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# Optimality Guidelines for the Fuzzy Multi-Objective Optimization under the Assumptions of Vector Granular Convexity and Differentiability

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**Abstract:** In this research, we investigate a novel class of granular type optimality guidelines for the fuzzy multi-objective optimizations based on guidelines of vector granular convexity and granular differentiability. Firstly, the concepts of vector granular convexity is introduced to the vector fuzzy-valued function. Secondly, several properties of vector granular convex fuzzy-valued functions are provided. Thirdly, the granular type Karush-Kuhn-Tucker(KKT) optimality guidelines are derived for the fuzzy multi-objective optimizations.

**Keywords:** optimality guidelines; granular convexity; fuzzy-valued functions; granular differentiability

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## 1. Introduction

The optimality guidelines can be used to identify the possible candidates for optimal solutions, which plays a significant effect in the field of optimization principle and has been researched for more than a century. However, since the data utilized in real-world issues sometimes include uncertain or inaccurate data due to measurement mistakes or other unforeseen factors, fuzzy optimization model is often used to deal with this actual optimization problem containing fuzzy data.

In the past two decades, many results have been achieved in the research of optimality guidelines for the fuzzy optimizations. As we all know, the comparison of fuzzy numbers is often in partial order, so the resulting fuzzy difference and the derivative of the fuzzy function are not as simple as the difference of the real function and the definition of the derivative. Differentiability and convexity are two important conditions that are indispensable in the research of optimality guidelines of fuzzy optimization. For example, under the Hukuhara difference and Hukuhara differentiability, the notion of convexity for real-valued functions has been extended by Wu to LU-convexity for fuzzy-valued functions.The KKT guidelines [1–4] for an optimization problem with a fuzzy-valued objective function based on the premise of LU-convexity were then created. By using generalized Hukuhara difference and generalized Hukuhara differentiability, Chalco-Cano et al. [5–7] researched the optimality guidelines for the fuzzy optimizations. Zhang, Liu and Li researched the optimality guidelines for the fuzzy optimizations based on the assumptions of univexity [8] and invexity [9].

However, the Hukuhara differentiability or generalized Hukuhara differentiability of the fuzzy optimizations have some disadvantages [10,11]. Recently, Mazandarani et al. [11] have introduced the notion of a new fuzzy function differentiation as granular differentiability (gr-differentiability). This result has brought many new results in the field of fuzzy dynamic systems [12–15] and fuzzy differential equations [16–19] . Compared with generalized Hukuhara differential, a very important advantage of gr-differential is that it is relatively simple in calculation. Zhang et al. [20] proposed the granular convexity for fuzzy functions and researched the optimality guidelines for the fuzzy optimizations. However, there are three essential differences between [20] and this article. Firstly, the object of study is different from that of this paper. The fuzzy single-objective optimisation problem is studied in [20], while the fuzzy multi-objective optimisation problem is studied in this paper. The objective value for fuzzy single-objective optimization is the fuzzy number, while the objective value for fuzzy multi-objective optimization is the fuzzy vector. Secondly, the concept of solution is different. The solution in [20] is the optimal solution, while the solution in this paper is the efficient solution. Thirdly, the convexity condition is different. The condition of granular convexity of fuzzy function is studied in the [20], but this paper is the condition of vector convexity of fuzzy function. For multi-objective fuzzy optimizations [21,22], the computational complexity brought by Hukuhara or generalized Hukuhara differentiation are particularly obvious.

Convexity is one of the important basic assumptions in the study of optimality conditions. For the proof of the validity of convexity, Kalsoom [23] have proved some new Ostrowsk's type inequalities for q-differentiable preinvex functions by using the newly offered identity. Butt [24] gave some new results by using convexity of exponential s-convex functions of any positive integer order differentiable function. Kızıl [25] obtained new results for strongly convex functions with the help of Atangana-Baleanu integral operators. Ekinci [26] obtained new inequalities for the class of functions whose absolute values of first derivatives are convex on [a, b].

The purposes of this paper is to propose the optimality guidelines to fuzzy multi-objective optimizations under the conditions of vector granular convexity and granular differentiability. Firstly, we present the concepts of vector granular convexity for a fuzzy function. Secondly, we present the attributes for the vector granular convexity for a fuzzy function. Thirdly, we recommend the optimality guidelines for the fuzzy multi-objective optimization issue based on the assumptions of vector granular convexity and granular differentiability. Several examples are given to motivate our studies.

## 2. Preliminaries

This section covers some fundamental notions and arithmetics of fuzzy-valued functions. The abbreviations used in this paper are collected in Table 1.

**Table 1.** The abbreviations have been used in this paper.

| The Abbreviations | The Full Name |
|---|---|
| KKT | Karush-Kuhn-Tucker |
| FNs | fuzzy numbers |
| gr-differentiability | granular differentiability |
| gr-differentiable | granular-differentiable |
| gr-convex | granular convex |
| gr-pseudoconvex | granular pseudoconvex |
| gr-quasiconvex | granular quasiconvex |
| HMF | horizontal membership function |
| FMOP | fuzzy multi-objective programming problem |
| FCMOP | fuzzy constrained multi-objective programming problem |
| ES | efficient solutions |
| WES | weakly efficient solutions |
| V-gr-convex | vector granular convex |
| V-gr-convexity | vector granular convexity |
| V-gr-pseudoconvex | vector granular pseudoconvex |
| V-gr-quasiconvex | vector granular quasiconvex |
| V-gr-pseudo-convexity | vector granular pseudoconvex convexity |
| SOP | scalar problem |
| KTCQ | Kuhn-Tucker constraint qualification |

*2.1. Some Notions of the Fuzzy-Valued Functions*

Assume that $R^n$ is the $n-$dimensional Euclidean space, and $E$ is the space representing all fuzzy numbers(FNs) on $R$. The parametric form can be expressed as $[u]^\mu = [\underline{u}^\mu, \overline{u}^\mu] : [a, b] \to [0, 1]$. The horizontal membership function(HMF) [27] of $\widetilde{u}(\delta) \in E$ is denoted by $\mathcal{H}(\widetilde{u})(\mu, \alpha_u) = u^{gr}(\mu, \alpha_u)$. For simplicity, $\mathcal{H}(\widetilde{u})(\mu, \alpha_u)$ can be abbreviated to $\mathcal{H}(\widetilde{u})$, otherwise it would be too long. For a triangular FN $\widetilde{u} = (\gamma, \epsilon, \zeta), \gamma \leq \epsilon \leq \zeta$, $\mathcal{H}(\widetilde{u}) = [\gamma + (\epsilon - \gamma)\mu] + [(1 - \mu)(\zeta - \gamma)]\alpha_u$. The trapezoidal FN $\widetilde{v} = (\gamma, \epsilon, \zeta, \eta)$, the HMF is $\mathcal{H}(\widetilde{v}) = [\gamma + (\epsilon - \gamma)\mu] + [(\eta - \gamma) - (\eta - \gamma + \epsilon - \zeta)\mu]\alpha_v$, for $\mu, \alpha_v, \alpha_u \in [0, 1]$.

There are two FNs $\widetilde{v}$ and $\widetilde{u}$. Then, $\widetilde{v} \succeq \widetilde{u} \Leftrightarrow \mathcal{H}(\widetilde{v}) \geq \mathcal{H}(\widetilde{u})$ for all $\alpha_v = \alpha_u \in [0, 1]$ , and $\mu \in [0, 1]$; $\widetilde{v} = \widetilde{u} \Leftrightarrow \mathcal{H}(\widetilde{v}) = \mathcal{H}(\widetilde{u})$ for all $\alpha_v = \alpha_u \in [0, 1]$ , and $\mu \in [0, 1]$.

**Remark 1** ([18]). *The HMF $\mathcal{H}$ is a linear map. The following guidelines are met for any $\widetilde{v} \in E$ , $\widetilde{u} \in E$ and any constant $c \in R$,*
*(1) $\mathcal{H}(\widetilde{v} \oplus \widetilde{u}) := v^{gr}(\mu, \alpha_v) + u^{gr}(\mu, \alpha_u) = \mathcal{H}(\widetilde{v}) + \mathcal{H}(\widetilde{u})$;*
*(2) $\mathcal{H}(c\widetilde{u}) := cu^{gr}(\mu, \alpha_u) = c\mathcal{H}(\widetilde{u})$.*

**Definition 1** ([11]). *The following is the definition of the granular distance between $\widetilde{v}$ and $\widetilde{u}$ in E.*

$$D^{gr}(\widetilde{v}, \widetilde{u}) = \sup_\mu \max_{\alpha_v, \alpha_u} |v^{gr}(\mu, \alpha_v) - u^{gr}(\mu, \alpha_u)|.$$

*Assume that $\widetilde{g} : [e, f] \to E$ is a fuzzy function with n distinct parameters $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$. The HMF of $\widetilde{g}(\zeta)$ is $\mathcal{H}(\widetilde{g}(\zeta)) \triangleq g^{gr}(\zeta, \mu, \alpha_g)$, $g^{gr} : [e, f] \times [0, 1]R \times [0, 1]^n \to [k, l] \subseteq R$, where $\alpha_g = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$.*

**Definition 2** ([11]). *The fuzzy function $\widetilde{g} : [e, f] \to E$ is regarded as granular-differentiable (gr-differentiable) at $\zeta_0 \in [e, f]$ if there exists an element $\frac{d^{gr}\widetilde{g}(\zeta_0)}{d\zeta} \in E$ such that the following limit,*

$$\lim_{\Delta\zeta \to 0} \frac{\widetilde{g}(\zeta_0 + \Delta\zeta) \ominus^{gr} \widetilde{g}(\zeta_0)}{\Delta\zeta} = \frac{d^{gr}\widetilde{g}(\zeta_0)}{d\zeta},$$

*exists for $\Delta\zeta$ adequately value close to 0. $\frac{d^{gr}\widetilde{g}(\zeta_0)}{d\zeta}$ is regarded as gr-derivative of $\widetilde{g}$ at $\zeta_0$. If the gr-derivative exists for $\zeta \in [e, f] \subseteq R$, the $\widetilde{g}$ is gr-differentiable on $[e, f] \subseteq R$. The space of fuzzy-valued functions of all constantly gr-differentiable on $U \subseteq R$ is defined as $C^1(U, E)$.*

**Theorem 1** ([11]). *The fuzzy-valued function $\widetilde{g} : [e, f] \subseteq R \to E$ is gr-differentiable at $\zeta \in [e, f] \Leftrightarrow$ at that point its HMF is differentiable with reference to $\zeta$. Furthermore,*

$$\mathcal{H}(\frac{d\widetilde{g}(\zeta)}{d\zeta}) = \frac{\partial g^{gr}(\zeta, \mu, \alpha_g)}{\partial\zeta}.$$

A multivariate fuzzy function's partial derivative is defined by Zhang et al. [20]. Assume $\widetilde{G}(\boldsymbol{\varsigma}) : K \subseteq R^n \to E$, $\boldsymbol{\varsigma} = (\varsigma_1, \varsigma_2, \cdots, \varsigma_n) \in K$, and which with $n$ distinct FNs $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$. We denote that $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ with respect to the distinct FNs $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$, and $\mu, \alpha_{u_i} \in [0, 1]$.

