\left({}_{x_1}\mathcal{I}^{\zeta}\hbar_2\right)(v) = \frac{1}{\Gamma(\zeta)}\int_{x_1}^{v}(v-\nu)^{\zeta-1}\hbar_2(\nu)d\nu,\ x_1 < v,$$

and

$$\left({}_{x_2}\mathcal{I}^{\zeta}\hbar_2\right)(v) = \frac{1}{\Gamma(\zeta)}\int_{v}^{x_2}(\nu-v)^{\zeta-1}\hbar_2(\nu)d\nu,\ x_2 > v.$$

Definition 5 ([27])**.** *The Prabhakar type fractional integral is defined by*

$$({}_{x_1}\mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\hbar_2)(v) = \int_{x_1}^{v}(v-\nu)^{\zeta-1}\mathcal{E}^{\gamma}_{\sigma,\zeta}(\lambda(v-\nu)^{\sigma})\hbar_2(\nu)d\nu.$$

Definition 6. *The one-sided Prabhakar type fractional integral is defined by*

$$(\mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\hbar_2)(v) = \int_{0}^{v}(v-\nu)^{\zeta-1}\mathcal{E}^{\gamma}_{\sigma,\zeta}(\lambda(v-\nu)^{\sigma})\hbar_2(\nu)d\nu. \tag{4}$$

Definition 7 ([28,32])**.** *The Prabhakar integral operator having multivariate M-L function in the kernel is defined by*

$$\begin{aligned}\left({}_{x_1}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2\right)(\xi) &= \left({}_{x_1}\mathcal{I}^{(\gamma_1,\cdots,\gamma_j),(\lambda_1,\cdots,\lambda_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}\hbar_2\right)(\xi)\\ &= \int_{x_1}^{\xi}(\xi-\nu)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\nu)^{\sigma_1}\cdots\lambda_j(\xi-\nu)^{\sigma_j})\hbar_2(\nu)d\nu,\end{aligned}$$

where $\zeta, \sigma_i, \lambda_i, \gamma_i \in \mathbb{C}$, $\Re(\sigma_i) > 0$, $\Re(\zeta) > 0$, $\Re(\gamma_i) > 0$ *for* $i = 1, 2, \ldots, j$.

Definition 8. *The one-sided Prabhakar integral operator having multivariate M-L function in the kernel is defined by*

$$\begin{aligned}\left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2\right)(\xi) &= \left(\mathcal{I}^{(\gamma_1,\cdots,\gamma_j),(\lambda_1,\cdots,\lambda_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}\hbar_2\right)(\xi)\\ &= \int_{0}^{\xi}(\xi-\nu)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\nu)^{\sigma_1}\cdots\lambda_j(\xi-\nu)^{\sigma_j})\hbar_2(\nu)d\nu,\end{aligned} \tag{5}$$

where $\zeta, \sigma_i, \lambda_i, \gamma_i \in \mathbb{C}$, $\Re(\sigma_i) > 0$, $\Re(\zeta) > 0$, $\Re(\gamma_i) > 0$ *for* $i = 1, 2, \ldots, j$.

Remark 1. *i. If we take* $m_i = 0$ *for* $i = 2, 3, \ldots, j$*, then* (5) *reduce to* (4)*. ii. If we consider one of* $\lambda_i = 0$*, then* (5) *reduce to the classical R-L fractional integral* $\Gamma(\zeta)\left({}_{x_1}\mathcal{I}^{\zeta}\hbar_2\right)(v)$.

The objective of this article is to establish integral inequalities such as Grüss-type and several other related inequalities by employing the generalized Prabhakar fractional integral (5). The mentioned inequalities via the Prabhakar operator containing the three parameters M-L function are discussed. Also, some examples and corollaries are discussed which are the special cases of our main results.

3. Grüss-Type Inequalities via Generalized Fractional Integral

In this section, we present generalization of certain inequalities by utilizing the integral operator (5) having the multi-parameters M-L function.

Theorem 2. *Let the function $\hbar_1$ be integrable on $[0,\infty)$. If the two functions $\aleph_1$ and $\aleph_2$ be integrable on $[0,\infty)$ such that*

$$\aleph_1(\xi) \leq \hbar_1(\xi) \leq \aleph_2(\xi),\ \xi \in [0,\infty). \tag{6}$$

Then, for $\xi \geq 0$, $\sigma_i, \zeta, \eta, \lambda_i > 0$ (where $i = 1,2,\dots,j$), we have

$$\begin{aligned}&\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\aleph_1(\xi)\\ \geq&\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\aleph_1(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi).\end{aligned} \tag{7}$$

Proof. Applying (6) for all $\varrho \geq 0$ and $v \geq 0$, we have

$$(\aleph_2(\varrho) - \hbar_1(\varrho))(\hbar_1(v) - \aleph_1(v)) \geq 0.$$

It follows that

$$\aleph_2(\varrho)\hbar_1(v) + \aleph_1(v)\hbar_1(\varrho) \geq \aleph_2(\varrho)\aleph_1(v) + \hbar_1(\varrho)\hbar_1(v). \tag{8}$$

