fractal and fractional

Article

Systems of Riemann–Liouville Fractional Differential Equations with ρ-Laplacian Operators and Nonlocal Coupled Boundary Conditions

Alexandru Tudorache [1] and Rodica Luca [2,*]

[1] Department of Computer Science and Engineering, Gh. Asachi Technical University, 700050 Iasi, Romania
[2] Department of Mathematics, Gh. Asachi Technical University, 700506 Iasi, Romania
* Correspondence: rluca@math.tuiasi.ro

Abstract: In this paper, we study the existence of positive solutions for a system of fractional differential equations with ρ-Laplacian operators, Riemann–Liouville derivatives of diverse orders and general nonlinearities which depend on several fractional integrals of differing orders, supplemented with nonlocal coupled boundary conditions containing Riemann–Stieltjes integrals and varied fractional derivatives. The nonlinearities from the system are continuous nonnegative functions and they can be singular in the time variable. We write equivalently this problem as a system of integral equations, and then we associate an operator for which we are looking for its fixed points. The main results are based on the Guo–Krasnosel'skii fixed point theorem of cone expansion and compression of norm type.

Keywords: Riemann–Liouville fractional differential equations; nonlocal coupled boundary conditions; singular functions; positive solutions; multiplicity

MSC: 34A08; 34B10; 34B16; 34B18

Citation: Tudorache, A.; Luca, R. Systems of Riemann–Liouville Fractional Differential Equations with ρ-Laplacian Operators and Nonlocal Coupled Boundary Conditions. *Fractal Fract.* **2022**, *6*, 610. https://doi.org/10.3390/fractalfract6100610

Academic Editor: Maria Rosaria Lancia

Received: 25 September 2022
Accepted: 14 October 2022
Published: 19 October 2022

1. Introduction

We consider the system of Riemann–Liouville fractional differential equations with ρ_1-Laplacian and ρ_2-Laplacian operators

$$\begin{cases} D_{0+}^{\delta_1}(\varphi_{\rho_1}(D_{0+}^{\gamma_1}x(t))) = \mathfrak{f}(t, x(t), y(t), I_{0+}^{\mu_1}x(t), I_{0+}^{\mu_2}y(t)), \ \ t\in(0,1), \\ D_{0+}^{\delta_2}(\varphi_{\rho_2}(D_{0+}^{\gamma_2}y(t))) = \mathfrak{g}(t, x(t), y(t), I_{0+}^{\nu_1}x(t), I_{0+}^{\nu_2}y(t)), \ \ t\in(0,1), \end{cases} \tag{1}$$

subject to the nonlocal coupled boundary conditions

$$\begin{cases} x^{(j)}(0) = 0, \ \ j = 0,\dots,p-2, \ \ D_{0+}^{\gamma_1}x(0) = 0, \\ \varphi_{\rho_1}(D_{0+}^{\gamma_1}x(1)) = \displaystyle\int_0^1 \varphi_{\rho_1}(D_{0+}^{\gamma_1}x(\tau))\,d\mathfrak{M}_0(\tau), \ \ D_{0+}^{\alpha_0}x(1) = \sum_{k=1}^{n}\int_0^1 D_{0+}^{\alpha_k}y(\tau)\,d\mathfrak{M}_k(\tau), \\ y^{(j)}(0) = 0, \ \ j = 0,\dots,q-2, \ \ D_{0+}^{\gamma_2}y(0) = 0, \\ \varphi_{\rho_2}(D_{0+}^{\gamma_2}y(1)) = \displaystyle\int_0^1 \varphi_{\rho_2}(D_{0+}^{\gamma_2}y(\tau))\,d\mathfrak{N}_0(\tau), \ \ D_{0+}^{\beta_0}y(1) = \sum_{k=1}^{m}\int_0^1 D_{0+}^{\beta_k}x(\tau)\,d\mathfrak{N}_k(\tau), \end{cases} \tag{2}$$

where $\delta_1, \delta_2 \in (1,2]$, $\gamma_1 \in (p-1,p]$, $p\in\mathbb{N}$, $p\geq 3$, $\gamma_2\in(q-1,q]$, $q\in\mathbb{N}$, $q\geq 3$, $n,m\in\mathbb{N}$, $\mu_1,\mu_2,\nu_1,\nu_2>0$, $\alpha_k\in\mathbb{R}$, $k=0,\dots,n$, $0\leq\alpha_1<\alpha_2<\cdots<\alpha_n\leq\beta_0<\gamma_2-1$, $\beta_0\geq 1$, $\beta_k\in\mathbb{R}$, $k=0,\dots,m$, $0\leq\beta_1<\beta_2<\cdots<\beta_m\leq\alpha_0<\gamma_1-1$, $\alpha_0\geq 1$, $\varphi_{\rho_i}(s)=|s|^{\rho_i-2}s$, $\varphi_{\rho_i}^{-1}=\varphi_{\varrho_i}$, $\varrho_i=\frac{\rho_i}{\rho_i-1}$, $i=1,2$, $\rho_i>1$, $i=1,2$, $\mathfrak{f},\mathfrak{g}:(0,1)\times\mathbb{R}_+^4\to\mathbb{R}_+$ are continuous functions, singular at $t=0$ and/or $t=1$, ($\mathbb{R}_+=[0,\infty)$), I_{0+}^{θ} is the Riemann–Liouville fractional integral of order θ (for $\theta=\mu_1,\mu_2,\nu_1,\nu_2$), D_{0+}^{θ} is the Riemann–Liouville fractional derivative of order θ (for $\theta=\delta_1,\gamma_1,\delta_2,\gamma_2,\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_m$), and the integrals from the boundary conditions (2) are Riemann–Stieltjes integrals with $\mathfrak{M}_i : [0,1] \to \mathbb{R},\ i = 0,\dots,n$ and $\mathfrak{N}_j : [0,1] \to \mathbb{R},\ j = 0,\dots,m$ functions of bounded variation. The present work was motivated by the applications of ρ-Laplacian operators in various fields such as nonlinear elasticity, glaciology, nonlinear electrorheological fluids, fluid flows through porous media, etc. see for details the paper [1] and its references.

In this paper, we present varied conditions for the functions $\mathfrak{f}$ and $\mathfrak{g}$ such that problem (1), (2) has a positive solution, and then it has two positive solutions. A positive solution of (1), (2) is a pair of functions $(x,y) \in (C([0,1],\mathbb{R}_+))^2$ satisfying the system (1) and the boundary conditions (2), with $x(s) > 0$ for all $s \in (0,1]$ or $y(s) > 0$ for all $s \in (0,1]$. We apply the Guo–Krasnosel'skii fixed point theorem of cone expansion and compression of norm type (see [2]) in the proof of our main results. Connected to our problem, we mention the following papers. In [3], the authors studied the existence of multiple positive solutions of the system of nonlinear fractional differential equations with p_1-Laplacian and p_2-Laplacian operators

$$\begin{cases} D_{0+}^{\beta_1}(\varphi_{p_1}(D_{0+}^{\alpha_1}x(s))) = \mathfrak{f}(s,x(s),y(s)), & s \in (0,1), \\ D_{0+}^{\beta_2}(\varphi_{p_2}(D_{0+}^{\alpha_2}y(s))) = \mathfrak{g}(s,x(s),y(s)), & s \in (0,1), \end{cases}$$

supplemented with the nonlocal uncoupled boundary conditions

$$\begin{cases} x(0) = 0,\ D_{0+}^{\gamma_1}x(1) = \sum\limits_{k=1}^{m-2} \xi_{1k} D_{0+}^{\gamma_1}x(\eta_{1k}), \\ D_{0+}^{\alpha_1}x(0) = 0,\ \varphi_{p_1}(D_{0+}^{\alpha_1}x(1)) = \sum_{k=1}^{m-2} \zeta_{1k}\varphi_{p_1}(D_{0+}^{\alpha_1}x(\eta_{1k})), \\ y(0) = 0,\ D_{0+}^{\gamma_2}y(1) = \sum\limits_{k=1}^{m-2} \xi_{2k} D_{0+}^{\gamma_2}y(\eta_{2k}), \\ D_{0+}^{\alpha_2}y(0) = 0,\ \varphi_{p_2}(D_{0+}^{\alpha_2}y(1)) = \sum_{k=1}^{m-2} \zeta_{2k}\varphi_{p_2}(D_{0+}^{\alpha_2}y(\eta_{2k})), \end{cases}$$

where $\alpha_i, \beta_i \in (1,2]$, $\gamma_i \in (0,1]$, $\alpha_i + \beta_i \in (3,4]$, $\alpha_i > \gamma_i + 1$, $i = 1,2$, $\xi_{1k}, \eta_{1k}, \zeta_{1k}, \xi_{2k}, \eta_{2k}, \zeta_{2k} \in (0,1)$ for $k = 1,\dots,m-2$, $p_1, p_2 > 1$, and $\mathfrak{f}$ and $\mathfrak{g}$ are nonnegative and nonsingular functions. They applied the Leray-Schauder alternative theorem, the Leggett-Williams fixed point theorem and the Avery-Henderson fixed point theorem in the proof of the existence results. In [4], the authors studied the existence and nonexistence of positive solutions for the system of Riemann–Liouville fractional differential equations with ϱ_1-Laplacian and ϱ_2-Laplacian operators