**Definition 3** ([20]). *Assume that $\mathbf{y}_0 = (\varsigma_1^0, \varsigma_2^0, \cdots, \varsigma_n^0) \in K$ is a fixed element and $h_i(\varsigma_i) = \widetilde{G}(\varsigma_1^0, \cdots, \varsigma_{i-1}^0, \varsigma_i, \varsigma_{i+1}^0, \cdots, \varsigma_n^0)$ be a fuzzy function. The $\widetilde{G}$ has the ith partial gr-derivative at $\mathbf{y}_0$, if $h_i$ is gr-differentiable at $\varsigma_i^0$, denoted by $\frac{\partial G^{gr}(\mathbf{y}_0, \mu, \alpha_G)}{\partial\varsigma_i}$ and $\frac{\partial G^{gr}(\mathbf{y}_0, \mu, \alpha_G)}{\partial\varsigma_i} = (h_i)'(\varsigma_i^0)$, where $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$.*

*$\widetilde{G}$ is gr-differentiable at $\mathbf{y}_0$, if all the partial gr-derivative $(\frac{\partial G^{gr}(\mathbf{y}_0, \mu, \alpha_G)}{\partial\varsigma_1}, \cdots, \frac{\partial G^{gr}(\mathbf{y}_0, \mu, \alpha_G)}{\partial\varsigma_n})$ exist on some neighborhood of $\mathbf{y}_0$ and are consecutive at $\mathbf{y}_0$.*

**Definition 4** ([20])**.** *We say that*

$$\bigtriangledown G^{gr}(\boldsymbol{\varsigma}^*, \mu, \alpha_G) = (\frac{\partial G^{gr}(\boldsymbol{\varsigma}^*, \mu, \alpha_G)}{\partial \varsigma_1^*}, \cdots, \frac{\partial G^{gr}(\boldsymbol{\varsigma}^*, \mu, \alpha_G)}{\partial \varsigma_n^*})^T$$

*is the granular gradient of $\widetilde{G}$ at $\boldsymbol{\varsigma}^*$, where $\frac{\partial G^{gr}(\boldsymbol{\varsigma}^*, \mu, \alpha_G)}{\partial \varsigma_j^*}$ is the jth partial gr-derivative of $\widetilde{G}$ at $\boldsymbol{\varsigma}^*$.*

From Definition 1, the fuzzy functions' distance measure is defined as follows:

**Definition 5.** *It is assumed that $\widetilde{G}, \widetilde{Q} : K \subseteq R^n \to E$, are two fuzzy-valued functions. The distance measure between $\widetilde{G}$ and $\widetilde{Q}$ can be defined by*

$$D^{gr}(\widetilde{G}, \widetilde{Q}) = \sup_{\boldsymbol{\zeta} \in K, \mu \in [0,1]} \max_{\alpha_G, \alpha_Q} |G^{gr}(\boldsymbol{\zeta}, \mu, \alpha_G) - Q^{gr}(\boldsymbol{\zeta}, \mu, \alpha_Q)|,$$

*where $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ and $\alpha_Q = (\alpha_{v_1}, \alpha_{v_2}, \cdots, \alpha_{v_m})$ with respect to the distinct FNs $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$ and $\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_m$, and $\mu, \alpha_{u_i}, \alpha_{v_i} \in [0, 1]$.*

At present, we cope with fuzzy vector mapping $\widetilde{\mathbf{G}} : K \to E^p$. For $\widetilde{\mathbf{G}} = (\widetilde{G}_1, \cdots, \widetilde{G}_p)$, with respect to the different FNs $\widetilde{u}_{i1}, \widetilde{u}_{i2}, \cdots, \widetilde{u}_{ij}, j = 1, \cdots, n$, and $\mu, \alpha_{u_{ij}} \in [0, 1]$, each $\widetilde{G}_i, i = 1, \cdots, p$ is a fuzzy function. The HMF of $\widetilde{\mathbf{G}}$ is $\mathcal{H}(\widetilde{\mathbf{G}}) = (\mathcal{H}(\widetilde{G}_1), \cdots, \mathcal{H}(\widetilde{G}_p))$, where $\mathcal{H}(\widetilde{G}_i) = G_i^{gr}(\boldsymbol{\zeta}, \mu, \alpha_{G_i})$ and $\alpha_{G_i} = (\alpha_{u_{i1}}, \cdots, \alpha_{u_{ij}})$.

**Definition 6.** *Consider a fuzzy vector mapping $\widetilde{\mathbf{G}} = (\widetilde{G}_1, \cdots, \widetilde{G}_p) : K \to E^p$. It can be said by us that $\widetilde{\mathbf{G}}$ is vector gr-differentiable at $\boldsymbol{\zeta}_0 \in K \Leftrightarrow \widetilde{G}_i$ is gr-differentiable at $\boldsymbol{\zeta}_0$ for all $i = 1, \cdots, p$.*

### 2.2. Solution Concepts

Firstly, we review the comparison of two real vectors. If $\mathbf{c} = (c_1, \cdots, c_n)^T$, $\mathbf{d} = (d_1, \cdots, d_n)^T \in R^n$, then

(i) $\mathbf{c} = \mathbf{d} \Leftrightarrow c_i = d_i$ for all $i = 1, 2, \cdots, n$;
(ii) $\mathbf{c} < \mathbf{b} \Leftrightarrow c_i < d_i$ for all $i = 1, 2, \cdots, n$;
(iii) $\mathbf{c} \leqq \mathbf{d} \Leftrightarrow c_i \leq d_i$ for all $i = 1, 2, \cdots, n$;
(iv) $\mathbf{c} \leq \mathbf{d} \Leftrightarrow \mathbf{c} \leqq \mathbf{d}$ and $\mathbf{c} \neq \mathbf{d}$.

Based on the partial order relations on $E^p, p = 1, \cdots, n$, we also define the comparison of two fuzzy vectors.

**Definition 7.** *Let $\widetilde{\mathbf{w}} = (\widetilde{w}_1, \cdots, \widetilde{w}_n)^T, \widetilde{\mathbf{z}} = (\widetilde{z}_1, \cdots, \widetilde{z}_n)^T \in E^n$, then*

*(i) $\widetilde{\mathbf{w}} = \widetilde{\mathbf{z}} \Leftrightarrow \widetilde{w}_i = \widetilde{z}_i$ for all $i = 1, \cdots, n$;*
*(ii) $\widetilde{\mathbf{w}} \prec \widetilde{\mathbf{z}} \Leftrightarrow \widetilde{w}_i \prec \widetilde{z}_i$ for all $i = 1, \cdots, n$;*
*(iii) $\widetilde{\mathbf{w}} \precapprox \widetilde{\mathbf{z}} \Leftrightarrow \widetilde{w}_i \preceq \widetilde{z}_i$ for all $i = 1, \cdots, n$;*
*(iv) $\widetilde{\mathbf{w}} \preceq \widetilde{\mathbf{z}} \Leftrightarrow \widetilde{w}_i \preceq \widetilde{z}_i$ for all $i = 1, \cdots, n$ and $\widetilde{\mathbf{w}} \neq \widetilde{\mathbf{z}}$.*

We can get the follows proposition for the fuzzy vector function.

**Proposition 1.** *Assume a fuzzy vector mapping $\widetilde{\mathbf{G}} = (\widetilde{G}_1, \cdots, \widetilde{G}_p) : K \to E^p$, and $\boldsymbol{\zeta}, \boldsymbol{\zeta}^* \in K$ with respect to distinct FNs $\widetilde{u}_{i1}, \widetilde{u}_{i2}, \cdots, \widetilde{u}_{ij}, j = 1, \cdots, n, i = 1, \cdots, p$, and $\mu, \alpha_{u_{ij}} \in [0, 1]$. Then, we have*

*(i) $\widetilde{\mathbf{G}}(\boldsymbol{\zeta}^*) = \widetilde{\mathbf{G}}(\boldsymbol{\zeta}) \Leftrightarrow \mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}^*)) = \mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}))$;*
*(ii) $\widetilde{\mathbf{G}}(\boldsymbol{\zeta}^*) \prec \widetilde{\mathbf{G}}(\boldsymbol{\zeta}) \Leftrightarrow \mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}^*)) < \mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}))$;*
*(iii) $\widetilde{\mathbf{G}}(\boldsymbol{\zeta}^*) \precapprox \widetilde{\mathbf{G}}(\boldsymbol{\zeta}) \Leftrightarrow \mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}^*)) \leqq \mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}))$;*
*(iv) $\widetilde{\mathbf{G}}(\boldsymbol{\zeta}^*) \preceq \widetilde{\mathbf{G}}(\boldsymbol{\zeta}) \Leftrightarrow \mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}^*)) \leqq \mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}))$ and $\mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}^*)) \neq \mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta}))$;*

**Proof.** It is easy to be proved by Remark 1 and Definition 7. □

It is considered by us that the following two classes of gr-differentiable fuzzy multi-objective programming problems:

$$\text{(FMOP)} \quad \min \quad \widetilde{\mathbf{G}}(\zeta) = (\widetilde{G}_1(\zeta), \widetilde{G}_2(\zeta), \cdots, \widetilde{G}_p(\zeta)) \\ \zeta \in K \subseteq R^n.$$

where $\widetilde{\mathbf{G}}(\zeta) : K \to E^p$ are (consecutively) gr-differentiable fuzzy functions, and $K$ is a open subset which is nonempty of $R^n$.