Taking the product of $(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})$ with (8) and then the integration of the obtained inequality with respect to ϱ from 0 to ξ gives

$$\begin{aligned}&\hbar_1(v)\int_0^\xi(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\aleph_2(\varrho)d\varrho\\ &+\aleph_1(v)\int_0^\xi(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\hbar_1(\varrho)d\varrho\\ \geq&\aleph_1(v)\int_0^\xi(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\aleph_2(\varrho)d\varrho\\ &+\hbar_1(v)\int_0^\xi(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\hbar_1(\varrho)d\varrho,\end{aligned}$$

which in view of (5) follows

$$\begin{aligned}&\hbar_1(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi) + \aleph_1(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\\ \geq&\aleph_1(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi) + \hbar_1(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi).\end{aligned} \tag{9}$$

Again, taking the product of $(\xi-v)^{\eta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-v)^{\sigma_1}\cdots\lambda_j(\xi-v)^{\sigma_j})$ with (9) and then the integration of the obtained inequality with respect to v from 0 to ξ gives

$$\begin{aligned}&\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\aleph_1(\xi)\\ \geq&\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\aleph_1(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\,^{\Psi}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi),\end{aligned} \tag{10}$$

which proves the inequality (7). □

Corollary 1. *Let the function $\hbar_1$ be defined and integrable on $\xi \in [0, \infty)$ and satisfying $\mathfrak{m} \leq \hbar_1(\xi) \leq \mathfrak{M}$, $\xi \in [0, \infty)$. Then, for $\xi \geq 0$, $\sigma_i, \zeta, \eta, \lambda_i > 0$ (where $i = 1, 2, \ldots, j$), we have*

$$\begin{aligned}
&\mathfrak{M}\xi^{\zeta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi) + \mathfrak{m}\xi^{\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi) \\
\geq &\mathfrak{m}\mathfrak{M}\xi^{\zeta+\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j}) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi).
\end{aligned}$$

Example 1. *Let the function $\hbar_1$ be integrable on $\xi \in [0, \infty)$ and satisfying $\xi \leq \hbar_1(\xi) \leq \xi + 1$, $\xi \in [0, \infty)$. Then, for $\xi \geq 0$, $\sigma_i, \zeta, \lambda_i > 0$ (where $i = 1, 2, \ldots, j$) and put $\zeta = \eta$ in Theorem 2, we have*

$$\begin{aligned}
&\left[2\xi^{\zeta+1}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+2}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j}) + \xi^{\zeta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\right]\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi) \\
\geq &\left[\xi^{\zeta+1}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+2}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j}) + \xi^{\zeta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\right]\left(\xi^{\zeta+1}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+2}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\right) \\
+ &\left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\right)^2.
\end{aligned}$$

Theorem 3. *Let the two functions $\hbar_1$ & $\hbar_2$ be positive and integrable on $[0, \infty)$. Assume that (6) holds and the functions Υ_1, Υ_2 are integrable on $[0, \infty)$ such that*

$$\Upsilon_1(\xi) \leq \hbar_2(\xi) \leq \Upsilon_2(\xi),\ \xi \in [0, \infty). \tag{11}$$

Then for $\xi \geq 0$ and $\zeta, \eta, \sigma_i, \lambda_i > 0$, then the following four inequalities hold:

$$\begin{aligned}
&\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\Upsilon_1(\xi) \\
\geq &\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\Upsilon_1(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi),
\end{aligned}\tag{12}$$

$$\begin{aligned}
&\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\aleph_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\Upsilon_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi) \\
\geq &\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\Upsilon_2(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi),
\end{aligned}\tag{13}$$

$$\begin{aligned}
&\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\Upsilon_2(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi) \\
\geq &\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\Upsilon_2(\xi),
\end{aligned}\tag{14}$$

$$\begin{aligned}
&\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\Upsilon_1(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi) \\
\geq &\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi) + \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\Upsilon_1(\xi).
\end{aligned}\tag{15}$$

Proof. To derive (12), we use (6) and (11) for $\varrho, \upsilon \in [0, \infty)$ yield

$$(\aleph_2(\varrho) - \hbar_1(\varrho))(\hbar_2(\upsilon) - \Upsilon_1(\upsilon)) \geq 0.$$

It follows that

$$\aleph_2(\varrho)\hbar_2(\upsilon) + \hbar_1(\varrho)\Upsilon_1(\upsilon) \geq \aleph_2(\varrho)\Upsilon_1(\upsilon) + \hbar_1(\varrho)\hbar_2(\upsilon). \tag{16}$$

Multiplying $(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})$ with (16) and then taking the integration with respect to ϱ from 0 to ξ, we have

$$\begin{aligned}
&\hbar_2(v)\int_0^{\xi}(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\aleph_2(\varrho)d\varrho\\
&+\Upsilon_1(v)\int_0^{\xi}(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\hbar_1(\varrho)d\varrho\\
&\geq\Upsilon_1(v)\int_0^{\xi}(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\aleph_2(\varrho)d\varrho\\
&+\hbar_2(v)\int_0^{\xi}(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\hbar_1(\varrho)d\varrho,
\end{aligned}$$

which in view of (5) follows

$$\hbar_2(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)+\Upsilon_1(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\geq\Upsilon_1(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)+\hbar_2(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi). \tag{17}$$