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\varrho_1}(D_{0+}^{\delta_1}x(s))) + \lambda\mathfrak{f}(s,x(s),y(s)) = 0, & s \in (0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\varrho_2}(D_{0+}^{\delta_2}y(s))) + \mu\mathfrak{g}(s,x(s),y(s)) = 0, & s \in (0,1), \end{cases} \tag{3}$$

subject to the coupled nonlocal boundary conditions

$$\begin{cases} x^{(j)}(0) = 0,\ j = 0,\dots,p-2;\ D_{0+}^{\delta_1}x(0) = 0,\ D_{0+}^{\alpha_0}x(1) = \sum\limits_{k=1}^{n} \int_0^1 D_{0+}^{\alpha_k}y(\zeta)\,d\mathfrak{M}_k(\zeta), \\ y^{(j)}(0) = 0,\ j = 0,\dots,q-2;\ D_{0+}^{\delta_2}y(0) = 0,\ D_{0+}^{\beta_0}y(1) = \sum\limits_{k=1}^{m} \int_0^1 D_{0+}^{\beta_k}x(\zeta)\,d\mathfrak{N}_k(\zeta), \end{cases} \tag{4}$$

where λ and μ are positive parameters, $\gamma_1,\ \gamma_2 \in (0,1]$, $\delta_1 \in (p-1,p]$, $\delta_2 \in (q-1,q]$, $p,\ q \in \mathbb{N}$, $p,\ q \geq 3$, $n,m \in \mathbb{N}$, $\alpha_k \in \mathbb{R}$ for all $k = 0,\dots,n$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq \beta_0 < \delta_2 - 1$, $\beta_0 \geq 1$, $\beta_k \in \mathbb{R}$ for all $k = 0,\dots,m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \alpha_0 < \delta_1 - 1$, $\alpha_0 \geq 1$, $\varrho_1,\ \varrho_2 > 1$, the functions $\mathfrak{f}, \mathfrak{g} \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and the functions $\mathfrak{M}_j$, $j = 1,\dots,n$ and $\mathfrak{N}_k$, $k = 1,\dots,m$ are bounded variation functions. They presented sufficient conditions on the functions $\mathfrak{f}$ and $\mathfrak{g}$, and intervals for the parameters λ and μ such that problem (3), (4) has positive solutions. In [5], the authors investigated the existence and multiplicity of positive solutions for the system (3) with $\lambda = \mu = 1$, supplemented with the uncoupled nonlocal boundary conditions

$$\begin{cases} x^{(j)}(0) = 0, \ j = 0, \ldots, p-2; \ D_{0+}^{\delta_1} x(0) = 0, \ D_{0+}^{\alpha_0} x(1) = \sum_{k=1}^{n} \int_0^1 D_{0+}^{\alpha_k} x(\zeta)\, d\mathfrak{M}_k(\zeta), \\ y^{(j)}(0) = 0, \ j = 0, \ldots, q-2; \ D_{0+}^{\delta_2} y(0) = 0, \ D_{0+}^{\beta_0} y(1) = \sum_{k=1}^{m} \int_0^1 D_{0+}^{\beta_k} y(\zeta)\, d\mathfrak{N}_k(\zeta), \end{cases}$$

where $n, m \in \mathbb{N}$, $\alpha_k \in \mathbb{R}$ for all $k = 0, 1, \ldots, n$, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le \alpha_0 < \delta_1 - 1$, $\alpha_0 \ge 1$, $\beta_k \in \mathbb{R}$ for all $k = 0, 1, \ldots, m$, $0 \le \beta_1 < \beta_2 < \cdots < \beta_m \le \beta_0 < \delta_2 - 1$, $\beta_0 \ge 1$, the functions $\mathfrak{f}$ and $\mathfrak{g}$ from system (3) are nonnegative and continuous, and they may be singular at $s = 0$ and/or $s = 1$, and $\mathfrak{M}_j$, $j = 1, \ldots, n$ and $\mathfrak{N}_k$, $k = 1, \ldots, m$ are functions of bounded variation. They applied the Guo–Krasnosel'skii fixed point theorem in the proof of the main existence results. In [6] the authors studied the existence and multiplicity of positive solutions for the system (1) subject to general uncoupled boundary conditions in the point $t = 1$. We mention that our problem (1), (2) is different than the problems from papers [4,6]. Indeed the orders of the first fractional derivatives in the system (3) (from [4]) are positive numbers less than or equal to 1, and in our system (1) the first fractional derivatives are numbers greater than 1 and less than or equal to 2. This difference conducts to the consideration of different boundary conditions (more precisely, for our problem, we have a bigger number of such boundary conditions)—see (2) and (4). Another differences are the presence of the parameters in system (3)—here, we do not have any parameters, and also the nonlinearities $\mathfrak{f}$ and $\mathfrak{g}$ from (3) which are nonsingular functions, as opposed to our problem in which the functions $\mathfrak{f}$ and $\mathfrak{g}$ are singular; so here is a more difficult case to study. On the other hand, the essential difference between the present problem (1), (2) and the problem studied in [6], is given by the boundary conditions. In [6] the last boundary conditions for the unknown functions are uncoupled in the point 1, and here in (2), the last boundary conditions for the unknown functions x and y are coupled in the point 1; that is, the fractional derivative of order α_0 of function x in the point 1 is dependent of varied fractional derivatives of function y, and the fractional derivative of order β_0 of function y in 1 is dependent of various fractional derivatives of function x. Hence the novelty of our problem (1), (2) is represented by a combination between the existence of ρ-Laplacian operators in system (1), the dependence of the nonlinearities in (1) on diverse fractional integrals, and the nature of the last boundary conditions in the point 1 which are coupled here. We also mention the recent papers [7–12] in which the authors study fractional differential equations and systems with ρ-Laplacian operators, and some recent monographs devoted to the investigation of boundary value problems for fractional differential equations and systems, namely [13–17].

The paper is organized in the following way. In Section 2, some auxiliary results which include the properties of the Green functions associated to our problem (1), (2) are given. In Section 3 we present the system of integral equations corresponding to our problem, and the main existence and multiplicity theorems for positive solutions of (1), (2), and Section 4 contains their proofs. Finally, two examples which illustrate our obtained results are presented in Section 5, and the conclusions are given in Section 6.

2. Auxiliary Results

In this section, we consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{\delta_1}(\varphi_{\rho_1}(D_{0+}^{\gamma_1} x(t))) = u(t), \ \ t \in (0,1), \\ D_{0+}^{\delta_2}(\varphi_{\rho_2}(D_{0+}^{\gamma_2} y(t))) = v(t), \ \ t \in (0,1), \end{cases} \tag{5}$$

with the coupled boundary conditions (2), where $u, v \in C(0,1) \cap L^1(0,1)$.

We denote $\varphi_{\rho_1}(D_{0+}^{\gamma_1}x(t)) = h(t)$, $\varphi_{\rho_2}(D_{0+}^{\gamma_2}y(t)) = k(t)$. Then problem (2), (5) is equivalent to the following three problems

$$\begin{cases} D_{0+}^{\delta_1}h(t) = u(t), \ \ t \in (0,1), \\ h(0) = 0, \ \ h(1) = \int_0^1 h(\tau)\, d\mathfrak{M}_0(\tau), \end{cases} \tag{6}$$

$$\begin{cases} D_{0+}^{\delta_2}k(t) = v(t), \ \ t \in (0,1), \\ k(0) = 0, \ \ k(1) = \int_0^1 k(\tau)\, d\mathfrak{N}_0(\tau), \end{cases} \tag{7}$$

and

$$\begin{cases} D_{0+}^{\gamma_1}x(t) = \varphi_{\varrho_1}(h(t)), \ \ t \in (0,1), \\ D_{0+}^{\gamma_2}y(t) = \varphi_{\varrho_2}(k(t)), \ \ t \in (0,1), \end{cases} \tag{8}$$

with the boundary conditions

$$\begin{cases} x^{(j)}(0) = 0, \ \ j = 0,\dots,p-2, \ \ D_{0+}^{\alpha_0}x(1) = \sum_{k=1}^{n}\int_0^1 D_{0+}^{\alpha_k}y(\tau)\, d\mathfrak{M}_k(\tau), \\ y^{(j)}(0) = 0, \ \ j = 0,\dots,q-2, \ \ D_{0+}^{\beta_0}y(1) = \sum_{k=1}^{m}\int_0^1 D_{0+}^{\beta_k}x(\tau)\, d\mathfrak{N}_k(\tau). \end{cases} \tag{9}$$

By Lemma 4.1.5 from [16], the unique solution $h \in C[0,1]$ of problem (6) is

$$h(t) = -\int_0^1 \mathfrak{G}_1(t,\tau)u(\tau)\, d\tau, \ \ t \in [0,1], \tag{10}$$

where

$$\mathfrak{G}_1(t,\tau) = \mathfrak{g}_1(t,\tau) + \frac{t^{\delta_1-1}}{\mathfrak{a}_1}\int_0^1 \mathfrak{g}_1(\zeta,\tau)\, d\mathfrak{M}_0(\zeta),$$
$$\mathfrak{g}_1(t,\tau) = \frac{1}{\Gamma(\delta_1)}\begin{cases} t^{\delta_1-1}(1-\tau)^{\delta_1-1} - (t-\tau)^{\delta_1-1}, \ \ 0 \le \tau \le t \le 1, \\ t^{\delta_1-1}(1-\tau)^{\delta_1-1}, \ \ 0 \le t \le \tau \le 1, \end{cases}$$

for $(t,\tau) \in [0,1]\times[0,1]$, with $\mathfrak{a}_1 = 1 - \int_0^1 \zeta^{\delta_1-1}\, d\mathfrak{M}_0(\zeta) \neq 0$.

By the same lemma (Lemma 4.1.5 from [16]), the unique solution $k \in C[0,1]$ of problem (7) is

$$k(t) = -\int_0^1 \mathfrak{G}_2(t,\tau)v(\tau)\, d\tau, \ \ t \in [0,1], \tag{11}$$

where

$$\mathfrak{G}_2(t,\tau) = \mathfrak{g}_2(t,\tau) + \frac{t^{\delta_2-1}}{\mathfrak{a}_2}\int_0^1 \mathfrak{g}_2(\zeta,\tau)\, d\mathfrak{N}_0(\zeta),$$
$$\mathfrak{g}_2(t,\tau) = \frac{1}{\Gamma(\delta_2)}\begin{cases} t^{\delta_2-1}(1-\tau)^{\delta_2-1} - (t-\tau)^{\delta_2-1}, \ \ 0 \le \tau \le t \le 1, \\ t^{\delta_2-1}(1-\tau)^{\delta_2-1}, \ \ 0 \le t \le \tau \le 1, \end{cases}$$

for $(t,\tau) \in [0,1]\times[0,1]$, with $\mathfrak{a}_2 = 1 - \int_0^1 \zeta^{\delta_2-1}\, d\mathfrak{N}_0(\zeta) \neq 0$.