The constrained gr-differentiable fuzzy multi-objective programming problem,

$$\text{(FCMOP)} \quad \min \quad \widetilde{\mathbf{G}}(\zeta) = (\widetilde{G}_1(\zeta), \widetilde{G}_2(\zeta), \cdots, \widetilde{G}_p(\zeta)) \\ \widetilde{q}_i(\zeta) \preceq \widetilde{0}, \ \ i \in J = \{1, \cdots, m\},$$

where $\widetilde{\mathbf{G}}(\zeta) : K \to E^p, \widetilde{q}_i : K \to E, i \in J$, are (consecutively) gr-differentiable fuzzy functions, and $K$ is a open subset which is nonempty of $R^n$. It is denoted by us that $S := \{\zeta \in K : \widetilde{q}_i(\zeta) \preceq \widetilde{0}, \ \ i \in J\}$ as the feasible set of (FCMOP), $J(\zeta^*) := \{i \in J, \widetilde{q}_i(\zeta^*) = \widetilde{0}\}$ as the set of indices of active constraints at $\zeta^* \in K$.

We'll employ the notions of efficient solutions (ES) and weakly efficient solutions (WES) of (FMOP), which were brought in by [10].

**Definition 8** ([22])**.** *Assume that $\widetilde{\mathbf{G}}(\widetilde{\zeta}) : S \to E^p$ is a p-dimensional fuzzy function. Reputedly, a point $\zeta^* \in S$ is:*

*(1) a strongly ES if there exists no $\zeta \in S$ such that $\widetilde{\mathbf{G}}(\zeta) \preceq \widetilde{\mathbf{G}}(\zeta^*)$ ;*

*(2) an ES if there exists no $\zeta \in S$ such that $\widetilde{\mathbf{G}}(\zeta) \preceq \widetilde{\mathbf{G}}(\zeta^*)$ and $\exists k$ such that $\widetilde{G}_k(\zeta) \prec \widetilde{G}_k(\zeta^*)$, $k \in \{1, \cdots, p\}$;*

*(3) a mildly WES if there exists no $\zeta \in S$ such that $\widetilde{G}_j(\zeta) \preceq \widetilde{G}_j(\zeta^*), \forall j = 1, \cdots, p$;*

*(4) a WES if there exists no $\zeta \in S$ such that $\widetilde{\mathbf{G}}(\zeta) \prec \widetilde{\mathbf{G}}(\zeta^*)$.*

Osuna-Gómez et al. have discussed the relationship between the above solutions, one can refer to [21,22].

## 3. Vector Granular Convexity of Fuzzy Vector Functions

The notion of convexity is crucial in optimization theory. Recently, the notion of convexity has been developed in a number of fields using new and imaginative approaches. Based on the HMF of fuzzy vector functions, we define the notion of vector granular convexity for fuzzy vector functions and suggest several aspects of this class of fuzzy vector functions in this section.

**Definition 9** ([20])**.** *Assume that $\widetilde{G}$ is a fuzzy function defined on a convex set $K \subseteq R^n$. It can be said by us that $\widetilde{G}$ is granular convex if*

$$\mathcal{H}(\widetilde{G}(\lambda\zeta_1 + \zeta_2 - \lambda\zeta_2)) \leq \lambda\mathcal{H}(\widetilde{G}(\zeta_1)) + \mathcal{H}(\widetilde{G}(\zeta_2)) - \lambda\mathcal{H}(\widetilde{G}(\zeta_2))$$

*for all $\lambda \in (0,1)$, $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ with respect to distinct FNs $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$, and each $\zeta_1, \zeta_2(\zeta_1 \neq \zeta_2) \in K$.*

*It can be said that $\widetilde{G}$ is granular strict convex if*

$$\mathcal{H}(\widetilde{G}(\lambda\zeta_1 + \zeta_2 - \lambda\zeta_2)) < \lambda\mathcal{H}(\widetilde{G}(\zeta_1)) + \mathcal{H}(\widetilde{G}(\zeta_2)) - \lambda\mathcal{H}(\widetilde{G}(\zeta_2))$$

*for all $\lambda \in (0,1)$, $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ with respect to distinct FNs $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$, and each $\zeta_1, \zeta_2(\zeta_1 \neq \zeta_2) \in K$.*

**Definition 10.** *A fuzzy vector function $\widetilde{\mathbf{G}} = (\widetilde{G}_1, \cdots, \widetilde{G}_p) : K \to E^p$ is regarded as vector granular convex (for short: V-gr-convex) if there exist function $r_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) : K \times K \times [0,1]^{j+1} \to R^+ - \{0\}$ such that for each $\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}} \in K$ and for $\alpha_{G_i} = (\alpha_{u_{i1}}, \cdots, \alpha_{u_{ij}}), i = 1, \cdots, p,$*

$$\mathcal{H}(\widetilde{G}_i(\boldsymbol{\zeta})) \geq \mathcal{H}(\widetilde{G}_i(\overline{\boldsymbol{\zeta}})) + r_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i})\langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle$$

*with respect to distinct FNs $\widetilde{u}_{i1}, \widetilde{u}_{i2}, \cdots, \widetilde{u}_{ij}, j = 1, \cdots, n$, and $\mu, \alpha_{u_{ij}} \in [0,1]$.*

**Remark 2.** *For $p = 1$ and $r_i^{gr} = 1$, the above definition reduces to the granular convex fuzzy function.*

**Definition 11.** *If each $\widetilde{G}_i$ and $\widetilde{q}_j$ is a V-gr-convex fuzzy function for $i = 1, \cdots, p$, and $j = 1, \cdots, m$, a fuzzy multi-objective programming problem (FCMOP) is considered V-gr-convex fuzzy multi-objective programming problem.*

**Remark 3.** *A gr-convex fuzzy multi-objective programming problem is necessarily a V-gr-convex fuzzy multi-objective programming problem, but not conversely. In other words, each $\widetilde{G}_i$ and $\widetilde{q}_j$ is a gr-convex fuzzy function for $i = 1, \cdots, p$, and $j = 1, \cdots, m$, the problem of (FCMOP) is also a V-gr-convex fuzzy multi-objective programming problem, but not conversely.*

*For the real-valued multi-objective programming problem, a good example has been given by Jeyakumar and Mond in [28]. A similar fuzzy multi-objective programming issue example is as follows.*

**Example 1.** *Let $\widetilde{c} = (1,2,3)$ and $\widetilde{d} = (0,1,2)$, It is considered by us that the following fuzzy multi-objective programming problem.*

$$\begin{aligned} minimize_{\zeta \in R^2} \quad & \widetilde{\mathbf{G}}(\boldsymbol{\zeta}) = (\widetilde{G}_1(\boldsymbol{\zeta}), \widetilde{G}_2(\boldsymbol{\zeta})) = (\widetilde{c} \cdot \frac{\zeta_1^2}{\zeta_2}, \widetilde{d} \cdot \frac{\zeta_2}{\zeta_1}) \\ Subject\ \ to \quad & \widetilde{q}_1(\boldsymbol{\zeta}) = 1 - \zeta_1 \leq 0, \\ & \widetilde{q}_2(\boldsymbol{\zeta}) = 1 - \zeta_2 \leq 0. \end{aligned}$$

*The following is the HMF of $\widetilde{\mathbf{G}}(\boldsymbol{\zeta})$ for $\mu, \alpha_c, \alpha_d \in [0,1]$.*

$$\mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta})) = (\mathcal{H}(\widetilde{G}_1(\boldsymbol{\zeta})), \mathcal{H}(\widetilde{G}_2(\boldsymbol{\zeta}))),$$

*where*

$$\mathcal{H}(\widetilde{G}_1(\boldsymbol{\zeta})) = [\mu + 1 + (2 - 2\mu)\alpha_c] \cdot \frac{\zeta_1^2}{\zeta_2},$$

*and*

$$\mathcal{H}(\widetilde{G}_2(\boldsymbol{\zeta})) = [\mu + (2 - 2\mu)\alpha_d] \cdot \frac{\zeta_2}{\zeta_1}.$$

*Seeing that this problem is a V-gr-convex fuzzy multi-objective programming problem with $r_{G_1}^{gr} = \frac{\overline{\zeta}_2}{\zeta_2}, r_{G_2}^{gr} = \frac{\overline{\zeta}_1}{\zeta_1}, r_{q_1}^{gr} = r_{q_2}^{gr} = 1$is easy, but this issue does not live up to the gr-convexity guidelines.*

*For example, for $\widetilde{G}_2(\boldsymbol{\zeta})$, we can check it satisfies the guideline of Definition 10 for $\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}} \in S$.*

$$\nabla G_2^{gr}(\boldsymbol{\zeta}, \mu, \alpha_{G_2}) = (\frac{-[\mu + (2 - 2\mu)\alpha_d] \cdot \zeta_2}{\zeta_1^2}, \frac{\mu + (2 - 2\mu)\alpha_d}{\zeta_1})^T,$$