Again, multiplying $(\xi-v)^{\eta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-v)^{\sigma_1}\cdots\lambda_j(\xi-v)^{\sigma_j})$ with (17), taking the integration with respect to v from 0 to ξ and utilizing (5) gives

$$\begin{aligned}
&\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi)+\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\Upsilon_1(\xi)\\
&\geq\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\aleph_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\Upsilon_1(\xi)+\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi)
\end{aligned}$$

which completes the inequality (12). Similarly, one can prove (13)–(15) by utilizing the following results:

$$(\Upsilon_2(\varrho)-\hbar_2(\varrho))(\hbar_1(v)-\aleph_1(v))\geq 0,$$

$$(\aleph_2(\varrho)-\hbar_1(\varrho))(\hbar_2(v)-\Upsilon_2(v))\leq 0$$

and

$$(\aleph_1(\varrho)-\hbar_1(\varrho))(\hbar_2(v)-\Upsilon_1(v))\leq 0,$$

respectively. □

Corollary 2. *Let the two functions $\hbar_1$ & $\hbar_2$ be integrable and positive on $[0,\infty)$ and satisfying $\mathfrak{m}\leq\hbar_1(\xi)\leq\mathfrak{M}$ and $\mathfrak{n}\leq\hbar_2(\xi)\leq\mathfrak{N}$; $\xi\in[0,\infty)$. Then, for $\xi\geq 0$, $\sigma_i,\zeta,\eta,\lambda_i>0$ (where $i=1,2,\ldots,j$), we have*

$$\begin{aligned}
&\mathfrak{M}\xi^{\zeta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi)+\mathfrak{n}\xi^{\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\\
&\geq\mathfrak{M}\mathfrak{n}\xi^{\zeta+\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})+\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi),
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{m}\xi^{\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2(\xi)+\mathfrak{N}\xi^{\zeta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\\
&\geq\mathfrak{m}\mathfrak{N}\xi^{\zeta+\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})+\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi),
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{M}\mathfrak{N}\xi^{\zeta+\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})+\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi)\\
&\geq\mathfrak{M}\xi^{\zeta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi)+\mathfrak{N}\xi^{\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi),
\end{aligned}$$

$$\mathfrak{m}\mathfrak{n}\xi^{\zeta+\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})+\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi)$$
$$\geq\mathfrak{m}\xi^{\zeta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi)+\mathfrak{n}\xi^{\eta}\mathcal{E}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta+1}(\lambda_1\xi^{\sigma_1}\cdots\lambda_j\xi^{\sigma_j})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi).$$

4. Some Other Related Inequalities via the Generalized Prabhakar Integral

In this section, we establish certain other inequalities which involve the generalized Prabhakar integral (5) containing multivariate perimeters.

Theorem 4. *Let the two functions $\hbar_1$ and $\hbar_2$ be positive and integrable on $[0,\infty)$. If $p_1,q_1>1$ be such that $\frac{1}{p_1}+\frac{1}{q_1}=1$. For $\xi\geq 0$, we have*

$$\frac{1}{p_1}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1^{p_1}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^{p_1}(\xi)+\frac{1}{q_1}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2^{q_1}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^{q_1}(\xi)$$
$$\geq\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\hbar_2(\xi)\,\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi)\hbar_1(\xi), \tag{18}$$

$$\frac{1}{p_1}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1^{p_1}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^{q_1}(\xi)+\frac{1}{q_1}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2^{q_1}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^{p_1}(\xi)$$
$$\geq\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^{q_1-1}(\xi)\hbar_1^{p_1-1}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\hbar_2(\xi), \tag{19}$$

$$\frac{1}{p_1}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1^{p_1}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^{2}(\xi)+\frac{1}{q_1}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2^{q_1}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^{2}(\xi)$$
$$\geq\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^{\frac{2}{p_1}}(\xi)\hbar_2^{\frac{2}{q_1}}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\hbar_2(\xi) \tag{20}$$

and

$$\frac{1}{p_1}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1^{2}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^{q_1}(\xi)+\frac{1}{q_1}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2^{2}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^{p_1}(\xi)$$
$$\geq\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^{p_1-1}(\xi)\hbar_2^{q_1-1}(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1^{\frac{2}{p_1}}(\xi)\hbar_2^{\frac{2}{q_1}}(\xi), \tag{21}$$

where $\sigma_i,\lambda_i,\zeta,\gamma_i,\eta>0$.