By Lemma 2.2 from [4], the unique solution $(x,y) \in (C[0,1])^2$ of problem (8), (9) is

$$\begin{cases} x(t) = -\int_0^1 \mathfrak{G}_3(t,\tau)\varphi_{\varrho_1}(h(\tau))\, d\tau - \int_0^1 \mathfrak{G}_4(t,\tau)\varphi_{\varrho_2}(k(\tau))\, d\tau, \ \ t \in [0,1], \\ y(t) = -\int_0^1 \mathfrak{G}_5(t,\tau)\varphi_{\varrho_1}(h(\tau))\, d\tau - \int_0^1 \mathfrak{G}_6(t,\tau)\varphi_{\varrho_2}(k(\tau))\, d\tau, \ \ t \in [0,1], \end{cases} \tag{12}$$

where

$$\begin{aligned}
\mathfrak{G}_3(t,\tau) &= \mathfrak{g}_3(t,\tau) + \frac{t^{\gamma_1-1}\mathfrak{b}_1}{\mathfrak{b}}\left(\sum_{i=1}^{m}\int_0^1 \mathfrak{g}_{3i}(\vartheta,\tau)\,d\mathfrak{N}_i(\vartheta)\right),\\
\mathfrak{G}_4(t,\tau) &= \frac{t^{\gamma_1-1}\Gamma(\gamma_2)}{\mathfrak{b}\Gamma(\gamma_2-\beta_0)}\sum_{i=1}^{n}\int_0^1 \mathfrak{g}_{4i}(\vartheta,\tau)\,d\mathfrak{M}_i(\vartheta),\\
\mathfrak{G}_5(t,\tau) &= \frac{t^{\gamma_2-1}\Gamma(\gamma_1)}{\mathfrak{b}\Gamma(\gamma_1-\alpha_0)}\sum_{i=1}^{m}\int_0^1 \mathfrak{g}_{3i}(\vartheta,\tau)\,d\mathfrak{N}_i(\vartheta),\\
\mathfrak{G}_6(t,\tau) &= \mathfrak{g}_4(t,\tau) + \frac{t^{\gamma_2-1}\mathfrak{b}_2}{\mathfrak{b}}\left(\sum_{i=1}^{n}\int_0^1 \mathfrak{g}_{4i}(\vartheta,\tau)\,d\mathfrak{M}_i(\vartheta)\right),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{g}_3(t,\tau) &= \frac{1}{\Gamma(\gamma_1)}\begin{cases} t^{\gamma_1-1}(1-\tau)^{\gamma_1-\alpha_0-1}-(t-\tau)^{\gamma_1-1}, & 0\le\tau\le t\le 1,\\ t^{\gamma_1-1}(1-\tau)^{\gamma_1-\alpha_0-1}, & 0\le t\le\tau\le 1,\end{cases}\\
\mathfrak{g}_{3i}(\vartheta,\tau) &= \frac{1}{\Gamma(\gamma_1-\beta_i)}\begin{cases} \vartheta^{\gamma_1-\beta_i-1}(1-\tau)^{\gamma_1-\alpha_0-1}-(\vartheta-\tau)^{\gamma_1-\beta_i-1}, & 0\le\tau\le\vartheta\le 1,\\ \vartheta^{\gamma_1-\beta_i-1}(1-\tau)^{\gamma_1-\alpha_0-1}, & 0\le\vartheta\le\tau\le 1,\end{cases}\\
\mathfrak{g}_4(t,\tau) &= \frac{1}{\Gamma(\gamma_2)}\begin{cases} t^{\gamma_2-1}(1-\tau)^{\gamma_2-\beta_0-1}-(t-\tau)^{\gamma_2-1}, & 0\le\tau\le t\le 1,\\ t^{\gamma_2-1}(1-\tau)^{\gamma_2-\beta_0-1}, & 0\le t\le\tau\le 1,\end{cases}\\
\mathfrak{g}_{4j}(\vartheta,\tau) &= \frac{1}{\Gamma(\gamma_2-\alpha_j)}\begin{cases} \vartheta^{\gamma_2-\alpha_j-1}(1-\tau)^{\gamma_2-\beta_0-1}-(\vartheta-\tau)^{\gamma_2-\alpha_j-1}, & 0\le\tau\le\vartheta\le 1,\\ \vartheta^{\gamma_2-\alpha_j-1}(1-\tau)^{\gamma_2-\beta_0-1}, & 0\le\vartheta\le\tau\le 1,\end{cases}
\end{aligned}$$

for all $t,\tau,\vartheta\in[0,1]$, $i=1,\dots,m$, $j=1,\dots,n$, and $\mathfrak{b}_1=\sum_{i=1}^{n}\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_i)}\int_0^1 \zeta^{\gamma_2-\alpha_i-1}\,d\mathfrak{M}_i(\zeta)$, $\mathfrak{b}_2=\sum_{i=1}^{m}\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1-\beta_i)}\int_0^1 \zeta^{\gamma_1-\beta_i-1}\,d\mathfrak{N}_i(\zeta)$, and $\mathfrak{b}=\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1-\alpha_0)\Gamma(\gamma_2-\beta_0)}-\mathfrak{b}_1\mathfrak{b}_2\neq 0$.

Combining the above Formulas (10)–(12) for $h(t),\ k(t),\ x(t),\ y(t),\ t\in[0,1]$, we obtain the following result.

Lemma 1. *If $\mathfrak{a}_1\neq 0$, $\mathfrak{a}_2\neq 0$ and $\mathfrak{b}\neq 0$, then the unique solution $(x,y)\in(C[0,1])^2$ of problem (5), (2) is given by*

$$\begin{aligned}
x(t) &= \int_0^1 \mathfrak{G}_3(t,\tau)\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)u(\zeta)\,d\zeta\right)d\tau\\
&\quad+\int_0^1 \mathfrak{G}_4(t,\tau)\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{G}_2(\tau,\zeta)v(\zeta)\,d\zeta\right)d\tau,\ \ \forall\, t\in[0,1],\\
y(t) &= \int_0^1 \mathfrak{G}_5(t,\tau)\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)u(\zeta)\,d\zeta\right)d\tau\\
&\quad+\int_0^1 \mathfrak{G}_6(t,\tau)\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{G}_2(\tau,\zeta)v(\zeta)\,d\zeta\right)d\tau,\ \ \forall\, t\in[0,1].
\end{aligned}$$

Now by using the properties of functions $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$, $\mathfrak{g}_{3i}$, $i=1,\dots,m$, $\mathfrak{g}_4$, $\mathfrak{g}_{4j}$, $j=1,\dots,n$ (see [14,16]), we deduce the following properties of the functions $\mathfrak{G}_i$, $i=1,\dots,6$.

Lemma 2. *We suppose that $\mathfrak{a}_1>0$, $\mathfrak{a}_2>0$ and $\mathfrak{b}>0$, $\mathfrak{M}_i$, $i=1,\dots,n$ and $\mathfrak{N}_j$, $j=0,\dots,m$ are nondecreasing functions. Then the functions $\mathfrak{G}_i$, $i=1,\dots,6$ have the properties:*

(a) $\mathfrak{G}_i:[0,1]\times[0,1]\to[0,\infty)$, $i=1,\dots,6$ *are continuous functions.*

(b) $\mathfrak{G}_1(t,\tau)\le\mathfrak{J}_1(\tau)$, *for all* $(t,\tau)\in[0,1]\times[0,1]$, *where*

$$\mathfrak{J}_1(\tau)=\mathfrak{h}_1(\tau)+\frac{1}{\mathfrak{a}_1}\int_0^1 \mathfrak{g}_1(\zeta,\tau)\,d\mathfrak{M}_0(\zeta),\ \ \forall\,\tau\in[0,1],$$

with $\mathfrak{h}_1(\tau)=\frac{1}{\Gamma(\delta_1)}(1-\tau)^{\delta_1-1},\ \tau\in[0,1]$.

(c) $\mathfrak{G}_2(t,\tau)\le\mathfrak{J}_2(\tau)$, *for all* $(t,\tau)\in[0,1]\times[0,1]$, *where*

$$\mathfrak{J}_2(\tau)=\mathfrak{h}_2(\tau)+\frac{1}{\mathfrak{a}_2}\int_0^1 \mathfrak{g}_2(\zeta,\tau)\,d\mathfrak{N}_0(\zeta),\ \ \forall\,\tau\in[0,1],$$

with $\mathfrak{h}_2(\tau) = \frac{1}{\Gamma(\delta_2)}(1-\tau)^{\delta_2-1},\ \tau \in [0,1]$.

(d) $\mathfrak{G}_3(t,\tau) \le \mathfrak{J}_3(\tau)$, *for all* $(t,\tau) \in [0,1] \times [0,1]$, *where*

$$\mathfrak{J}_3(\tau) = \mathfrak{h}_3(\tau) + \frac{\mathfrak{b}_1}{\mathfrak{b}} \left(\sum_{i=1}^{m} \int_0^1 \mathfrak{g}_{3i}(\vartheta,\tau)\, d\mathfrak{N}_i(\vartheta) \right),\ \ \forall\, \tau \in [0,1],$$

with $\mathfrak{h}_3(\tau) = \frac{1}{\Gamma(\gamma_1)}(1-\tau)^{\gamma_1-\alpha_0-1}(1-(1-\tau)^{\alpha_0}),\ \ \tau \in [0,1]$.

(e) $\mathfrak{G}_3(t,\tau) \ge t^{\gamma_1-1}\mathfrak{J}_3(\tau)$, *for all* $(t,\tau) \in [0,1] \times [0,1]$.

(f) $\mathfrak{G}_4(t,\tau) \le \mathfrak{J}_4(\tau)$, *for all* $(t,\tau) \in [0,1] \times [0,1]$, *where*

$$\mathfrak{J}_4(\tau) = \frac{\Gamma(\gamma_2)}{\mathfrak{b}\Gamma(\gamma_2-\beta_0)} \sum_{i=1}^{n} \int_0^1 \mathfrak{g}_{4i}(\vartheta,\tau)\, d\mathfrak{M}_i(\vartheta),\ \ \forall\, \tau \in [0,1].$$

(g) $\mathfrak{G}_4(t,\tau) = t^{\gamma_1-1}\mathfrak{J}_4(\tau)$, *for all* $(t,\tau) \in [0,1] \times [0,1]$.