*and*

$$r_{G_2}^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_2})\langle \nabla G_2^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_2}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle = \frac{[\mu + (2 - 2\mu)\alpha_d] \cdot [\overline{\zeta}_1 \zeta_2 - \overline{\zeta}_2 \zeta_1]}{\overline{\zeta}_1 \zeta_1},$$

*and*

$$\mathcal{H}(\widetilde{G}_2(\boldsymbol{\zeta})) - \mathcal{H}(\widetilde{G}_2(\overline{\boldsymbol{\zeta}})) = \frac{[\mu + (2-2\mu)\alpha_d] \cdot [\overline{\zeta}_1\zeta_2 - \overline{\zeta}_2\zeta_1]}{\overline{\zeta}_1\zeta_1}.$$

*The above equations means that*

$$\mathcal{H}(\widetilde{G}_2(\boldsymbol{\zeta})) - \mathcal{H}(\widetilde{G}_2(\overline{\boldsymbol{\zeta}})) = r^{gr}_{G_2}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_2})\langle \nabla G^{gr}_2(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_2}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}}\rangle,$$

*So, $\widetilde{G}_2(\boldsymbol{\zeta})$ satisfies the guideline of Definition 10. We also can check that $\widetilde{G}_1$ and $\widetilde{q}_j$ is a V-gr-convex fuzzy function for $j = 1, 2$. From Definition 11, this issue is a V-gr-convex fuzzy multi-objective programming issue. But this issue does not live up to the gr-convexity guidelines.*

**Proposition 2.** *Assume that $\varphi : R \to R$ is differentiable and convex with positive derivative everywhere and $\widetilde{g} : K \to E^p$ is a V-gr-convex fuzzy vector function. Then, $\widetilde{Q}(\boldsymbol{\zeta}) = (\varphi(\widetilde{g}_1(\boldsymbol{\zeta})), \cdots, \varphi(\widetilde{g}_p(\boldsymbol{\zeta})))$ is also a V-gr-convex fuzzy vector function.*

**Proof.** Let $\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}} \in K$. According to the monotonicity of $\varphi$ and V-gr-convexity of $\widetilde{g}$, we get

$$\begin{aligned} \varphi(\mathcal{H}(\widetilde{g}_i(\boldsymbol{\zeta}))) &\geq \varphi[\mathcal{H}(\widetilde{g}_i(\overline{\boldsymbol{\zeta}})) + r^{gr}_i(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{g_i})\langle \nabla G^{gr}_i(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{g_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}}\rangle] \\ &\geq \varphi[\mathcal{H}(\widetilde{g}_i(\overline{\boldsymbol{\zeta}}))] + \varphi'[\mathcal{H}(\widetilde{g}_i(\overline{\boldsymbol{\zeta}}))][r^{gr}_i\langle \nabla G^{gr}_i(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{g_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}}\rangle]. \end{aligned}$$

Therefore, $\widetilde{Q}(\boldsymbol{\zeta})$ is also a V-gr-convex fuzzy vector function. □

**Definition 12.** *A viable point $\overline{\boldsymbol{\zeta}} \in K$ is denoted as a vector critical point to (FMOP) if there exists a vector $\boldsymbol{\lambda}^{gr} \in R^p$ with $\lambda^{gr}_i > 0$ such that*

$$\sum_{i=1}^{p} \lambda^{gr}_i \nabla G^{gr}_i(\overline{\boldsymbol{\zeta}}, \mu, \alpha_G) = \mathbf{0}$$

*for $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ with regard to distinct FNs $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$.*

*The vector critical point of (FMOP) is defined as the point $\overline{\boldsymbol{\zeta}}$ where a non-negative linear combination of the granular gradient vectors of each component fuzzy objective function equal to zero.*

**Example 2.** *Let $\widetilde{v}_1 = (-1, 1, 3), \widetilde{v}_2 = (0, 1, 2), \widetilde{v}_3 = (1, 2, 4), \widetilde{v}_4 = (3, 4, 5), \widetilde{v}_5 = (1, 2, 3)$. It is considered by us that the unconstrained fuzzy multi-objective programming issue*

$$\begin{array}{ll} \textit{minimize} & \widetilde{\mathbf{G}}(\boldsymbol{\zeta}) = (\widetilde{G}_1(\boldsymbol{\zeta}), \widetilde{G}_2(\boldsymbol{\zeta})) = (\widetilde{v}_1\zeta_1^2 + \widetilde{v}_2\zeta_1\zeta_2 + \widetilde{v}_3\zeta_2^2, \widetilde{v}_4\zeta_1^2 + \widetilde{v}_5\zeta_2^2) \\ \textit{Subject} & \textit{to } \boldsymbol{\zeta} \in R^2. \end{array}$$

*The following is the HMF of $\widetilde{\mathbf{G}}(\boldsymbol{\zeta})$for $\mu, \alpha_{v_i} \in [0, 1]$ and $i = 1, 2, 3, 4, 5$.*

$$\mathcal{H}(\widetilde{\mathbf{G}}(\boldsymbol{\zeta})) = (\mathcal{H}(\widetilde{G}_1(\boldsymbol{\zeta})), \mathcal{H}(\widetilde{G}_2(\boldsymbol{\zeta}))),$$

*where*

$$\mathcal{H}(\widetilde{G}_1(\boldsymbol{\zeta})) = [-1 + 2\mu + (4-4\mu)\alpha_{v_1}]\zeta_1^2 + [\mu + (2-2\mu)\alpha_{v_2}]\zeta_1\zeta_2 + [1 + \mu + (3-3\mu)\alpha_{v_3}]\zeta_2^2,$$

*and*

$$\mathcal{H}(\widetilde{G}_2(\boldsymbol{\zeta})) = [3 + \mu + (2-2\mu)\alpha_{v_4}]\zeta_1^2 + [1 + \mu + (2-2\mu)\alpha_{v_5}]\zeta_2^2.$$

*The granular gradient of $\widetilde{\mathbf{G}}$ as follows,*

$$\begin{aligned} \nabla G^{gr}_1(\boldsymbol{\zeta}, \mu, \alpha_{G_1}) = \quad & (2[-1 + 2\mu + (4-4\mu)\alpha_{v_1}]\zeta_1 + [\mu + (2-2\mu)\alpha_{v_2}]\zeta_2, \\ & [\mu + (2-2\mu)\alpha_{v_2}]\zeta_1 + 2[1 + \mu + 3 - 3\mu)\alpha_{v_3}]\zeta_2)^T, \end{aligned}$$

*and*

$$\nabla G_2^{gr}(\boldsymbol{\zeta}, \mu, \alpha_{G_2}) = (2[3 + \mu + (2 - 2\mu)\alpha_{v_4}]\zeta_1, 2[1 + \mu + (2 - 2\mu)\alpha_{v_5}]\zeta_2)^T.$$

*We can see that* $\overline{\boldsymbol{\zeta}} = (0, 0)^T$ *is a vector critical point to this unconstrained fuzzy multi-objective programming problem, there exists a vector* $\boldsymbol{\lambda}^{gr} \in R^2$ *with* $\lambda_i^{gr} > 0$ *such that* $\sum_{i=1}^{p} \lambda_i^{gr} \nabla G_i^{gr}$ $(\overline{\boldsymbol{\zeta}}, \mu, \alpha_G) = \mathbf{0}$ *for* $\alpha_G = (\alpha_{v_1}, \alpha_{v_2}, \cdots, \alpha_{v_5})$ *and* $i = 1, 2$.

Based on the HMF of the fuzzy function, we may delimit the notions of granular pseudoconvex and granular pseudoconcave fuzzy functions, which are comparable to the definition of real-valued generalized convex functions.

**Definition 13.** *Assume that* $\widetilde{G} : K \to E$ *is a gr-differentiable fuzzy function defined on an open convex set* $K \subseteq R^n$. $\widetilde{G}$ *is denoted as granular pseudoconvex (gr-pseudoconvex) at* $\boldsymbol{\zeta} \in K$, *if for all* $\boldsymbol{\delta} \in K$, *one has*

$$\langle \nabla G^{gr}(\boldsymbol{\zeta}, \mu, \alpha_G), \boldsymbol{\delta} - \boldsymbol{\zeta} \rangle \geq 0 \Rightarrow \mathcal{H}(\widetilde{G}(\boldsymbol{\delta}) \geq \mathcal{H}(\widetilde{G}(\boldsymbol{\zeta}));$$

*or equivalently,*

$$\mathcal{H}(\widetilde{G}(\boldsymbol{\delta}) < \mathcal{H}(\widetilde{G}(\boldsymbol{\zeta})) \Rightarrow \langle \nabla G^{gr}(\boldsymbol{\zeta}, \mu, \alpha_G), \boldsymbol{\delta} - \boldsymbol{\zeta} \rangle < 0$$

*for* $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ *with respect to distinct FNs* $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$ *and* $\mu, \alpha_{u_i} \in [0, 1]$.

*The* $\widetilde{G} : K \to E$ *is called gr-pseudoconvex on K, if the above property is contented for all* $\boldsymbol{\zeta} \in K$. *The* $\widetilde{G}$ *is called gr-pseudoconcave on K, if* $-\widetilde{G}$ *is gr-pseudoconvex on K. The* $\widetilde{G}$ *is called gr-pseudolinear on K, if* $\widetilde{G}$ *is both gr-pseudoconvex and gr-pseudoconcave on K.*

**Definition 14.** *Assume that* $\widetilde{G} : K \to E$ *is a gr-differentiable fuzzy function defined on an open convex set* $K \subseteq R^n$. *The* $\widetilde{G}$ *is regarded as strictly granular pseudoconvex (gr-pseudoconvex) at* $\boldsymbol{\zeta} \in K$, *if for all* $\boldsymbol{\delta} \in K, \boldsymbol{\zeta} \neq \boldsymbol{\delta}$ *one has*

$$\mathcal{H}(\widetilde{G}(\boldsymbol{\delta}) \leq \mathcal{H}(\widetilde{G}(\boldsymbol{\zeta})) \Rightarrow \langle \nabla G^{gr}(\boldsymbol{\zeta}, \mu, \alpha_G), \boldsymbol{\delta} - \boldsymbol{\zeta} \rangle < 0$$

*for* $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ *with respect to distinct FNs* $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$ *and* $\mu, \alpha_{u_i} \in [0, 1]$.