Proof. To prove (18), we use the following Young's inequality [33] given by:

$$\frac{1}{p_1}u^{p_1}+\frac{1}{q_1}v^{q_1}\geq uv,\ u,v>0,\ \frac{1}{p_1}+\frac{1}{q_1}=1. \tag{22}$$

Applying (22) for $u=\hbar_1(\varrho)\hbar_2(v)$ and $v=\hbar_1(v)\hbar_2(\varrho)$, $\varrho,v>0$, we have

$$\frac{1}{p_1}(\hbar_1(\varrho)\hbar_2(v))^{p_1}+\frac{1}{q_1}(\hbar_1(v)\hbar_2(\varrho))^{q_1}\geq(\hbar_1(\varrho)\hbar_2(v))(\hbar_1(v)\hbar_2(\varrho)). \tag{23}$$

Multiplying $(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})$ with (23) and taking the integration with respect to ϱ from 0 to ξ gives

$$\frac{\hbar_2^{p_1}(v)}{p_1}\int_0^{\xi}\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\hbar_1^{p_1}(\varrho)d\varrho$$
$$+\frac{\hbar_1^{q_1}(v)}{q_1}\int_0^{\xi}\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\hbar_2^{q_1}(\varrho)d\varrho$$
$$\geq\hbar_1(v)\hbar_2(v)\int_0^{\xi}\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\hbar_1(\varrho)\hbar_2(\varrho)d\varrho$$

which in view of (5) follows

$$\frac{\hbar_2^{p_1}(\upsilon)}{p_1}\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{p_1}(\xi)+\frac{\hbar_1^{q_1}(\upsilon)}{q_1}\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{q_1}(\xi)\geq\hbar_1(\upsilon)\hbar_2(\upsilon)\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_1(\xi)\hbar_2(\xi). \tag{24}$$

Now, multiplying $(\xi-\upsilon)^{\eta-1}\mathcal{E}_{(\sigma_1,\cdots,\sigma_j),\eta}^{(\gamma_1,\cdots,\gamma_j)}(\lambda_1(\xi-\upsilon)^{\sigma_1}\cdots\lambda_j(\xi-\upsilon)^{\sigma_j})$ with (24), taking the integration with respect to υ from 0 to ξ and using (5), we have

$$\begin{aligned}&\frac{1}{p_1}\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{p_1}(\xi)\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{p_1}(\xi)+\frac{1}{q_1}\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{q_1}(\xi)\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{q_1}(\xi)\\ \geq&\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_1(\xi)\hbar_2(\xi)\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_1(\xi)\hbar_2(\xi),\end{aligned}$$

which gives the inequality (18). The inequalities (19), (20) and (21) can be easily derived by substituting the following identities in (22), respectively.

$$u=\frac{\hbar_1(\varrho)}{\hbar_1(\upsilon)},\ \ \upsilon=\frac{\hbar_2(\varrho)}{\hbar_2(\upsilon)},\ \hbar_1(\upsilon),\hbar_2(\upsilon)\neq 0, \tag{25}$$

$$u=\hbar_1(\varrho)\hbar_2^{\frac{2}{p_1}}(\upsilon),\ \ \upsilon=\hbar_1^{\frac{2}{q_1}}(\upsilon)\hbar_2(\varrho) \tag{26}$$

and

$$u=\hbar_1^{\frac{2}{p_1}}(\varrho)\hbar_1(\upsilon),\ \ \upsilon=\hbar_2^{\frac{2}{q_1}}(\varrho)\hbar_2(\upsilon). \tag{27}$$

□

Theorem 5. *Let the two functions $\hbar_1$ and $\hbar_2$ be positive and integrable on $[0,\infty)$. If $p_1,q_1>1$ be such that $\frac{1}{p_1}+\frac{1}{q_1}=1$. Then for $\xi\geq 0$ and $\eta,\zeta>0$, the following inequalities hold:*

$$\begin{aligned}&p_1\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_1(\xi)\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_2(\xi)+q_1\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_2(\xi)\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_1(\xi)\\ \geq&\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\left(\hbar_1^{p_1}(\xi)\hbar_2^{q_1}(\xi)\right)\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\left(\hbar_1^{q_1}(\xi)\hbar_2^{p_1}(\xi)\right),\end{aligned} \tag{28}$$

$$\begin{aligned}&p_1\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{p_1-1}(\xi)\,\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\left(\hbar_1(\xi)\hbar_2^{q_1}(\xi)\right)+q_1\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{q_1-1}(\xi)\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\left(\hbar_1^{q_1}(\xi)\hbar_2(\xi)\right)\\ \geq&\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{q}(\xi)\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{p}(\xi),\end{aligned} \tag{29}$$

$$\begin{aligned}&p_1\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_1(\xi)\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{\frac{2}{p_1}}(\xi)+q_1\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{q_1}(\xi)\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{\frac{2}{q_1}}(\xi)\\ \geq&\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{p_1}(\xi)\hbar_2(\xi)\,\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{q_1}(\xi)\hbar_1^{2}(\xi)\end{aligned} \tag{30}$$

and

$$\begin{aligned}&p_1\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{\frac{2}{p_1}}(\xi)\hbar_2^{q_1}(\xi)\,\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{q_1-1}(\xi)+q_1\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{q_1-1}(\xi)\,\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{\frac{2}{q_1}}(\xi)\hbar_2^{p_1}(\xi)\\ \geq&\mathcal{I}_{(\sigma)_j,\eta}^{(\gamma)_j,(\lambda)_j}\hbar_1^{2}(\xi)\mathcal{I}_{(\sigma)_j,\zeta}^{(\gamma)_j,(\lambda)_j}\hbar_2^{2}(\xi).\end{aligned} \tag{31}$$