(h) $\mathfrak{G}_5(t,\tau) \le \mathfrak{J}_5(\tau)$, *for all* $(t,\tau) \in [0,1] \times [0,1]$, *where*

$$\mathfrak{J}_5(\tau) = \frac{\Gamma(\gamma_1)}{\mathfrak{b}\Gamma(\gamma_1-\alpha_0)} \sum_{i=1}^{m} \int_0^1 \mathfrak{g}_{3i}(\vartheta,\tau)\, d\mathfrak{N}_i(\vartheta),\ \ \forall\, \tau \in [0,1].$$

(i) $\mathfrak{G}_5(t,\tau) = t^{\gamma_2-1}\mathfrak{J}_5(\tau)$, *for all* $(t,\tau) \in [0,1] \times [0,1]$.

(j) $\mathfrak{G}_6(t,\tau) \le \mathfrak{J}_6(\tau)$, *for all* $(t,\tau) \in [0,1] \times [0,1]$, *where*

$$\mathfrak{J}_6(\tau) = \mathfrak{h}_4(\tau) + \frac{\mathfrak{b}_2}{\mathfrak{b}} \left(\sum_{i=1}^{n} \int_0^1 \mathfrak{g}_{4i}(\vartheta,\tau)\, d\mathfrak{M}_i(\vartheta) \right),\ \ \forall\, \tau \in [0,1],$$

with $\mathfrak{h}_4(\tau) = \frac{1}{\Gamma(\gamma_2)}(1-\tau)^{\gamma_2-\beta_0-1}(1-(1-\tau)^{\beta_0}),\ \ \tau \in [0,1]$.

(k) $\mathfrak{G}_6(t,\tau) \ge t^{\gamma_2-1}\mathfrak{J}_6(\tau)$, *for all* $(t,\tau) \in [0,1] \times [0,1]$.

Under the assumptions of Lemma 2, we find that $\mathfrak{J}_i(\tau) \ge 0$ for all $\tau \in [0,1]$ and $i = 1,\ldots,6$, and $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \mathfrak{J}_6 \not\equiv 0$. In addition, $\mathfrak{J}_4 \equiv 0$ if all the functions $\mathfrak{M}_i,\ i = 1,\ldots,n$ are constant, and $\mathfrak{J}_5 \equiv 0$ if all the functions $\mathfrak{N}_j,\ j = 1,\ldots,m$ are constant.

We also deduce easily the next lemma.

Lemma 3. *We suppose that* $\mathfrak{a}_1 > 0$, $\mathfrak{a}_2 > 0$ *and* $\mathfrak{b} > 0$, $\mathfrak{M}_i,\ i = 1,\ldots,n$ *and* $\mathfrak{N}_j,\ j = 0,\ldots,m$ *are nondecreasing functions,* $u, v \in C(0,1) \cap L^1(0,1)$ *with* $u(s) \ge 0$, $v(s) \ge 0$ *for all* $s \in (0,1)$. *Then the solution* (x,y) *of problem (5), (2) satisfies the inequalities* $x(s) \ge 0$, $y(s) \ge 0$ *for all* $s \in [0,1]$, *and* $x(s) \ge s^{\gamma_1-1}x(\tau)$ *and* $y(s) \ge s^{\gamma_2-1}y(\tau)$ *for all* $s,\tau \in [0,1]$.

3. Main Theorems

By using Lemma 1, the pair of functions (x,y) is a solution of problem (1), (2) if and only if (x,y) is a solution of the system

$$\begin{aligned}
x(t) = {} & \int_0^1 \mathfrak{G}_3(t,\tau)\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta))\, d\zeta \right) d\tau \\
& + \int_0^1 \mathfrak{G}_4(t,\tau)\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{G}_2(\tau,\zeta)\mathfrak{g}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1}x(\zeta), I_{0+}^{\nu_2}y(\zeta))\, d\zeta \right) d\tau, \\
y(t) = {} & \int_0^1 \mathfrak{G}_5(t,\tau)\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta))\, d\zeta \right) d\tau \\
& + \int_0^1 \mathfrak{G}_6(t,\tau)\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{G}_2(\tau,\zeta)\mathfrak{g}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1}x(\zeta), I_{0+}^{\nu_2}y(\zeta))\, d\zeta \right) d\tau,
\end{aligned}$$

for all $t \in [0,1]$. We introduce the Banach space $\mathfrak{U} = C[0,1]$ with supremum norm $\|x\| = \sup_{s\in[0,1]} |x(s)|$, and the Banach space $\mathfrak{V} = \mathfrak{U} \times \mathfrak{U}$ with the norm $\|(x,y)\|_{\mathfrak{V}} = \|x\| + \|y\|$. We define the cone

$$\mathfrak{Q} = \{(x,y) \in \mathfrak{V},\ \ x(s) \geq 0,\ y(s) \geq 0,\ \forall s \in [0,1]\}.$$

We also define the operators $\mathfrak{E}_1, \mathfrak{E}_2 : \mathfrak{V} \to \mathfrak{U}$ and $\mathfrak{E} : \mathfrak{V} \to \mathfrak{V}$ by

$$\begin{aligned}
\mathfrak{E}_1(x,y)(t) &= \int_0^1 \mathfrak{G}_3(t,\tau)\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta,x(\zeta),y(\zeta),I_{0+}^{\mu_1}x(\zeta),I_{0+}^{\mu_2}y(\zeta))\,d\zeta\right)d\tau \\
&+ \int_0^1 \mathfrak{G}_4(t,\tau)\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{G}_2(\tau,\zeta)\mathfrak{g}(\zeta,x(\zeta),y(\zeta),I_{0+}^{\nu_1}x(\zeta),I_{0+}^{\nu_2}y(\zeta))\,d\zeta\right)d\tau, \\
\mathfrak{E}_2(x,y)(t) &= \int_0^1 \mathfrak{G}_5(t,\tau)\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta,x(\zeta),y(\zeta),I_{0+}^{\mu_1}x(\zeta),I_{0+}^{\mu_2}y(\zeta))\,d\zeta\right)d\tau \\
&+ \int_0^1 \mathfrak{G}_6(t,\tau)\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{G}_2(\tau,\zeta)\mathfrak{g}(\zeta,x(\zeta),y(\zeta),I_{0+}^{\nu_1}x(\zeta),I_{0+}^{\nu_2}y(\zeta))\,d\zeta\right)d\tau,
\end{aligned}$$

for all $t \in [0,1]$ and $(x,y) \in \mathfrak{V}$, and $\mathfrak{E}(x,y) = (\mathfrak{E}_1(x,y), \mathfrak{E}_2(x,y)),\ (x,y) \in \mathfrak{V}$. We remark that (x,y) is a solution of problem (1), (2) if and only if (x,y) is a fixed point of operator $\mathfrak{E}$.

We define the constants: $\Xi_i = \int_0^1 \mathfrak{J}_i(\tau)\xi_i(\tau)\,d\tau$, $i=1,2$, $\Xi_j = \int_0^1 \mathfrak{J}_j(\tau)\,d\tau$, $j=3,\ldots,6$, and for $\sigma_1,\sigma_2 \in (0,1)$, $\sigma_1 < \sigma_2$, $\Xi_7 = \int_{\sigma_1}^{\sigma_2} \mathfrak{J}_3(\tau)\left(\int_{\sigma_1}^{\tau} \mathfrak{G}_1(\tau,\zeta)\,d\zeta\right)^{\varrho_1-1} d\tau$, $\Xi_8 = \int_{\sigma_1}^{\sigma_2} \mathfrak{J}_6(\tau) \left(\int_{\sigma_1}^{\tau} \mathfrak{G}_2(\tau,\zeta)\,d\zeta\right)^{\varrho_2-1} d\tau$.

We now present the assumptions that we will use in our theorems.

$(H1)$ $\delta_1,\delta_2 \in (1,2]$, $\gamma_1 \in (p-1,p]$, $p \in \mathbb{N}$, $p \geq 3$, $\gamma_2 \in (q-1,q]$, $q \in \mathbb{N}$, $q \geq 3$, $n,m \in \mathbb{N}$, $\mu_1,\mu_2,\nu_1,\nu_2 > 0$, $\alpha_k \in \mathbb{R}$, $k=0,\ldots,n$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq \beta_0 < \gamma_2 - 1$, $\beta_0 \geq 1$, $\beta_k \in \mathbb{R}$, $k=0,\ldots,m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \alpha_0 < \gamma_1 - 1$, $\alpha_0 \geq 1$, $\mathfrak{M}_i : [0,1] \to \mathbb{R}$, $i=0,\ldots,n$, and $\mathfrak{N}_j : [0,1] \to \mathbb{R}$, $j=0,\ldots,m$ are nondecreasing functions, $\varphi_{\rho_i}(\tau) = |\tau|^{\rho_i-2}\tau$, $\varphi_{\rho_i}^{-1} = \varphi_{\varrho_i}$, $\varrho_i = \frac{\rho_i}{\rho_i-1}$, $i=1,2$, $\rho_i > 1$, $i=1,2$, $\mathfrak{a}_1 > 0$, $\mathfrak{a}_2 > 0$, $\mathfrak{b} > 0$ (given in Section 2).

$(H2)$ The functions $\mathfrak{f},\ \mathfrak{g} \in C((0,1)\times\mathbb{R}_+^4,\mathbb{R}_+)$ and there exist the functions $\xi_1,\xi_2 \in C((0,1),\mathbb{R}_+)$ and $\psi_1,\psi_2 \in C([0,1]\times\mathbb{R}_+^4,\mathbb{R}_+)$ with $M_1 = \int_0^1 (1-t)^{\delta_1-1}\xi_1(t)\,dt \in (0,\infty)$, $M_2 = \int_0^1 (1-t)^{\delta_2-1}\xi_2(t)\,dt \in (0,\infty)$, such that

$$\begin{aligned}
&\mathfrak{f}(t,w_1,w_2,w_3,w_4) \leq \xi_1(t)\psi_1(t,w_1,w_2,w_3,w_4), \\
&\mathfrak{g}(t,w_1,w_2,w_3,w_4) \leq \xi_2(t)\psi_2(t,w_1,w_2,w_3,w_4),
\end{aligned}$$

for any $t \in (0,1),\ w_i \in \mathbb{R}_+,\ i=1,\ldots,4$.