*The fuzzy function* $\widetilde{G} : K \to E$ *is called strictly gr-pseudoconvex on K, if the above property is contented for all* $\boldsymbol{\zeta} \in K$.

*It is clear that every strictly gr-pseudoconvex fuzzy function is also a gr-pseudoconvex fuzzy function. However, in conclusion, the reverse is not true.*

**Proposition 3.** *Assume that* $\widetilde{G} : K \to E$ *is a gr-differentiable fuzzy function defined on an open convex set* $K \subseteq R^n$. *If the fuzzy function* $\widetilde{G}$ *is gr-convex on K, then it also is gr-pseudoconvex on K.*

**Proof.** From Definition 10 and Remark 2, it can be easily proved. □

The Proposition 3's converse is not true.

**Example 3.** *Let* $\widetilde{u}_1 = (1, 2, 3), \widetilde{u}_2 = (0, 1, 2)$, *It is considered by us that* $\widetilde{g}(\zeta) = \widetilde{u}_1\zeta + \widetilde{u}_2\zeta^3$. *The following is the granular gradient of* $\widetilde{g}$,

$$\nabla g^{gr}(\zeta, \mu, \alpha_g) = \quad [1 + \mu + (2 - 2\mu)\alpha_{u_1}] + 3[\mu + (2 - 2\mu)\alpha_{u_2}]\zeta^2 > 0,$$

*Then, we own*

$$\langle \nabla g^{gr}(\zeta, \mu, \alpha_g), \delta - \zeta \rangle \geq 0 \Rightarrow \mathcal{H}(\widetilde{g}(\delta) \geq \mathcal{H}(\widetilde{g}(\zeta)),$$

*which signifies* $\widetilde{g}(\zeta)$ *is a gr-pseudoconvex fuzzy function, but it is not a gr-convex fuzzy function.*

**Definition 15.** *A fuzzy vector function* $\widetilde{\mathbf{G}} = (\widetilde{G}_1, \cdots, \widetilde{G}_p) : K \to E^p$ *is denoted as vector granular pseudoconvex (for short: V-gr-pseudoconvex) if there exist function* $\beta_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) :$

*$K \times K \times [0,1]^{j+1} \to R^{+} - \{0\}$ and $\tau_i > 0$ such that for each $\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}} \in K$ and for $\alpha_{G_i} = (\alpha_{u_{i1}}, \cdots, \alpha_{u_{ij}}), i = 1, \cdots, p,$*

$$\sum_{i=1}^{p} \tau_i \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle \geq 0$$
$$\Rightarrow \sum_{i=1}^{p} \beta_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) \mathcal{H}(\widetilde{G}_i(\boldsymbol{\zeta})) \geq \sum_{i=1}^{p} \beta_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) \mathcal{H}(\widetilde{G}_i(\overline{\boldsymbol{\zeta}}))$$

*with respect to distinct FNs $\widetilde{u}_{i1}, \widetilde{u}_{i2}, \cdots, \widetilde{u}_{ij}, j = 1, \cdots, n$, and $\mu, \alpha_{u_{ij}} \in [0,1]$.*

*We can also define the notions of granular quasiconvex and granular quasiconcave fuzzy functions.*

**Definition 16.** *Assume that $\widetilde{G} : K \to E$ is a gr-differentiable fuzzy function defined on an open convex set $K \subseteq R^n$. The $\widetilde{G}$ is denoted as granular quasiconvex (gr-quasiconvex) at $\boldsymbol{\zeta} \in K$, if for all $\boldsymbol{\delta} \in K$, and $\lambda \in [0,1]$*

$$\mathcal{H}(\widetilde{G}(\lambda\boldsymbol{\zeta} + \boldsymbol{\delta} - \lambda\boldsymbol{\delta})) \leq max\{\mathcal{H}(\widetilde{G}(\boldsymbol{\delta})), \mathcal{H}(\widetilde{G}(\boldsymbol{\zeta}))\}$$

*for $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ with regard to distinct FNs $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$ and $\mu, \alpha_{u_i} \in [0,1]$.*

*The $\widetilde{G}$ is called gr-quasiconcave on K, if $-\widetilde{G}$ is gr-quasiconvex on K. The $\widetilde{G}$ is referred to as strictly gr-quasiconvex if*

$$\mathcal{H}(\widetilde{G}(\lambda\boldsymbol{\zeta} + \boldsymbol{\delta} - \lambda\boldsymbol{\delta})) < max\{\mathcal{H}(\widetilde{G}(\boldsymbol{\delta})), \mathcal{H}(\widetilde{G}(\boldsymbol{\zeta}))\}$$

*is contented for $\boldsymbol{\zeta} \neq \boldsymbol{\delta}$ and $\lambda \in [0,1]$.*

**Example 4.** *The fuzzy function*

$$\widetilde{g}(\zeta) = \begin{cases} \frac{\widetilde{a}|\zeta|}{\zeta}, & if \ \zeta \neq 0, \\ 0, & if \ \zeta = 0, \end{cases} \tag{1}$$

*and $\widetilde{a} = (1,2,3)$.*

*It is clear that $\widetilde{g}(\zeta)$ is a gr-quasiconvex fuzzy function on K.*

For real-valued functions, the following theorem characterizes differentiable quasiconvex functions. The proof can be found in [29].

**Theorem 2** ([29])**.** *Assume that $K \subseteq R^n$ is an open convex set and assume that $g : K \to R$ is a differentiable function on K. Then, g is quasiconvex on K, $\Leftrightarrow$ The following implication is true.*

$$g(\boldsymbol{\delta}) \leq g(\boldsymbol{\zeta}) \Rightarrow \langle \nabla g(\boldsymbol{\zeta}), \boldsymbol{\delta} - \boldsymbol{\zeta} \rangle \leq 0, \forall \boldsymbol{\delta}, \boldsymbol{\zeta} \in K.$$

Since the HMF of a fuzzy function is a real-valued function, we can propose the following proposition for the gr-differentiable quasiconvex fuzzy function by Theorem 2.

**Proposition 4.** *Assume that $\widetilde{G} : K \to E$ is a gr-differentiable fuzzy function defined on an open convex set $K \subseteq R^n$. Then, $\widetilde{G}$ is gr-quasiconvex on K $\Leftrightarrow$ The following implication is true.*

$$\mathcal{H}(\widetilde{G}(\boldsymbol{\delta})) \leq \mathcal{H}(\widetilde{G}(\boldsymbol{\zeta})) \Rightarrow \langle \nabla G^{gr}(\boldsymbol{\zeta}, \mu, \alpha_G), \boldsymbol{\delta} - \boldsymbol{\zeta} \rangle \leq 0, \forall \boldsymbol{\zeta}, \boldsymbol{\delta} \in K,$$

*for $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ with respect to distinct FNs $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$ and $\mu, \alpha_{u_i} \in [0,1]$.*

**Proof.** Since $\mathcal{H}(\widetilde{G}(\boldsymbol{\delta}))$ and $\mathcal{H}(\widetilde{G}(\boldsymbol{\zeta}))$ are two real-valued functions. From Theorem 2, we can easily prove this proposition. □

**Definition 17.** *A fuzzy vector function $\widetilde{\mathbf{G}} = (\widetilde{G}_1, \cdots, \widetilde{G}_p) : K \to E^p$ is denoted as vector granular quasiconvex (for short: V-gr-quasiconvex) if there exist function $\delta_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) : K \times K \times [0,1]^{j+1} \to R^+ - \{0\}$ and $\lambda_i > 0$ such that for each $\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}} \in K$ and for $\alpha_{G_i} = (\alpha_{u_{i1}}, \cdots, \alpha_{u_{ij}}), i = 1, \cdots, p,$*

$$\sum_{i=1}^{p} \delta_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) \mathcal{H}(\widetilde{G}_i(\boldsymbol{\zeta})) \leq \sum_{i=1}^{p} \delta_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) \mathcal{H}(\widetilde{G}_i(\overline{\boldsymbol{\zeta}})),$$
$$\Rightarrow \sum_{i=1}^{p} \lambda_i \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle \leq 0$$

*with respect to distinct FNs $\widetilde{u}_{i1}, \widetilde{u}_{i2}, \cdots, \widetilde{u}_{ij}, j = 1, \cdots, n$, and $\mu, \alpha_{u_{ij}} \in [0,1]$.*

## 4. The Karush-Kuhn-Tucker Optimality Guidelines

In this part, we will present the KKT optimality guidelines for the problems of (FMOP) and (FCMOP) based on the gr-convexity and gr-differentiability.

### *4.1. Optimality Guidelines for the Issue of (FMOP)*

The following Gordan's alternative theorem will be used to establishing the essential optimality guidelines. Mangasarian [29] provides evidence for this.

**Theorem 3.** *Assume that $\mathbf{C}$ is a $p \times n$ matrix, then either*
*(i) $\mathbf{C}\boldsymbol{\zeta} < 0$ has a result $\boldsymbol{\zeta} \in R^n$;*
*or*
*(ii) $\mathbf{C}^T \delta = b, \delta_i \geq 0, i = 1, \cdots, p$ for some nonzero $\delta \in R^p$, but never both.*

At present time, we present the following KKT optimality guidelines for the issues of (FMOP).