Proof. Recall the arithmetic mean and geometric mean inequality (AM-GM) given by

$$p_1u+q_1\upsilon\geq u^{p_1}\upsilon^{q_1},\ \forall\, u,\upsilon\geq 0,\ p_1+q_1=1. \tag{32}$$

By substituting $u = \hbar_1(\varrho)\hbar_2(v)$ and $v = \hbar_1(v)\hbar_2(\varrho)$, $\varrho, v > 0$ in (32), we have

$$p_1\hbar_1(\varrho)\hbar_2(v) + q_1\hbar_1(v)\hbar_2(\varrho) \geq (\hbar_1(\varrho)\hbar_2(v))^{p_1}(\hbar_1(v)\hbar_2(\varrho))^{q_1}. \tag{33}$$

Multiplying $(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})$ with (33) and taking the integration with respect to ϱ from 0 to ξ yields

$$\begin{aligned}&p_1\hbar_2(v)\int_0^\xi(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\hbar_1(\varrho)d\varrho\\
&+q_1\hbar_1(v)\int_0^\xi(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\hbar_2(\varrho)d\varrho\\
&\geq\hbar_2^{p_1}(v)\hbar_1^{q_1}(v)\int_0^\xi(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})\Psi'(\varrho)\hbar_1^{p_1}(\varrho)\hbar_2^{q_1}(\varrho)d\varrho,\end{aligned}$$

which in view of (5) follows,

$$p_1\hbar_2(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi) + q_1\hbar_1(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2(\xi) \geq \hbar_2^{p_1}(v)\hbar_1^{q_1}(v)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\left(\hbar_1^{p_1}(\xi)\hbar_2^{q_1}(\xi)\right). \tag{34}$$

Again, multiplying $(\xi-v)^{\eta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-v)^{\sigma_1}\cdots\lambda_j(\xi-v)^{\sigma_j})$ with (34), taking the integration with respect to v from 0 to ξ and using (5), we obtain (28) as,

$$\begin{aligned}&p_1\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_1(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2(\xi) + q_1\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\hbar_2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\\
&\geq\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta}\left(\hbar_1^{p_1}(\xi)\hbar_2^{q_1}(\xi)\right)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\left(\hbar_2^{p_1}(\xi)\hbar_1^{q_1}(\xi)\right),\end{aligned}$$

which completes the desired inequality (28). One can derive the inequalities (29), (30) and (31) by substituting the following identities in (32), respectively.

$$u = \frac{\hbar_1(v)}{\hbar_1(\varrho)},\ \ v = \frac{\hbar_2(\varrho)}{\hbar_2(v)},\ \hbar_1(\varrho),\hbar_2(v) \neq 0, \tag{35}$$

$$u = \hbar_1(\varrho)\hbar_1^{\frac{2}{p_1}}(v),\ \ v = \hbar_1^{\frac{2}{q_1}}(v)\hbar_2(\varrho) \tag{36}$$

and

$$u = \frac{\hbar_1^{\frac{2}{p_1}}(\varrho)}{\hbar_2(v)},\ \ v = \frac{\hbar_1^{\frac{2}{q_1}}(v)}{\hbar_2(\varrho)},\ \hbar_2(\varrho),\hbar_2(v) \neq 0. \tag{37}$$

□

Theorem 6. *Let the two functions $\hbar_1$ and $\hbar_2$ be positive and integrable on $[0,\infty)$. If $p_1, q_1 > 1$ be such that $\frac{1}{p_1}+\frac{1}{q_1}=1$. Suppose*

$$\mathcal{K} := \min_{0\leq\varrho\leq\xi}\frac{\hbar_1(\varrho)}{\hbar_2(\varrho)}\qquad \mathcal{H} := \max_{0\leq\varrho\leq\xi}\frac{\hbar_1(\varrho)}{\hbar_2(\varrho)}. \tag{38}$$

Then for $\xi \geq 0$, the following inequalities hold:

$$0 \leq \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^2(\xi) \leq \frac{(\mathcal{K}+\mathcal{H})^2}{4\mathcal{K}\mathcal{H}}\left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\hbar_2(\xi)\right)^2, \tag{39}$$

$$0 \leq \sqrt{\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^2(\xi)\,\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^2(\xi)} - \left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\hbar_2(\xi)\right) \leq \frac{\sqrt{\mathcal{H}}-\sqrt{\mathcal{K}}}{2\sqrt{\mathcal{KH}}}\left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\hbar_2(\xi)\right) \tag{40}$$

and

$$0 \leq \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^2(\xi) - \left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\hbar_2(\xi)\right)^2 \leq \frac{\mathcal{H}-\mathcal{K}}{4\mathcal{KH}}\left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\hbar_2(\xi)\right)^2. \tag{41}$$