$(H3)$ There exist $l_i \geq 0,\ i=1,\ldots,4$ with $\sum_{i=1}^4 l_i > 0$, $m_i \geq 0,\ i=1,\ldots,4$ with $\sum_{i=1}^4 m_i > 0$, and $\theta_1 \geq 1,\ \theta_2 \geq 1$ such that

$$\begin{aligned}
\psi_{10} &= \limsup_{\sum_{i=1}^4 l_i w_i \to 0} \max_{t\in[0,1]} \frac{\psi_1(t,w_1,w_2,w_3,w_4)}{\varphi_{\rho_1}((l_1w_1+l_2w_2+l_3w_3+l_4w_4)^{\theta_1})} < c_1, \\
\text{and}\ \ \psi_{20} &= \limsup_{\sum_{i=1}^4 m_i w_i \to 0} \max_{t\in[0,1]} \frac{\psi_2(t,w_1,w_2,w_3,w_4)}{\varphi_{\rho_2}((m_1w_1+m_2w_2+m_3w_3+m_4w_4)^{\theta_2})} < c_2,
\end{aligned}$$

where

$c_1 = \left\{\min\left\{\left(4^{\rho_1-1}\Xi_1\Xi_3^{\rho_1-1}d_1^{\theta_1(\rho_1-1)}\right)^{-1}, \left(4^{\rho_1-1}\Xi_1\Xi_5^{\rho_1-1}d_1^{\theta_1(\rho_1-1)}\right)^{-1}\right\},\ \text{if } \Xi_5 \neq 0;\ \ \left(4^{\rho_1-1}\Xi_1\Xi_3^{\rho_1-1}d_1^{\theta_1(\rho_1-1)}\right)^{-1},\ \text{if } \Xi_5 = 0\right\}$,

$c_2 = \left\{\min\left\{\left(4^{\rho_2-1}\Xi_2\Xi_4^{\rho_2-1}d_2^{\theta_2(\rho_2-1)}\right)^{-1}, \left(4^{\rho_2-1}\Xi_2\Xi_6^{\rho_2-1}d_2^{\theta_2(\rho_2-1)}\right)^{-1}\right\},\ \text{if } \Xi_4 \neq 0;\ \ \left(4^{\rho_2-1}\Xi_2\Xi_6^{\rho_2-1}d_2^{\theta_2(\rho_2-1)}\right)^{-1},\ \text{if } \Xi_4 = 0\right\}$, with $d_1 = 2\max\left\{l_1, l_2, \frac{l_3}{\Gamma(\mu_1+1)}, \frac{l_4}{\Gamma(\mu_2+1)}\right\}$, $d_2 = 2\max\left\{m_1, m_2, \frac{m_3}{\Gamma(\nu_1+1)}, \frac{m_4}{\Gamma(\nu_2+1)}\right\}$.

$(H4)$ There exist $s_i \geq 0,\ i = 1, \ldots, 4$ with $\sum_{i=1}^4 s_i > 0$, $t_i \geq 0,\ i = 1, \ldots, 4$ with $\sum_{i=1}^4 t_i > 0$, $\sigma_1, \sigma_2 \in (0,1)$, $\sigma_1 < \sigma_2$ and $\eta_1 > 1$, $\eta_2 > 1$ such that

$$\mathfrak{f}_\infty = \liminf_{\sum_{i=1}^4 s_i w_i \to \infty} \min_{t \in [\sigma_1, \sigma_2]} \frac{\mathfrak{f}(t, w_1, w_2, w_3, w_4)}{\varphi_{\rho_1}(s_1 w_1 + s_2 w_2 + s_3 w_3 + s_4 w_4)} > c_3,$$
$$\text{or} \quad \mathfrak{g}_\infty = \liminf_{\sum_{i=1}^4 t_i w_i \to \infty} \min_{t \in [\sigma_1, \sigma_2]} \frac{\mathfrak{g}(t, w_1, w_2, w_3, w_4)}{\varphi_{\rho_2}(t_1 w_1 + t_2 w_2 + t_3 w_3 + t_4 w_4)} > c_4,$$

where $c_3 = \eta_1 \left(2 d_3 \Xi_7 \sigma_1^{\gamma_1 - 1}\right)^{1-\rho_1}$, $c_4 = \eta_2 \left(2 d_4 \Xi_8 \sigma_1^{\gamma_2 - 1}\right)^{1-\rho_2}$ with $d_3 = \min\left\{s_1 \sigma_1^{\gamma_1 - 1},\ s_2 \sigma_1^{\gamma_2 - 1}, s_3 \frac{\sigma_1^{\mu_1 + \gamma_1 - 1}\Gamma(\gamma_1)}{\Gamma(\gamma_1 + \mu_1)}, s_4 \frac{\sigma_1^{\mu_2 + \gamma_2 - 1}\Gamma(\gamma_2)}{\Gamma(\gamma_2 + \mu_2)}\right\}$, $d_4 = \min\left\{t_1 \sigma_1^{\gamma_1 - 1}, t_2 \sigma_1^{\gamma_2 - 1}, t_3 \frac{\sigma_1^{\nu_1 + \gamma_1 - 1}\Gamma(\gamma_1)}{\Gamma(\gamma_1 + \nu_1)},\ t_4 \frac{\sigma_1^{\nu_2 + \gamma_2 - 1}\Gamma(\gamma_2)}{\Gamma(\gamma_2 + \nu_2)}\right\}$.

$(H5)$ There exist $u_i \geq 0,\ i = 1, \ldots, 4$ with $\sum_{i=1}^4 u_i > 0$, $v_i \geq 0,\ i = 1, \ldots, 4$ with $\sum_{i=1}^4 v_i > 0$ such that

$$\psi_{1\infty} = \limsup_{\sum_{i=1}^4 u_i w_i \to \infty} \max_{t \in [0,1]} \frac{\psi_1(t, w_1, w_2, w_3, w_4)}{\varphi_{\rho_1}(u_1 w_1 + u_2 w_2 + u_3 w_3 + u_4 w_4)} < e_1,$$
$$\text{and} \quad \psi_{2\infty} = \limsup_{\sum_{i=1}^4 v_i w_i \to \infty} \max_{t \in [0,1]} \frac{\psi_2(t, w_1, w_2, w_3, w_4)}{\varphi_{\rho_2}(v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4)} < e_2,$$

where $e_1 < \left[2 \Xi_1^{\varrho_1 - 1}(\Xi_3 + \Xi_5)\Lambda_1 k_1\right]^{1-\rho_1}$, $e_2 < \left[2 \Xi_2^{\varrho_2 - 1}(\Xi_4 + \Xi_6)\Lambda_2 k_2\right]^{1-\rho_2}$, with $\Lambda_1 = \max\{2^{\varrho_1 - 2}, 1\}$, $\Lambda_2 = \max\{2^{\varrho_2 - 2}, 1\}$, $k_1 = 2\max\left\{u_1, u_2, \frac{u_3}{\Gamma(\mu_1 + 1)}, \frac{u_4}{\Gamma(\mu_2 + 1)}\right\}$, $k_2 = 2\max\left\{v_1, v_2, \frac{v_3}{\Gamma(\nu_1 + 1)}, \frac{v_4}{\Gamma(\nu_2 + 1)}\right\}$.

$(H6)$ There exist $p_i \geq 0,\ i = 1, \ldots, 4$ with $\sum_{i=1}^4 p_i > 0$, $q_i \geq 0,\ i = 1, \ldots, 4$ with $\sum_{i=1}^4 q_i > 0$, $\sigma_1, \sigma_2 \in (0,1)$, $\sigma_1 < \sigma_2$ and $\varsigma_1 \in (0,1]$, $\varsigma_2 \in (0,1]$, $\eta_3 \geq 1$, $\eta_4 \geq 1$ such that

$$\mathfrak{f}_0 = \liminf_{\sum_{i=1}^4 p_i w_i \to 0} \min_{t \in [\sigma_1, \sigma_2]} \frac{\mathfrak{f}(t, w_1, w_2, w_3, w_4)}{\varphi_{\rho_1}((p_1 w_1 + p_2 w_2 + p_3 w_3 + p_4 w_4)^{\varsigma_1})} > e_3,$$
$$\text{or} \quad \mathfrak{g}_0 = \liminf_{\sum_{i=1}^4 q_i w_i \to 0} \min_{t \in [\sigma_1, \sigma_2]} \frac{\mathfrak{g}(t, w_1, w_2, w_3, w_4)}{\varphi_{\rho_2}((q_1 w_1 + q_2 w_2 + q_3 w_3 + q_4 w_4)^{\varsigma_2})} > e_4,$$

where $e_3 = \left(\sigma_1^{\gamma_1 - 1} 2^{\varsigma_1} k_3^{\varsigma_1} \Xi_7\right)^{1-\rho_1}$, $e_4 = \left(\sigma_1^{\gamma_2 - 1} 2^{\varsigma_2} k_4^{\varsigma_2} \Xi_8\right)^{1-\rho_2}$, with $k_3 = \min\Big\{p_1 \sigma_1^{\gamma_1 - 1}$, $p_2 \sigma_1^{\gamma_2 - 1}, p_3 \frac{\sigma_1^{\mu_1 + \gamma_1 - 1}\Gamma(\gamma_1)}{\Gamma(\gamma_1 + \mu_1)}, p_4 \frac{\sigma_1^{\mu_2 + \gamma_2 - 1}\Gamma(\gamma_2)}{\Gamma(\gamma_2 + \mu_2)}\Big\}$, $k_4 = \min\Big\{q_1 \sigma_1^{\gamma_1 - 1}, q_2 \sigma_1^{\gamma_2 - 1}$, $q_3 \frac{\sigma_1^{\nu_1 + \gamma_1 - 1}\Gamma(\gamma_1)}{\Gamma(\gamma_1 + \nu_1)}, q_4 \frac{\sigma_1^{\nu_2 + \gamma_2 - 1}\Gamma(\gamma_2)}{\Gamma(\gamma_2 + \nu_2)}\Big\}$.