**Theorem 4.** *Assume that $\widetilde{\mathbf{G}}$ is a V-gr-convex fuzzy vector function, then $\overline{\boldsymbol{\zeta}}$ is a WES for the issue of (FMOP) $\Leftrightarrow$ $\overline{\boldsymbol{\zeta}}$ is a vector critical point to the problem of (FMOP).*

**Proof.** ($\Leftarrow$) Since $\overline{\boldsymbol{\zeta}}$ is a vector critical point, then there exists a vector $\boldsymbol{\lambda}^{gr} \in R^p$ with $\lambda_i^{gr} > 0$ such that
$$\sum_{i=1}^{p} \lambda_i^{gr} \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_G) = \mathbf{0}, i = 1, \cdots, p$$
for $\alpha_G = (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ with respect to distinct FNs $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n$.

Suppose that the point $\overline{\boldsymbol{\zeta}}$ is not a WES for the issue of (FMOP). Then there exists $\boldsymbol{\zeta} \in K$ such that
$$\widetilde{\mathbf{G}}(\boldsymbol{\zeta}) \prec \widetilde{\mathbf{G}}(\overline{\boldsymbol{\zeta}}).$$

From Definition 7 and Proposition 1, we get
$$\mathcal{H}(\widetilde{G}_i(\boldsymbol{\zeta})) < \mathcal{H}(\widetilde{G}_i(\overline{\boldsymbol{\zeta}})), i = 1, \cdots, p.$$

Since $\widetilde{\mathbf{G}}$ is a V-gr-convex fuzzy vector function, there exist function $r_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) : K \times K \times [0,1]^{j+1} \to R^+ - \{0\}$ such that for each $\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}} \in K$ and for $\alpha_{G_i} = (\alpha_{u_{i1}}, \cdots, \alpha_{u_{ij}})$, $i = 1, \cdots, p$,
$$\mathcal{H}(\widetilde{G}_i(\boldsymbol{\zeta})) \geq \mathcal{H}(\widetilde{G}_i(\overline{\boldsymbol{\zeta}})) + r_i^{gr}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle.$$

According to the two inequations above, we obtain
$$r_i^{gr} \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle < 0, i = 1, \cdots, p,$$

and for $\lambda_i^{gr} = r_i^{gr} > 0$, we have

$$\sum_{i=1}^{p} \lambda_i^{gr} \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle < 0, i = 1, \cdots, p,$$

which is a conflict. So the point $\overline{\boldsymbol{\zeta}}$ is a WES for the issue of (FMOP).

($\Rightarrow$) Assume that the point $\overline{\boldsymbol{\zeta}}$ is a WES for the problem of (FMOP), then there exists no $\boldsymbol{\zeta} \in K$ such that

$$\widetilde{\mathbf{G}}(\boldsymbol{\zeta}) \prec \widetilde{\mathbf{G}}(\overline{\boldsymbol{\zeta}}).$$

According to Definition 7 and Proposition 1, obtain

$$\mathcal{H}(\widetilde{G}_i(\boldsymbol{\zeta})) < \mathcal{H}(\widetilde{G}_i(\overline{\boldsymbol{\zeta}})), i = 1, \cdots, p.$$

From the V-gr-convexity of the fuzzy vector function, we have

$$0 > \mathcal{H}(\widetilde{G}_i(\boldsymbol{\zeta})) - \mathcal{H}(\widetilde{G}_i(\overline{\boldsymbol{\zeta}})) \geq r_i^{gr} \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle,$$

which means

$$\sum_{i=1}^{p} r_i^{gr} \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle < 0.$$

According to Gordan's alternative theorem and the above inequation, there exists a vector $\boldsymbol{\lambda}^{gr} \in R^p$ with $\lambda_i^{gr} > 0$ such that

$$\sum_{i=1}^{p} \lambda_i^{gr} \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_G) = \mathbf{0}, i = 1, \cdots, p.$$

So, the issue of (FMOP) has a vector critical point called $\overline{\boldsymbol{\zeta}}$. □

As is known to all that for real-valued multi-objective optimizations, based on the convexity hypothesis, the vector critical point, the WES and the solutions with optimality for weighting scalar issues coincide. And Osuna-Gómez et al. [30] have proved this results under the assumptions of invexity. We can also shown that the results are contented under the V-gr-convexity hypothesis for (FMOP).

It is considered by us that the following gr-differentiable weighting optimization issue:

$$\begin{array}{lll} \text{(SOP)} & \min & \lambda_1 G_1^{gr}(\boldsymbol{\zeta}, \mu, \alpha_{G_1}) + \lambda_2 G_2^{gr}(\boldsymbol{\zeta}, \mu, \alpha_{G_2}) + \cdots + \lambda_p G_p^{gr}(\boldsymbol{\zeta}, \mu, \alpha_{G_p}) \\ & & \boldsymbol{\zeta} \in K \subseteq R^n. \end{array}$$

where $\lambda = (\lambda_1, \cdots, \lambda_p)^T \in R^p$.

**Theorem 5.** *Assume that $\widetilde{\mathbf{G}}$ is a V-gr-convex fuzzy vector function on an open set K, then all WES of (FMOP) work out the weighting scalar problem (SOP) with $\lambda \geq \mathbf{0}$.*

**Proof.** Assume that $\overline{\boldsymbol{\zeta}}$ is a WES of (FMOP), from Theorem 4, then there exists $\lambda^{gr} = (\lambda_1^{gr}, \cdots, \lambda_p^{gr})^T \in R^p$ with $\lambda \geq \mathbf{0}$ so that

$$\sum_{i=1}^{p} \lambda_i^{gr} \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_G) = \mathbf{0}, i = 1, \cdots, p.$$

Since $\widetilde{\mathbf{G}}$ is a V-gr-convex fuzzy vector function, then

$$\mathcal{H}(\widetilde{G}_i(\boldsymbol{\zeta})) - \mathcal{H}(\widetilde{G}_i(\overline{\boldsymbol{\zeta}})) \geq r_i^{gr} \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \boldsymbol{\zeta} - \overline{\boldsymbol{\zeta}} \rangle,$$

and
$$\sum_{i=1}^{p}[\mathcal{H}(\lambda_i^{gr}\widetilde{G}_i(\zeta)) - \mathcal{H}(\lambda_i^{gr}\widetilde{G}_i(\overline{\zeta}))] \geq \sum_{i=1}^{p}\lambda_i^{gr} r_i^{gr}\langle \nabla G_i^{gr}(\overline{\zeta},\mu,\alpha_{G_i}), \zeta - \overline{\zeta}\rangle = 0.$$

Then, we have
$$\sum_{i=1}^{p}[\mathcal{H}(\lambda_i^{gr}\widetilde{G}_i(\zeta)) - \mathcal{H}(\lambda_i^{gr}\widetilde{G}_i(\overline{\zeta}))] \geq 0.$$

From the above inequation and Remark 1, we obtain
$$\mathcal{H}(\sum_{i=1}^{p}\lambda_i^{gr}\widetilde{G}_i(\zeta)) - \mathcal{H}(\sum_{i=1}^{p}\lambda_i^{gr}\widetilde{G}_i(\overline{\zeta})) \geq 0,$$

which means
$$\sum_{i=1}^{p}\lambda_i^{gr} G_i^{gr}(\zeta,\mu,\alpha_{G_i}) \geq \sum_{i=1}^{p}\lambda_i^{gr} G_i^{gr}(\overline{\zeta},\mu,\alpha_{G_i}).$$

Therefore, $\overline{\zeta}$ is a solutions with optimality for (SOP) with $\lambda^{gr} \geq \mathbf{0}$. □

**Theorem 6.** *Assume that $\overline{\zeta}$ is a vector critical point for (FMOP), and $\widetilde{G}_i, i = 1,\cdots,p$ is a gr-pseudoconvex fuzzy function at $\overline{\zeta}$. Then $\overline{\zeta}$ is a WES of (FMOP).*

**Proof.** Assume that $\overline{\zeta}$ is a vector critical point for (FMOP), then there exists $\lambda^{gr} \geq \mathbf{0}$ such that
$$\sum_{i=1}^{p}\lambda_i^{gr}\nabla G_i^{gr}(\overline{\zeta},\mu,\alpha_G) = \mathbf{0}, i = 1,\cdots,p.$$

If there exists another $\zeta \in K$ such that $\widetilde{\mathbf{G}}(\zeta) \prec \widetilde{\mathbf{G}}(\overline{\zeta})$, from Proposition 1, which means $\mathcal{H}(\widetilde{G}_i(\zeta) < \mathcal{H}(\widetilde{G}_i(\overline{\zeta}))$ for $i = 1,\cdots,p$. According to the gr-pseudoconvexity of $\widetilde{G}_i, i = 1,\cdots,p$, we have
$$\langle \nabla G_i^{gr}(\overline{\zeta},\mu,\alpha_G), \zeta - \overline{\zeta}\rangle < 0,$$

and
$$\sum_{i=1}^{p}\langle \lambda_i^{gr}\nabla G_i^{gr}(\overline{\zeta},\mu,\alpha_G), \zeta - \overline{\zeta}\rangle < 0.$$

According to Gordan's alternative theorem, this system
$$\sum_{i=1}^{p}\lambda_i^{gr}\nabla G_i^{gr}(\overline{\zeta},\mu,\alpha_G) = \mathbf{0}, i = 1,\cdots,p.$$

have no solutions, which contradicts $\overline{\zeta}$ is a vector critical point for (FMOP). So, $\overline{\zeta}$ is a WES of (FMOP). □

**Example 5.** *Consider Example 2, $\widetilde{G}_i(\zeta)$ is a gr-pseudoconvex fuzzy function, so the vector critical point $\zeta^* = (0,0)^T$ is a WES of (FMOP) based on Theorem 6.*

### *4.2. Optimality Guidelines for the Problem of (FCMOP)*

The following Motzkin's alternative theorem and Kuhn-Tucker constraint qualification (KTCQ) will be used to establishing the necessary optimality guidelines. These results can be found in Mangasarian [29].