Proof. From (38), we have

$$\left(\frac{\hbar_1(\varrho)}{\hbar_2(\varrho)} - \mathcal{K}\right)\left(\mathcal{H} - \frac{\hbar_1(\varrho)}{\hbar_2(\varrho)}\right)\hbar_2^2(\varrho) \geq 0, 0 \leq \varrho \leq \xi.$$

It follows that

$$\hbar_1^2(\varrho) + \mathcal{KH}\hbar_1^2(\varrho) \leq (\mathcal{K}+\mathcal{H})\hbar_1(\varrho)\hbar_2(\varrho). \tag{42}$$

Multiplying $(\xi-\varrho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\cdots,\gamma_j)}_{(\sigma_1,\cdots,\sigma_j),\zeta}(\lambda_1(\xi-\varrho)^{\sigma_1}\cdots\lambda_j(\xi-\varrho)^{\sigma_j})$ with (42) and taking the integration with respect to ϱ from 0 to ξ and using (5), we get

$$\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^2(\xi) + \mathcal{KH}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^2(\xi) \leq (\mathcal{K}+\mathcal{H})\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\hbar_2(\xi). \tag{43}$$

Now, since $\mathcal{KH} > 0$ and

$$\left(\sqrt{\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^2(\xi)} - \sqrt{\mathcal{KH}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^2(\xi)}\right)^2 \geq 0,$$

which follows that

$$2\sqrt{\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^2(\xi)}\sqrt{\mathcal{KH}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^2(\xi)} \leq \mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^2(\xi) + \mathcal{KH}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^2(\xi). \tag{44}$$

Hence, by using (43) and (44), we have

$$4\mathcal{KH}\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^2(\xi) \leq (\mathcal{K}+\mathcal{H})^2\left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\hbar_2(\xi)\right)^2, \tag{45}$$

which gives the inequality (39).

Now, from (45), we have

$$\sqrt{\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1^2(\xi)\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_2^2(\xi)} \leq \frac{\mathcal{K}+\mathcal{H}}{2\sqrt{\mathcal{KH}}}\left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\hbar_2(\xi)\right). \tag{46}$$

Now, subtracting $\left(\mathcal{I}^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\eta}\hbar_1(\xi)\hbar_2(\xi)\right)$ from (46) gives the inequality (40). Also, one can easily derive the inequality (41) by applying (39). □

5. Special Cases

In this section, we present certain new inequalities via the Prabhakar fractional integral which are the special cases of inequalities proved in Sections 3 and 4 by applying certain conditions on parameters.

If we consider $m_i = 0$ for $i = 2, 3, \ldots, j$ in Theorem 2, we get the following inequality for (4) having the three parameters M-L function.

Theorem 7. *Let the function $\hbar_1$ be positive and integrable on $[0,\infty)$ and let the two functions $\aleph_1$ and $\aleph_2$ be integrable on $[0,\infty)$ such that*

$$\aleph_1(\xi) \leq \hbar_1(\xi) \leq \aleph_2(\xi),\ \xi \in [0,\infty).$$

Then, for $\xi \geq 0$, $\sigma, \lambda > 0$ (where $i = 1,2,\ldots,j$), we have

$$\begin{aligned}&\mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\aleph_2(\xi)\mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\hbar_1(\xi) + \mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\hbar_1(\xi)\mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\aleph_1(\xi)\\ \geq&\mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\aleph_2(\xi)\mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\aleph_1(\xi) + \mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\hbar_1(\xi)\mathcal{I}^{\gamma,\lambda}_{\sigma,\zeta}\hbar_1(\xi).\end{aligned}$$

Corollary 3. *Let the function $\hbar_1$ be positive and integrable on $[0,\infty)$ and satisfying $\mathfrak{m} \leq \hbar_1(\xi) \leq \mathfrak{M}$, $\xi \in [0,\infty)$. Then, for $\xi \geq 0$, $\sigma, \zeta, \eta, \lambda > 0$, we have the following inequality for the Prabhakar fractional integral (4) as:*

$$\begin{aligned}&\mathfrak{M}\xi^{\zeta}\mathcal{E}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta+1}(\lambda_1\xi^{\sigma_1})\mathcal{I}^{\gamma_1,\lambda_1}_{\sigma_1,\eta}\hbar_1(\xi) + \mathfrak{m}\xi^{\eta}\mathcal{E}^{\gamma_1,\lambda_1}_{\sigma_1,\eta+1}(\lambda_1\xi^{\sigma_1})\mathcal{I}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta}\hbar_1(\xi)\\ \geq&\mathfrak{m}\mathfrak{M}\xi^{\zeta+\eta}\mathcal{E}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta+1}(\lambda_1\xi^{\sigma_1})\mathcal{E}^{\gamma_1,\lambda_1}_{\sigma_1,\eta+1}(\lambda_1\xi^{\sigma_1}) + \mathcal{I}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta}\hbar_1(\xi)\mathcal{I}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta}\hbar_1(\xi).\end{aligned}$$