$(H7)$ $A_0^{\varrho_1 - 1}\Xi_3 \Xi_1^{\varrho_1 - 1} < \frac{1}{4}$, $A_0^{\varrho_2 - 1}\Xi_4 \Xi_2^{\varrho_2 - 1} < \frac{1}{4}$, $A_0^{\varrho_1 - 1}\Xi_5 \Xi_1^{\varrho_1 - 1} < \frac{1}{4}$, $A_0^{\varrho_2 - 1}\Xi_6 \Xi_2^{\varrho_2 - 1} < \frac{1}{4}$, where $A_0 = \max\left\{\max_{t \in [0,1],\, w_i \in [0,\varpi],\, i=1,\ldots,4} \psi_1(t, w_1, w_2, w_3, w_4), \max_{t \in [0,1],\, w_i \in [0,\varpi],\, i=1,\ldots,4} \psi_2(t, w_1, w_2, w_3, w_4)\right\}$, with $\varpi = \max\left\{1, \frac{1}{\Gamma(\mu_1 + 1)}, \frac{1}{\Gamma(\mu_2 + 1)}, \frac{1}{\Gamma(\nu_1 + 1)}, \frac{1}{\Gamma(\nu_2 + 1)}\right\}$.

Lemma 4. *We suppose that $(H1)$ and $(H2)$ hold. Then $\mathfrak{E} : \mathfrak{Q} \to \mathfrak{Q}$ is a completely continuous operator.*

We introduce now the cone

$$\mathfrak{Q}_0 = \{(x, y) \in \mathfrak{Q},\ x(\tau) \geq \tau^{\gamma_1 - 1}\|x\|,\ y(\tau) \geq \tau^{\gamma_2 - 1}\|y\|,\ \forall\, \tau \in [0,1]\}.$$

If $(H1)$ and $(H2)$ are satisfied, then by Lemma 3 we obtain $\mathfrak{E}(\mathfrak{Q}) \subset \mathfrak{Q}_0$ and then the operator $\mathfrak{E}|_{\mathfrak{Q}_0} : \mathfrak{Q}_0 \to \mathfrak{Q}_0$ (which we will denote again by $\mathfrak{E}$) is completely continuous. For

$\kappa > 0$ we denote by B_κ the open ball centered at zero of radius κ, and by $\overline{B}_\kappa$ and ∂B_κ its closure and its boundary, respectively.

Our main existence results are the following theorems.

Theorem 1. *We suppose that assumptions* $(H1)$–$(H4)$ *hold. Then there exists a positive solution* $(x(t), y(t)),\ t \in [0,1]$ *of problem (1), (2).*

Theorem 2. *We suppose that assumptions* $(H1)$, $(H2)$, $(H5)$, $(H6)$ *hold. Then there exists a positive solution* $(x(t), y(t)),\ t \in [0,1]$ *of problem (1), (2).*

Theorem 3. *We suppose that assumptions* $(H1)$, $(H2)$, $(H4)$, $(H6)$ *and* $(H7)$ *hold. Then there exist two positive solutions* $(x_1(t), y_1(t)),\ (x_2(t), y_2(t)),\ t \in [0,1]$ *of problem (1), (2).*

4. Proofs of the Results

Proof of Lemma 4. By $(H2)$, we have $\Xi_1 = \int_0^1 \mathfrak{J}_1(\tau)\xi_1(\tau)\,d\tau > 0$ and $\Xi_2 = \int_0^1 \mathfrak{J}_2(\tau)\xi_2(\tau)\,d\tau > 0$. In addition, by using Lemma 2.2 we find

$$\Xi_1 \le \frac{M_1}{\Gamma(\delta_1)}\left[1 + \frac{1}{\mathfrak{a}_1}\left(\int_0^1 \zeta^{\delta_1 - 1}\,d\mathfrak{M}_0(\zeta)\right)\right] < \infty,$$
$$\Xi_2 \le \frac{M_2}{\Gamma(\delta_2)}\left[1 + \frac{1}{\mathfrak{a}_2}\left(\int_0^1 \zeta^{\delta_2 - 1}\,d\mathfrak{N}_0(\zeta)\right)\right] < \infty.$$

Using now Lemma 3, we deduce that the operator $\mathfrak{E}$ maps $\mathfrak{Q}$ into $\mathfrak{Q}$.

Next, we will show that $\mathfrak{E}$ transforms the bounded sets into relatively compact sets. Let $\mathcal{S} \subset \mathfrak{Q}$ be a bounded set. So there exists $L_1 > 0$ such that $\|(x,y)\|_{\mathfrak{Y}} \le L_1$ for all $(x,y) \in \mathcal{S}$. Because ψ_1 and ψ_2 are continuous functions, we find that there exists $L_2 > 0$ such that $L_2 = \max\Big\{\sup_{\tau\in[0,1],\, w_i\in[0,\Lambda],\, i=1,\dots,4} \psi_1(\tau, w_1, w_2, w_3, w_4), \sup_{\tau\in[0,1],\, w_i\in[0,\Lambda],\, i=1,\dots,4} \psi_2(\tau, w_1, w_2, w_3, w_4)\Big\}$, where $\Lambda = L_1 \max\Big\{1, \frac{1}{\Gamma(\mu_1+1)}, \frac{1}{\Gamma(\mu_2+1)}, \frac{1}{\Gamma(\nu_1+1)}, \frac{1}{\Gamma(\nu_2+1)}\Big\}$. Because $|I_{0+}^{\omega} z(t)| \le \frac{\|z\|}{\Gamma(\omega+1)}$ for $\omega > 0$ and $z \in C[0,1]$, by Lemma 2 we obtain that for any $(x,y) \in \mathcal{S}$ and $t \in [0,1]$

$$\begin{aligned}
\mathfrak{E}_1(x,y)(t) &\le \int_0^1 \mathfrak{J}_3(\tau)\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta)\xi_1(\zeta)\psi_1(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta))\,d\zeta\right)d\tau \\
&\quad + \int_0^1 \mathfrak{J}_4(\tau)\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta)\xi_2(\zeta)\psi_2(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1}x(\zeta), I_{0+}^{\nu_2}y(\zeta))\,d\zeta\right)d\tau \\
&\le L_2^{\varrho_1 - 1}\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta)\xi_1(\zeta)\,d\zeta\right)\int_0^1 \mathfrak{J}_3(\tau)\,d\tau \\
&\quad + L_2^{\varrho_2 - 1}\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta)\xi_2(\zeta)\,d\zeta\right)\int_0^1 \mathfrak{J}_4(\tau)\,d\tau \\
&= L_2^{\varrho_1 - 1}\Xi_1^{\varrho_1 - 1}\Xi_3 + L_2^{\varrho_2 - 1}\Xi_2^{\varrho_2 - 1}\Xi_4.
\end{aligned}$$

In a similar way we have

$$\mathfrak{E}_2(x,y)(t) \le L_2^{\varrho_1 - 1}\Xi_1^{\varrho_1 - 1}\Xi_5 + L_2^{\varrho_2 - 1}\Xi_2^{\varrho_2 - 1}\Xi_6.$$

Therefore

$$\|\mathfrak{E}_1(x,y)\| \le L_2^{\varrho_1 - 1}\Xi_1^{\varrho_1 - 1}\Xi_3 + L_2^{\varrho_2 - 1}\Xi_2^{\varrho_2 - 1}\Xi_4,$$
$$\|\mathfrak{E}_2(x,y)\| \le L_2^{\varrho_1 - 1}\Xi_1^{\varrho_1 - 1}\Xi_5 + L_2^{\varrho_2 - 1}\Xi_2^{\varrho_2 - 1}\Xi_6,$$

for all $(x,y) \in \mathcal{S}$, and then $\mathfrak{E}_1(\mathcal{S})$, $\mathfrak{E}_2(\mathcal{S})$ and $\mathfrak{E}(\mathcal{S})$ are bounded.

In what follows, we prove that $\mathfrak{E}(\mathcal{S})$ is equicontinuous. By Lemma 1, for $(x,y) \in \mathcal{S}$ and $t \in [0,1]$ we find

$$
\begin{aligned}
\mathfrak{E}_1(x,y)(t) = &\int_0^1 \left[\mathfrak{g}_3(t,\tau) + \frac{t^{\gamma_1-1}\mathfrak{b}_1}{\mathfrak{b}}\left(\sum_{i=1}^m \int_0^1 \mathfrak{g}_{3i}(\vartheta,\tau)\, d\mathfrak{N}_i(\vartheta)\right)\right] \\
&\times \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}\Big(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta)\Big)d\zeta\right)d\tau \\
&+ \int_0^1 \frac{t^{\gamma_1-1}\Gamma(\gamma_2)}{\mathfrak{b}\Gamma(\gamma_2-\beta_0)}\left(\sum_{i=1}^n \int_0^1 \mathfrak{g}_{4i}(\vartheta,\tau)\, d\mathfrak{M}_i(\vartheta)\right) \\
&\times \varphi_{\varrho_2}\left(\int_0^1 \mathfrak{G}_2(\tau,\zeta)\mathfrak{g}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1}x(\zeta), I_{0+}^{\nu_2}y(\zeta))d\zeta\right)d\tau \\
= &\int_0^t \frac{1}{\Gamma(\gamma_1)}\Big[t^{\gamma_1-1}(1-\tau)^{\gamma_1-\alpha_0-1} - (t-\tau)^{\gamma_1-1}\Big] \\
&\times \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}\Big(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta)\Big)\, d\zeta\right)d\tau \\
&+ \int_t^1 \frac{1}{\Gamma(\gamma_1)} t^{\gamma_1-1}(1-\tau)^{\gamma_1-\alpha_0-1} \\
&\times \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta))\, d\zeta\right)d\tau \\
&+ \frac{t^{\gamma_1-1}\mathfrak{b}_1}{\mathfrak{b}}\int_0^1 \left(\sum_{i=1}^m \int_0^1 \mathfrak{g}_{3i}(\vartheta,\tau)\, d\mathfrak{N}_i(\vartheta)\right) \\
&\times \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta))\, d\zeta\right)d\tau \\
&+ \frac{t^{\gamma_1-1}\Gamma(\gamma_2)}{\mathfrak{b}\Gamma(\gamma_2-\beta_0)}\int_0^1 \left(\sum_{i=1}^n \int_0^1 \mathfrak{g}_{4i}(\vartheta,\tau)\, d\mathfrak{M}_i(\vartheta)\right) \\
&\times \varphi_{\varrho_2}\left(\int_0^1 \mathfrak{G}_2(\tau,\vartheta)\mathfrak{g}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1}x(\zeta), I_{0+}^{\nu_2}y(\zeta))\, d\zeta\right)d\tau.
\end{aligned}
$$