The (KTCQ) is shown below.

**Definition 18.** *Assume that the constraint functions $\widetilde{q}_j, j = 1,\cdots,m$ of the (FCMOP) is continuously gr-differentiable at $\overline{\zeta} \in S$. The (FCMOP) is said to be satisfy the (KTCQ) at $\overline{\zeta}$, if for any $d \in R^n$ such that*
$$\langle \nabla q_j^{gr}(\overline{\zeta},\mu,\alpha_{q_j}), d\rangle \leq 0, \forall j \in J(\overline{\zeta}),$$

*there exists a vector function* $\beta : [0,1] \to R^n$*, which is continuously differentiable at 0 and some real number* $\gamma > 0$ *such that*

$$\beta(0) = \overline{\boldsymbol{\zeta}}, \mathcal{H}[\widetilde{q}_j(\beta(\varsigma))] \leq 0, \forall \varsigma \in [0,1] \text{ and } \beta^{'}(\varsigma) = \gamma d.$$

The Motzkin's alternative theorem as follows.

**Theorem 7.** *Assume that* $\mathbf{X}, \mathbf{Y}$ *and* $\mathbf{Z}$ *is given* $p^1 \times n, p^2 \times n$ *and* $p^3 \times n$ *matrices, with* $\mathbf{X}$ *being nonvacuous. Then,*

*(i)* $\mathbf{X}\boldsymbol{\varsigma} > 0, \mathbf{Y}\boldsymbol{\varsigma} \geq 0, \mathbf{Z}\boldsymbol{\varsigma} = 0$*, has a solution* $\boldsymbol{\varsigma} \in R^n$*,*

*or*

*(ii)* $\mathbf{X}^T \boldsymbol{\o}_1 + \mathbf{Y}^T \boldsymbol{\o}_2 + \mathbf{Z}^T \boldsymbol{\o}_3 = 0, \boldsymbol{\o}_1 \geq 0, \boldsymbol{\o}_3 \geqq 0$ *has a solution* $\boldsymbol{\o}_1, \boldsymbol{\o}_2, \boldsymbol{\o}_3$*,*

*but never both.*

At present, we create the optimality guidelines for the issue of (FCMOP) based on vector granular convexity and granular differentiability. The necessary optimality guidelines as follows.

**Theorem 8.** *Assume that* $\widetilde{\mathbf{G}}(\boldsymbol{\zeta}) : K \to E^p$ *and* $\widetilde{\mathbf{q}}(\boldsymbol{\zeta}) : K \to E^m$ *is continuously vector gr-differentiable fuzzy functions at* $\overline{\boldsymbol{\zeta}} \in K \subseteq R^n$*. Assume that the (KTCQ) is contented at* $\overline{\boldsymbol{\zeta}}$*. Then, a guideline with necessity for* $\overline{\boldsymbol{\zeta}}$ *to be a WES for (FCMOP) is that there exist multipliers* $\lambda^{gr} \in R^p$ *and* $r^{gr} \in R^m$*, such that*

$$\sum_{i=1}^{p} \lambda_i^{gr} \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) + \sum_{j=1}^{m} r_j^{gr} \nabla q_j^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{q_j}) = \mathbf{0}, \tag{2}$$

$$r_j^{gr} q_j^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{q_j}) = 0, j = 1, \cdots, m, \tag{3}$$

$$\lambda_i^{gr} \geq 0, \boldsymbol{\lambda}^{gr} \neq \mathbf{0}, r_j^{gr} \geq 0, i = 1, \cdots, p.$$

**Proof.** Suppose $\overline{\boldsymbol{\zeta}}$ is a WES for (FCMOP). First of all, we evidence the follows system has no result $d \in R^n$.

$$\langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), d \rangle < 0, \forall i = 1, \cdots, p. \tag{4}$$

$$\langle \nabla q_j^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{q_j}), d \rangle \leq 0, \forall j \in J(\overline{\boldsymbol{\zeta}}). \tag{5}$$

If the above system has a result $d \in R^n$. From the (KTCQ), there exists a function $\beta : [0,1] \to R^n$ which is a continuously differentiable function at 0 and some real number $\gamma > 0$ such that

$$\beta(0) = \overline{\boldsymbol{\zeta}}, \mathcal{H}[\widetilde{q}_j(\alpha(\varsigma))] \leq 0, \forall \varsigma \in [0,1] \text{ and } \alpha^{'}(\varsigma) = \gamma d.$$

Since $\widetilde{\mathbf{G}}(\boldsymbol{\zeta}) : K \to E^p$ is a continuously vector gr-differentiable fuzzy functions at $\overline{\boldsymbol{\zeta}}$, then for every $\widetilde{G}_i$, we obtain

$$\begin{aligned} \mathcal{H}[\widetilde{G}_i(\beta(\varsigma))] &= \mathcal{H}[\widetilde{G}_i(\overline{\boldsymbol{\zeta}})] + \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \beta(\varsigma) - \overline{\boldsymbol{\zeta}} \rangle + \|\beta(\varsigma) - \overline{\boldsymbol{\zeta}}\| \varphi(\beta(\varsigma), \overline{\boldsymbol{\zeta}}) \\ &= \mathcal{H}[\widetilde{G}_i(\overline{\boldsymbol{\zeta}})] + \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \beta(\varsigma) - \beta(0) \rangle + \|\beta(\varsigma) - \beta(0)\| \varphi(\beta(\varsigma), \beta(0)) \\ &= \mathcal{H}[\widetilde{G}_i(\overline{\boldsymbol{\zeta}})] + \varsigma \langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), \frac{\beta(0+\varsigma) - \beta(0)}{\varsigma} \rangle \\ &\quad + \|\beta(\varsigma) - \beta(0)\| \varphi(\beta(\varsigma), \beta(0)), \end{aligned} \tag{6}$$

where, for $\varsigma \to 0$, we get

$$lim_{\varsigma \to 0} \|\beta(\varsigma) - \beta(0)\| = 0,$$

and
$$\frac{\beta(0+\varsigma)-\beta(0)}{\varsigma} \to \beta^{'}(0) = \gamma d.$$

From (6) and the assumption $\langle \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}), d \rangle < 0, \forall i = 1, \cdots, p$, we have
$$\mathcal{H}[\widetilde{G}_i(\beta(\varsigma))] < \mathcal{H}[\widetilde{G}_i(\overline{\boldsymbol{\zeta}})], \forall i = 1, \cdots, p,$$
for sufficiently small $\varsigma > 0$.

According to Proposition 1 and $\mathcal{H}[\widetilde{q}_j(\beta(\varsigma))] \le 0$, we get
$$\widetilde{G}_i(\beta(\varsigma)) \prec \widetilde{G}_i(\overline{\boldsymbol{\zeta}}), \forall i = 1, \cdots, p,$$
and
$$\widetilde{q}_j(\beta(\varsigma)) \preceq \widetilde{0}, \forall j = 1, \cdots, m,$$
which conflicts $\overline{\boldsymbol{\zeta}}$ is a WES for (FCMOP). So, the system (4)–(5) has no solution. From Motzkin's alternative theorem, there exist $\lambda_i^{gr} \ge 0, \boldsymbol{\lambda}^{gr} \ne \mathbf{0}, r_j^{gr} \ge 0, i = 1, \cdots, p$ and $j \in J(\overline{\boldsymbol{\zeta}})$, such that
$$\sum_{i=1}^{p} \lambda_i^{gr} \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) + \sum_{j \in J(\overline{\boldsymbol{\zeta}})}^{m} r_j^{gr} \nabla q_j^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{q_j}) = \mathbf{0}.$$

If we setting $r_j^{gr} = 0$ for all $j \in \{1, \cdots, m\} \setminus J(\overline{\boldsymbol{\zeta}})$, we obtain
$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i^{gr} \nabla G_i^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{G_i}) + \sum_{j=1}^{m} r_j^{gr} \nabla q_j^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{q_j}) = \mathbf{0}, \\
&r_j^{gr} q_j^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{q_j}) = 0, j = 1, \cdots, m, \\
&\lambda_i^{gr} \ge 0, \boldsymbol{\lambda}^{gr} \ne \mathbf{0}, r_j^{gr} \ge 0, i = 1, \cdots, p.
\end{aligned}$$
are contented at $\overline{\boldsymbol{\zeta}}$. □

By the gr-convexity of objective and constraint functions, we obtain the guidelines with sufficient optimality as follows.