Example 2. *Let the function $\hbar_1$ be positive and integrable on $[0,\infty)$ and satisfying $\xi \leq \hbar_1(\xi) \leq \xi+1$, $\xi \in [0,\infty)$. Then, for $\xi \geq 0$, $\sigma_i, \zeta, \lambda_i > 0$ (where $i = 1,2,\ldots,j$) and put $\zeta = \eta$ in Theorem 7, we have*

$$\begin{aligned}&\left[2\xi^{\zeta+1}\mathcal{E}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta+2}(\lambda_1\xi^{\sigma_1}) + \xi^{\zeta}\mathcal{E}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta+1}(\lambda_1\xi^{\sigma_1})\right]\mathcal{I}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta}\hbar_1(\xi)\\ \geq&\left[\xi^{\zeta+1}\mathcal{E}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta+2}(\lambda_1\xi^{\sigma_1}) + \xi^{\zeta}\mathcal{E}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta+1}(\lambda_1\xi^{\sigma_1})\right]\left(\xi^{\zeta+1}\mathcal{E}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta+2}(\lambda_1\xi^{\sigma_1})\right)\\ &+\left(\mathcal{I}^{\gamma_1,\lambda_1}_{\sigma_1,\zeta}\hbar_1(\xi)\right)^2.\end{aligned}$$

Remark 2. *i. If we take $m_i = 0$ for $i = 2,3,\ldots,j$ in Theorems 3–6, we get the inequalities via the Prabhakar fractional integral (4) having the three parameters M-L function. ii. If we consider one of $\lambda_i = 0$ for $i = 2,3,\ldots,j$, then we get the result derived by Tariboon et al. [34].*

6. Conclusions

In this present investigation, we established certain new Grüss type and other AM-GM inequalities for the generalized fractional integral having multivariate M-L function in the kernel. We also presented the mentioned inequalities for the fractional integral containing the three parameters M-L function. Additionally, we discussed some special cases and support our finding with examples. In any case, we hope that these results continue to sharpen our understanding of the nature of fractional calculus and their applications in different fields. For future developments, we will derive several new interesting inequalities via Hölder–İşcan, Chebyshev, Markov, Young and Minkowski inequalities using fractional calculus for the generalized fractional integral having multivariate M-L function in the kernel. Moreover, the interested reader can consider the mathematical equivalence (see e.g., [35]) among these proposed results.