Then for any $t \in (0,1)$, we obtain

$$
\begin{aligned}
(\mathfrak{E}_1(x,y))'(t) = &\int_0^t \frac{1}{\Gamma(\gamma_1)}\Big[(\gamma_1-1)t^{\gamma_1-2}(1-\tau)^{\gamma_1-\alpha_0-1} - (\gamma_1-1)(t-\tau)^{\gamma_1-2}\Big] \\
&\times \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta))d\zeta\right)d\tau \\
&+ \int_t^1 \frac{1}{\Gamma(\gamma_1)}(\gamma_1-1)t^{\gamma_1-2}(1-\tau)^{\gamma_1-\alpha_0-1} \\
&\times \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta))\, d\zeta\right)d\tau \\
&+ \frac{(\gamma_1-1)t^{\gamma_1-2}\mathfrak{b}_1}{\mathfrak{b}}\int_0^1 \left(\sum_{i=1}^m \int_0^1 \mathfrak{g}_{3i}(\vartheta,\tau)\, d\mathfrak{N}_i(\vartheta)\right) \\
&\times \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta))\, d\zeta\right)d\tau \\
&+ \frac{(\gamma_1-1)t^{\gamma_1-2}\Gamma(\gamma_2)}{\mathfrak{b}\Gamma(\gamma_2-\beta_0)}\int_0^1 \left(\sum_{i=1}^n \int_0^1 \mathfrak{g}_{4i}(\vartheta,\tau)\, d\mathfrak{M}_i(\vartheta)\right) \\
&\times \varphi_{\varrho_2}\left(\int_0^1 \mathfrak{G}_2(\tau,\zeta)\mathfrak{g}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1}x(\zeta), I_{0+}^{\nu_2}y(\zeta))\, d\zeta\right)d\tau.
\end{aligned}
$$

So for any $t \in (0,1)$ we deduce

$$
\begin{aligned}
|(\mathfrak{E}_1(x,y))'(t)| &\le \frac{1}{\Gamma(\gamma_1-1)} \int_0^t \Big[t^{\gamma_1-2}(1-\tau)^{\gamma_1-\alpha_0-1} + (t-\tau)^{\gamma_1-2}\Big] \\
&\quad \times \varphi_{\varrho_1}\bigg(\int_0^1 \mathfrak{J}_1(\zeta)\xi_1(\zeta)\psi_1\Big(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta)\Big)d\zeta\bigg)d\tau \\
&\quad + \frac{1}{\gamma_1-1}\int_t^1 t^{\gamma_1-2}(1-\tau)^{\gamma_1-\alpha_0-1} \\
&\quad \times \varphi_{\varrho_1}\bigg(\int_0^1 \mathfrak{J}_1(\zeta)\xi_1(\zeta)\psi_1\Big(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta)\Big)d\zeta\bigg)d\tau \\
&\quad + \frac{(\gamma_1-1)t^{\gamma_1-2}\mathfrak{b}_1}{\mathfrak{b}} \int_0^1 \left(\sum_{i=1}^m \int_0^1 \mathfrak{g}_{3i}(\vartheta,\tau)\, d\mathfrak{N}_i(\vartheta)\right) \\
&\quad \times \varphi_{\varrho_1}\bigg(\int_0^1 \mathfrak{J}_1(\zeta)\xi_1(\zeta)\psi_1\Big(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1}x(\zeta), I_{0+}^{\mu_2}y(\zeta)\Big)\, d\zeta\bigg)d\tau \\
&\quad + \frac{(\gamma_1-1)t^{\gamma_1-2}\Gamma(\gamma_2)}{\mathfrak{b}\Gamma(\gamma_2-\beta_0)} \int_0^1 \left(\sum_{i=1}^n \int_0^1 \mathfrak{g}_{4i}(\vartheta,\tau)\, d\mathfrak{M}_i(\vartheta)\right) \\
&\quad \times \varphi_{\varrho_2}\bigg(\int_0^1 \mathfrak{J}_2(\zeta)\xi_2(\zeta)\psi_2(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1}x(\zeta), I_{0+}^{\nu_2}y(\zeta))\, d\zeta\bigg)d\tau \\
&\le L_2^{\varrho_1-1}\Xi_1^{\varrho_1-1}\bigg\{\frac{1}{\Gamma(\gamma_1-1)}\int_0^t \Big[t^{\gamma_1-2}(1-\tau)^{\gamma_1-\alpha_0-1} + (t-\tau)^{\gamma_1-2}\Big]\,d\tau \\
&\quad + \frac{1}{\Gamma(\gamma_1-1)}\int_t^1 t^{\gamma_1-2}(1-\tau)^{\gamma_1-\alpha_0-1}d\tau \\
&\quad + \frac{(\gamma_1-1)t^{\gamma_1-2}\mathfrak{b}_1}{\mathfrak{b}} \int_0^1 \left(\sum_{i=1}^m \int_0^1 \mathfrak{g}_{3i}(\vartheta,\tau)\, d\mathfrak{N}_i(\vartheta)\right)d\tau\bigg\} \\
&\quad + L_2^{\varrho_2-1}\Xi_2^{\varrho_2-1}\frac{(\gamma_1-1)t^{\gamma_1-2}\Gamma(\gamma_2)}{\mathfrak{b}\Gamma(\gamma_2-\beta_0)} \int_0^1 \left(\sum_{i=1}^n \int_0^1 \mathfrak{g}_{4i}(\vartheta,\tau)\, d\mathfrak{M}_i(\vartheta)\right)d\tau.
\end{aligned}
$$

Hence for any $t \in (0,1)$ we find

$$
\begin{aligned}
|(\mathfrak{E}_1(x,y))'(t)| &\le L_2^{\varrho_1-1}\Xi_1^{\varrho_1-1}\bigg\{\frac{1}{\Gamma(\gamma_1-1)}\left(\frac{t^{\gamma_1-2}}{\gamma_1-\alpha_0} + \frac{t^{\gamma_1-1}}{\gamma_1-1}\right) \\
&\quad + \frac{(\gamma_1-1)t^{\gamma_1-2}\mathfrak{b}_1}{\mathfrak{b}} \int_0^1 \left(\sum_{i=1}^m \left(\int_0^1 \frac{1}{\Gamma(\gamma_1-\beta_i)}\vartheta^{\gamma_1-\beta_i-1}(1-\tau)^{\gamma_1-\alpha_0-1}\, d\mathfrak{N}_i(\vartheta)\right)\right)d\tau\bigg\} \\
&\quad + L_2^{\varrho_2-1}\Xi_2^{\varrho_2-1}\frac{(\gamma_1-1)t^{\gamma_1-2}\Gamma(\gamma_2)}{\mathfrak{b}\Gamma(\gamma_2-\beta_0)} \\
&\quad \times \int_0^1 \left(\sum_{i=1}^n \left(\int_0^1 \frac{1}{\Gamma(\gamma_2-\alpha_i)}\vartheta^{\gamma_2-\alpha_i-1}(1-\tau)^{\gamma_2-\beta_0-1}\, d\mathfrak{M}_i(\vartheta)\right)\right)d\tau \\
&= L_2^{\varrho_1-1}\Xi_1^{\varrho_1-1}\left[\frac{1}{\Gamma(\gamma_1-1)}\left(\frac{t^{\gamma_1-2}}{\gamma_1-\alpha_0} + \frac{t^{\gamma_1-1}}{\gamma_1-1}\right) + \frac{(\gamma_1-1)t^{\gamma_1-2}\mathfrak{b}_1\mathfrak{b}_2}{\mathfrak{b}(\gamma_1-\alpha_0)\Gamma(\gamma_1)}\right] \\
&\quad + L_2^{\varrho_2-1}\Xi_2^{\varrho_2-1}\frac{(\gamma_1-1)t^{\gamma_1-2}\mathfrak{b}_1}{\mathfrak{b}\Gamma(\gamma_2-\beta_0+1)} \\
&= L_2^{\varrho_1-1}\Xi_1^{\varrho_1-1}\left[\frac{1}{\Gamma(\gamma_1-1)}\left(\frac{t^{\gamma_1-2}}{\gamma_1-\alpha_0} + \frac{t^{\gamma_1-1}}{\gamma_1-1}\right) + \frac{t^{\gamma_1-2}\mathfrak{b}_1\mathfrak{b}_2}{\mathfrak{b}(\gamma_1-\alpha_0)\Gamma(\gamma_1-1)}\right] \\
&\quad + L_2^{\varrho_2-1}\Xi_2^{\varrho_2-1}\frac{(\gamma_1-1)t^{\gamma_1-2}\mathfrak{b}_1}{\mathfrak{b}\Gamma(\gamma_2-\beta_0+1)} \\
&= L_2^{\varrho_1-1}\Xi_1^{\varrho_1-1}\left[\frac{(\mathfrak{b}+\mathfrak{b}_1\mathfrak{b}_2)t^{\gamma_1-2}}{\mathfrak{b}(\gamma_1-\alpha_0)\Gamma(\gamma_1-1)} + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}\right] + L_2^{\varrho_2-1}\Xi_2^{\varrho_2-1}\frac{(\gamma_1-1)t^{\gamma_1-2}\mathfrak{b}_1}{\mathfrak{b}\Gamma(\gamma_2-\beta_0+1)}.
\end{aligned}
$$