**Theorem 9.** *Assume that $\overline{\boldsymbol{\zeta}}$ is a feasible result for the (FCMOP) and $\widetilde{\mathbf{G}}(\boldsymbol{\zeta}) : K \to E^p$ is a V-gr-pseudoconvex and vector gr-differentiable fuzzy function, and $\widetilde{\mathbf{q}}(\boldsymbol{\zeta}) : K \to E^m$ is a V-gr-quasiconvex and vector gr-differentiable fuzzy function at $\overline{\boldsymbol{\zeta}}$. If there exist multipliers $\mathbf{0} < \boldsymbol{\lambda}^{gr} \in R^p$ and $\mathbf{0} \le \boldsymbol{r}^{gr} \in R^m$, such that (2)–(3) are satisfied. Then, $\overline{\boldsymbol{\zeta}}$ is an ES for the (FCMOP).*

**Proof.** Assume that $\boldsymbol{\zeta} \in S$ is a feasible result of the issue (FCMOP). Then,
$$\widetilde{q}_j(\boldsymbol{\zeta}) \preceq \widetilde{0}, \forall j = 1, \cdots, m,$$
from Proposition 1, which means
$$\mathcal{H}[\widetilde{q}_j(\boldsymbol{\zeta})] \le 0, \forall j = 1, \cdots, m.$$

Since (2)–(3) are satisfied at $\overline{\boldsymbol{\zeta}}$, then for $\mathbf{0} \le \boldsymbol{r}^{gr} \in R^m$, we obtain
$$r_j^{gr} q_j^{gr}(\overline{\boldsymbol{\zeta}}, \mu, \alpha_{q_j}) = 0, j = 1, \cdots, m,$$
which means

$$\mathcal{H}[r_j^{gr}\widetilde{q}_j(\overline{\zeta})] = 0, j = 1, \cdots, m.$$

Then, from Remark 1, we obtain

$$\sum_{j=1}^{m} r_j^{gr}\mathcal{H}[\widetilde{q}_j(\zeta)] \leq \sum_{j=1}^{m} r_j^{gr}\mathcal{H}[\widetilde{q}_j(\overline{\zeta})].$$

By the V-gr-quasiconvexity of $\widetilde{\mathbf{q}}(\zeta)$ and the above inequality, we obtain

$$\sum_{j=1}^{m} r_j^{gr}\langle \nabla q_j^{gr}(\overline{\zeta}, \mu, \alpha_{q_j}), \zeta - \overline{\zeta}\rangle \leq 0.$$

Consequently, from (2), we obtain

$$\sum_{i=1}^{p} \lambda_i^{gr}\langle \nabla G_i^{gr}(\overline{\zeta}, \mu, \alpha_{G_i}), \zeta - \overline{\zeta}\rangle \geq 0.$$

By the V-gr-pseudoconvexity of $\widetilde{\mathbf{G}}(\zeta)$ and the inequality above, we obtain

$$\sum_{i=1}^{p} \beta_i^{gr}(\zeta, \overline{\zeta}, \mu, \alpha_{G_i})\mathcal{H}(\widetilde{G}_i(\zeta)) \geq \sum_{i=1}^{p} \beta_i^{gr}(\zeta, \overline{\zeta}, \mu, \alpha_{G_i})\mathcal{H}(\widetilde{G}_i(\overline{\zeta})), \tag{7}$$

is contented for $\beta_i^{gr}(\zeta, \overline{\zeta}, \mu, \alpha_{G_i}) : K \times K \times [0,1]^{j+1} \to R^+ - \{0\}$.

On the contrary, assume that $\overline{\zeta}$ is not an ES for the (FCMOP). Then, there exists some point $\zeta \in S$, such that

$$\widetilde{\mathbf{G}}(\zeta) \preceq \widetilde{\mathbf{G}}(\overline{\zeta}),$$

and there exists $k$ such that

$$\widetilde{G}_k(\zeta) \prec \widetilde{G}_k(\overline{\zeta}), k \in \{1, \cdots, p\}.$$

By Proposition 1 and $0 < \beta_i^{gr}$, we have

$$\sum_{i=1}^{p} \beta_i^{gr}\mathcal{H}(\widetilde{G}_i(\zeta)) < \sum_{i=1}^{p} \beta_i^{gr}\mathcal{H}(\widetilde{G}_i(\overline{\zeta})),$$

which is a conflict to (7). So, $\overline{\zeta}$ is an ES for the (FCMOP). □

**Example 6.** *Consider the example as follows.*

$$\begin{aligned}
&\textit{minimize} \quad \widetilde{\mathbf{G}}(\zeta) = (\widetilde{G}_1(\zeta), \widetilde{G}_2(\zeta)) \\
&\textit{S.t.}\ \ \widetilde{G}_1(\zeta) = \widetilde{v}_1(\zeta_1 - 1)^2 + \widetilde{v}_2(\zeta_2 - 1)^2, \\
&\qquad\ \ \widetilde{G}_2(\zeta) = \widetilde{v}_3(\zeta_1 - \zeta_2)^2, \\
&\qquad\ \ \widetilde{q}_1(\zeta) = \zeta_1 + \widetilde{v}_4\zeta_2 - 6 \succeq \widetilde{0}, \\
&\qquad\ \ \widetilde{q}_2(\zeta) = \widetilde{v}_5\zeta_1 + \zeta_2 - 6 \succeq \widetilde{0},
\end{aligned}$$

*where* $\widetilde{v}_1 = (0,1,2), \widetilde{v}_2 = (0,1,2), \widetilde{v}_3 = (3,4,5), \widetilde{v}_4 = (1,2,3)$ *and* $\widetilde{v}_5 = (1,2,3)$.

*The HMF of* $\widetilde{G}_1(\zeta)$ *and* $\widetilde{G}_2(\zeta)$ *as follows,*

$$\mathcal{H}(\widetilde{G}_1(\zeta)) = [\mu + (2 - 2\mu)\alpha_1](\zeta_1 - 1)^2 + [\mu + (2 - 2\mu)\alpha_2](\zeta_2 - 1)^2,$$

*and*

$$\mathcal{H}(\widetilde{G}_2(\zeta)) = [3 + \mu + (2 - 2\mu)\alpha_3](\zeta_1 - \zeta_2)^2.$$

*The HMF of $\widetilde{q}_1(\boldsymbol{\zeta})$ and $\widetilde{q}_2(\boldsymbol{\zeta})$ as follows,*

$$\mathcal{H}(\widetilde{q}_1(\boldsymbol{\zeta})) = \zeta_1 + [1 + \mu + (2 - 2\mu)\alpha_4]\zeta_2 - 6,$$

*and*

$$\mathcal{H}(\widetilde{q}_2(\boldsymbol{\zeta})) = [1 + \mu + (2 - 2\mu)\alpha_5]\zeta_1 + \zeta_2 - 6.$$

*Then, we have*

$$\nabla G_1^{gr}(\boldsymbol{\zeta}, \mu, \alpha_{G_1}) = (2[\mu + (2 - 2\mu)\alpha_1](\zeta_1 - 1), 2[\mu + (2 - 2\mu)\alpha_2](\zeta_2 - 1))^T,$$

*and*

$$\nabla G_2^{gr}(\boldsymbol{\zeta}, \mu, \alpha_3) = (2[3 + \mu + (2 - 2\mu)\alpha_3](\zeta_1 - \zeta_2), -2[3 + \mu + (2 - 2\mu)\alpha_3](\zeta_1 - \zeta_2))^T,$$

*where $\alpha_{G_1} = (\alpha_1, \alpha_2)$.*

$$\nabla q_1^{gr}(\boldsymbol{\zeta}, \mu, \alpha_4) = (1, [1 + \mu + (2 - 2\mu)\alpha_4])^T,$$

*and*

$$\nabla q_2^{gr}(\boldsymbol{\zeta}, \mu, \alpha_5) = ([1 + \mu + (2 - 2\mu)\alpha_5], 1)^T.$$

*By the optimality guidelines, we have*

$$\begin{aligned}
&2\lambda_1^{gr}[\mu + (2 - 2\mu)\alpha_1](\zeta_1 - 1) + 2\lambda_2^{gr}[3 + \mu + (2 - 2\mu)\alpha_3](\zeta_1 - \zeta_2) \\
&\quad - r_1^{gr} - r_2^{gr}[1 + \mu + (2 - 2\mu)\alpha_5] = 0, \\
&2\lambda_1^{gr}[\mu + (2 - 2\mu)\alpha_2](\zeta_2 - 1) - 2\lambda_2^{gr}[3 + \mu + (2 - 2\mu)\alpha_3](\zeta_1 - \zeta_2) \\
&\quad - r_1^{gr}[1 + \mu + (2 - 2\mu)\alpha_4] - r_2^{gr} = 0, \\
&r_1^{gr}\{\zeta_1 + [1 + \mu + (2 - 2\mu)\alpha_4]\zeta_2 - 6\} = 0, \\
&r_2^{gr}\{[1 + \mu + (2 - 2\mu)\alpha_5]\zeta_1 + \zeta_2 - 6\} = 0, \\
&\lambda_i^{gr} \geq 0, r_j^{gr} \geq 0, i = 1, 2, j = 1, 2, \\
&\zeta_1 + [1 + \mu + (2 - 2\mu)\alpha_4]\zeta_2 - 6 \geq 0, \\
&[1 + \mu + (2 - 2\mu)\alpha_5]\zeta_1 + \zeta_2 - 6 \geq 0.
\end{aligned}$$

*where $\mu, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in [0, 1]$.*

*By calculating, we obtain $\boldsymbol{\zeta}^* = (2, 2)^T$ is a KKT point when $\mu = 1, \alpha_\sigma = 0, \sigma = 1, 2, 3, 4, 5$ and $\lambda_1^{gr} = 1, r_1^{gr} = r_2^{gr} = \frac{2}{3}$. Seeing the vector fuzzy function $\widetilde{\mathbf{G}}(\boldsymbol{\zeta})$ is a V-gr-convex vector fuzzy function and the vector fuzzy function $\widetilde{\mathbf{q}}(\boldsymbol{\zeta})$ is a linear vector fuzzy function is easy. So, this problem satisfy the assumptions of Theorem 9. Then, $\boldsymbol{\zeta}^*$ is an ES for this issue.*

## 5. Conclusions

In this thesiss, for vector fuzzy-valued functions, we recommended the notions of V-gr-convexity, V-gr-pseudo-convexity, and V-gr-quasiconvexity. These are the vector real-valued functions' extensions of convexity, pseudoconvexity, and quasiconvexity. V-gr-convexity, V-gr-pseudoconvexity, V-gr-quasiconvexity, and V-gr-differentiability for vector fuzzy-valued functions were also explored. Under the assumptions of V-gr-convexity and V-gr-differentiability, we also certificated the optimality guidelines. Our findings extended the previous findings to the (FMOP) and (FCMOP).

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