Author Contributions: Conceptualization, Y.S., G.R., Y.E. and M.S.; methodology, G.R. and M.S.; software, G.R., M.S.; validation, Y.S., Y.E.; formal analysis, Y.S., G.R., Y.E., M.S. and K.N.; investigation, G.R., M.S.; data curation, A.K., K.N.; writing—original draft preparation, G.R., M.S.; writing—review and editing, A.K., K.N.; visualization, A.K.; supervision, A.K.; project administration; funding acquisition, K.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research has received funding support from the National Science, Research and Innovation Fund (NSRF), Thailand and the authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through the Large Groups Project under grant number (RGP.2/44/43).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kilbas, A.A.; Sarivastava, H.M.; Trujillo, J.J. *Theory and Application of Fractional Differential Equation*; North-Holland Mathematics Studies; Elsevier Sciences B.V.: Amsterdam, The Netherlands, 2006.
2. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives, Theory and Applications*; Edited and with a foreword by S. M. Nikol'skĭ, Translated from the 1987 Russian original, Revised by the authors; Gordon and Breach Science Publishers: Yverdon, Switzerland, 1993.
3. Podlubny, I. *Fractional Differential Equations*; Academic Press: London, UK, 1999.
4. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
5. Khalil, R.; Horani, M.A.; Yousef, A.; Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **2014**, *264*, 65–70. [CrossRef]
6. Abdeljawad, T. On conformable fractional calculus. *J. Comput. Appl. Math.* **2015**, *279*, 57–66. [CrossRef]
7. Jarad, F.; Ugurlu, E.; Abdeljawad, T.; Baleanu, D. On a new class of fractional operators. *Adv. Differ. Equ.* **2017**, *2017*, 247. [CrossRef]
8. Anderson, D.R.; Ulness, D.J. Newly defined conformable derivatives. *Adv. Dyn. Syst. Appl.* **2015**, *10*, 109–137.
9. Abdeljawad, T.; Baleanu, D. Monotonicity results for fractional difference operators with discrete exponential kernels. *Adv. Differ. Equ.* **2017**, *2017*, 78. [CrossRef]
10. Abdeljawad, T.; Baleanu, D. On Fractional derivatives with exponential kernel and their discrete versions. *Rep. Math. Phys.* **2017**, *80*, 11–27. [CrossRef]
11. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Thermal Sci.* **2016**, *20*, 763–769. [CrossRef]
12. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.* **2015**, *1*, 73–85.
13. Chu, Y.M.; Wang, M.K.; Qiu, S.L. Optimal combinations bounds of root-square and arithmetic means for Toader mean. *Proc. Indian Acad. Sci. Math. Sci.* **2012**, *122*, 41–51. [CrossRef]
14. Wang, M.K.; Chu, Y.M.; Jiang, Y.P. Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions. *Rocky Mt. J. Math.* **2016**, *46*, 670–691. [CrossRef]
15. Qian, W.M.; Zhang, X.H.; Chu, Y.M. Sharp bounds for the Toader–Qi mean in terms of harmonic and geometric means. *J. Math. Inequal.* **2017**, *11*, 121–127. [CrossRef]
16. Vanterler da C. Sousa, J.; Oliveira, D.S.; Capelas de Oliveira, E. Grüss-type inequalities by means of generalized fractional integrals. *Bull. Braz. Math. Soc., New Ser.* **2019**, *50*, 1029–1047 . [CrossRef]
17. Rahman, G.; Nisar, K.S.; Qi, F. Some new inequalities of the Grüss type for conformable fractional integrals. *AIMS Math.* **2018**, *3*, 575–583. [CrossRef]
18. Rahman, G.; Ullah, Z.; Khan, A.; Set, E.; Nisar, K.S. Certain Chebyshev type inequalities involving fractional conformable integral operators. *Mathematics* **2019**, *7*, 364. [CrossRef]
19. Kacar, E.; Kacar, Z.; Yildirim, H. Integral inequalities for Riemann–Liouville fractional integrals of a function with respect to another function. *Iran. J. Math. Sci. Inform.* **2018**, *13*, 1–13.
20. Alzabut, J.; Abdeljawad, T.; Jarad, F.; Sudsutad, W. A Gronwall inequality via the generalized proportional fractional derivative with applications. *J. Inequal. Appl.* **2019**, *2019*, 101. [CrossRef]
21. Rahman, G.; Khan, A.; Abdeljawad, T.; Nisar, K.S. The Minkowski inequalities via generalized proportional fractional integral operators. *Adv. Differ. Equ.* **2019**, *2019*, 287. [CrossRef]
22. Rahman, G.; Abdeljawad, T.; Khan, A.; Nisar, K.S. Some fractional proportional integral inequalities. *J. Inequal. Appl.* **2019**, *2019*, 244. [CrossRef]
23. Rahman, G.; Nisar, K.S.; Ghaffar, A.; Qi, F. Some inequalities of the Grüss type for conformable k-fractional integral operators. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM* **2020**, *114*, 614. [CrossRef]
24. Grüss, G.U. Das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b \hbar_1(\xi)\hbar_2(\xi)d\xi - \frac{1}{(b-a)^2}\int_a^b \hbar_1(\xi)d\xi \int_a^b \hbar_2(\xi)d\xi$. *Math. Z.* **1935**, *39*, 215–226. [CrossRef]
25. Yildirim, H.; Kirtay, Z. Ostrowski inequality for generalized fractional integral and related inequalities. *Malaya J. Mat.* **2014**, *2*, 322–329.
26. Katugampola, U.N. Approach to a generalized fractional integral. *Appl. Math. Comput.* **2011**, *218*, 860–865. [CrossRef]
27. Prabhakar, T.R. A singular integral equation with a generalized Mittag–Leffler function in the kernel. *Yokohama Math. J.* **1971**, *19*, 7–15.
28. Saxena, R.K.; Kalla, S.L.; Saxena, R. Multivariate analogue of generalized Mittag–Leffler function. *Integral Transform. Spec. Funct.* **2011**, *22*, 533–548. [CrossRef]

29. Firas, G.; Salaheddine, B.; Alaa, A.H. Certain implementations in fractional calculus operators involving Mittag-Leffler-confluent hypergeometric functions. *Proc. R. Soc. A.* **2022**, *478* . [CrossRef]
30. Sarivastava, H.M.; Kumar, A.; Das, S.; Mehrez, K. Geometric properties of a certain class of Mittag—Leffler-type functions. *Fractal Fract.* **2022**, *6*, 54. [CrossRef]
31. Goyal, R.; Agarwal, P.; Oros, G.I.; Jain, S. Extended Beta and Gamma matrix functions via 2-parameter Mittag–Leffler matrix function. *Mathematics* **2022**, *10*, 892. [CrossRef]
32. Özarslan, M.A.; Fernandez, A. On the fractional calculus of multivariate Mittag–Leffler functions. *Int. J. Comput. Math.* **2022**, *99*, 247–273. [CrossRef]
33. Kreyszig, E. *Introductory Functional Analysis with Applications*, Wiley: New York, NY, USA, 1989.
34. Tariboon, J.; Ntouyas, S.K.; Sudsutad, W. Some new Riemann–Liouville fractional integral inequalities. *Int. J. Math. Sci.* **2014**, *2014*, 869434. [CrossRef]
35. Li, Y.; Gu, X.M.; Zhao, J. The weighted arithmetic mean-geometric mean inequality is equivalent to the Hölder inequality. *Symmetry* **2018**, *10*, 380. [CrossRef]