We denote by

$$
\Theta_1(t) = \frac{(\mathfrak{b}+\mathfrak{b}_1\mathfrak{b}_2)t^{\gamma_1-2}}{\mathfrak{b}(\gamma_1-\alpha_0)\Gamma(\gamma_1-1)} + \frac{t^{\gamma_1-1}}{\Gamma(\gamma_1)}, \;\; \Theta_2(t) = \frac{(\gamma_1-1)t^{\gamma_1-2}\mathfrak{b}_1}{\mathfrak{b}\Gamma(\gamma_2-\beta_0+1)}, \;\; t \in (0,1).
$$

Then for any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $(x,y) \in \mathcal{S}$, we deduce

$$\begin{aligned} &|\mathfrak{E}_1(x,y)(t_1) - \mathfrak{E}_1(x,y)(t_2)| = \left| \int_{t_1}^{t_2} (\mathfrak{E}_1(x,y))'(\tau)\, d\tau \right| \\ &\leq L_2^{\varrho_1-1} \Xi_1^{\varrho_1-1} \int_{t_1}^{t_2} \Theta_1(\tau)\, d\tau + L_2^{\varrho_2-1} \Xi_2^{\varrho_2-1} \int_{t_1}^{t_2} \Theta_2(\tau)\, d\tau. \end{aligned} \tag{13}$$

Because Θ_1, $\Theta_2 \in L^1(0,1)$, by (13), we conclude that $\mathfrak{E}_1(\mathcal{S})$ is equicontinuous. By using a similar technique, we deduce that $\mathfrak{E}_2(\mathcal{S})$ is also equicontinuous, and so $\mathfrak{E}(\mathcal{S})$ is equicontinuous. We apply now the Arzela-Ascoli theorem and we obtain that $\mathfrak{E}_1(\mathcal{S})$ and $\mathfrak{E}_2(\mathcal{S})$ are relatively compact sets, and then $\mathfrak{E}(\mathcal{S})$ is relatively compact, too. In addition, we can prove that $\mathfrak{E}_1$, $\mathfrak{E}_2$ and $\mathfrak{E}$ are continuous operators on $\mathfrak{Q}$ (see Lemma 1.4.1 from [16]). Therefore, the operator $\mathfrak{E}$ is completely continuous on $\mathfrak{Q}$. □

Proof of Theorem 1. From $(H3)$ we deduce that there exists $r \in (0,1)$ such that

$$\begin{aligned} &\psi_1(t, w_1, w_2, w_3, w_4) \leq c_1 \varphi_{\rho_1}((l_1 w_1 + l_2 w_2 + l_3 w_3 + l_4 w_4)^{\theta_1}), \\ &\psi_2(t, w_1, w_2, w_3, w_4) \leq c_2 \varphi_{\rho_2}((m_1 w_1 + m_2 w_2 + m_3 w_3 + m_4 w_4)^{\theta_2}), \end{aligned} \tag{14}$$

for all $t \in [0,1]$, $w_i \geq 0$, $i = 1, \ldots, 4$ with $\sum_{i=1}^4 l_i w_i \leq r$ and $\sum_{i=1}^4 m_i w_i \leq r$. We consider firstly the case $\Xi_4 \neq 0$ and $\Xi_5 \neq 0$. We define $r_1 \leq \min\{r/d_1, r/d_2, r\}$. For any $(x,y) \in \overline{B}_{r_1} \cap \mathfrak{Q}$ and $\tau \in [0,1]$ we find

$$\begin{aligned} &l_1 x(\tau) + l_2 y(\tau) + l_3 I_{0+}^{\mu_1} x(\tau) + l_4 I_{0+}^{\mu_2} y(\tau) \\ &\quad \leq 2 \max\left\{ l_1, l_2, \tfrac{l_3}{\Gamma(\mu_1+1)}, \tfrac{l_4}{\Gamma(\mu_2+1)} \right\} \|(x,y)\|_{\mathfrak{Y}} = d_1 \|(x,y)\|_{\mathfrak{Y}} \leq d_1 r_1 \leq r, \\ &m_1 x(\tau) + m_2 y(\tau) + m_3 I_{0+}^{\nu_1} x(\tau) + m_4 I_{0+}^{\nu_2} y(\tau) \\ &\quad \leq 2 \max\left\{ m_1, m_2, \tfrac{m_3}{\Gamma(\nu_1+1)}, \tfrac{m_4}{\Gamma(\nu_2+1)} \right\} \|(x,y)\|_{\mathfrak{Y}} = d_2 \|(x,y)\|_{\mathfrak{Y}} \leq d_2 r_1 \leq r. \end{aligned}$$

Therefore by (14) and Lemma 2, for any $(x,y) \in \partial B_{r_1} \cap \mathfrak{Q}_0$ and $t \in [0,1]$ we deduce

$$\begin{aligned} &\mathfrak{E}_1(x,y)(t) \leq \int_0^1 \mathfrak{J}_3(\tau) \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta) \mathfrak{f}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1} x(\zeta), I_{0+}^{\mu_2} y(\zeta))\, d\zeta \right) d\tau \\ &\quad + \int_0^1 \mathfrak{J}_4(\tau) \varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta) \mathfrak{g}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1} x(\zeta), I_{0+}^{\nu_2} y(\zeta))\, d\zeta \right) d\tau \\ &= \Xi_3 \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta) \mathfrak{f}\left(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1} x(\zeta), I_{0+}^{\mu_2} y(\zeta)\right) d\zeta \right) \\ &\quad + \Xi_4 \varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta) \mathfrak{g}(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1} x(\zeta), I_{0+}^{\nu_2} y(\zeta)) d\zeta \right) \\ &\leq \Xi_3 \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta) \xi_1(\zeta) \psi_1\left(\zeta, x(\zeta), y(\zeta), I_{0+}^{\mu_1} x(\zeta), I_{0+}^{\mu_2} y(\zeta)\right) d\zeta \right) \\ &\quad + \Xi_4 \varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta) \xi_2(\zeta) \psi_2(\zeta, x(\zeta), y(\zeta), I_{0+}^{\nu_1} x(\zeta), I_{0+}^{\nu_2} y(\zeta)) d\zeta \right) \\ &\leq \Xi_3 \varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta) \xi_1(\zeta) c_1 \varphi_{\rho_1}\left(\left(l_1 x(\zeta) + l_2 y(\zeta) + l_3 I_{0+}^{\mu_1} x(\zeta) + l_4 I_{0+}^{\mu_2} y(\zeta) \right)^{\theta_1} \right) d\zeta \right) \\ &\quad + \Xi_4 \varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta) \xi_2(\zeta) c_2 \varphi_{\rho_2}\left((m_1 x(\zeta) + m_2 y(\zeta) + m_3 I_{0+}^{\nu_1} x(\zeta) + m_4 I_{0+}^{\nu_2} y(\zeta))^{\theta_2} \right) d\zeta \right) \\ &\leq \Xi_3 \varphi_{\varrho_1}\left(\varphi_{\rho_1}\left((d_1 \|(x,y)\|_{\mathfrak{Y}})^{\theta_1} \right) \right) \varphi_{\varrho_1}(c_1) \varphi_{\varrho_1}(\Xi_1) \\ &\quad + \Xi_4 \varphi_{\varrho_2}\left(\varphi_{\rho_2}\left((d_2 \|(x,y)\|_{\mathfrak{Y}})^{\theta_2} \right) \right) \varphi_{\varrho_2}(c_2) \varphi_{\varrho_2}(\Xi_2) \\ &= \Xi_3 \Xi_1^{\varrho_1-1} c_1^{\varrho_1-1} d_1^{\theta_1} \|(x,y)\|_{\mathfrak{Y}}^{\theta_1} + \Xi_4 \Xi_2^{\varrho_2-1} c_2^{\varrho_2-1} d_2^{\theta_2} \|(x,y)\|_{\mathfrak{Y}}^{\theta_2} \\ &\leq \Xi_3 \Xi_1^{\varrho_1-1} c_1^{\varrho_1-1} d_1^{\theta_1} \|(x,y)\|_{\mathfrak{Y}} + \Xi_4 \Xi_2^{\varrho_2-1} c_2^{\varrho_2-1} d_2^{\theta_2} \|(x,y)\|_{\mathfrak{Y}} \\ &\leq \tfrac{1}{4} \|(x,y)\|_{\mathfrak{Y}} + \tfrac{1}{4} \|(x,y)\|_{\mathfrak{Y}} = \tfrac{1}{2} \|(x,y)\|_{\mathfrak{Y}}. \end{aligned}$$

In a similar manner we obtain

$$\begin{aligned}&\mathfrak{E}_2(x,y)(t) \le \Xi_5\Xi_1^{\varrho_1-1}c_1^{\varrho_1-1}d_1^{\theta_1}\|(x,y)\|_{\mathfrak{Y}} + \Xi_6\Xi_2^{\varrho_2-1}c_2^{\varrho_2-1}d_2^{\theta_2}\|(x,y)\|_{\mathfrak{Y}}\\ &\le \tfrac{1}{4}\|(x,y)\|_{\mathfrak{Y}} + \tfrac{1}{4}\|(x,y)\|_{\mathfrak{Y}} = \tfrac{1}{2}\|(x,y)\|_{\mathfrak{Y}}.\end{aligned}$$

Then we conclude

$$\|\mathfrak{E}(x,y)\|_{\mathfrak{Y}} = \|\mathfrak{E}_1(x,y)\| + \|\mathfrak{E}_2(x,y)\| \le \|(x,y)\|_{\mathfrak{Y}},\ \ \forall\,(x,y)\in \partial B_{r_1}\cap\mathfrak{Q}_0. \tag{15}$$

If $\Xi_4 = 0$ or $\Xi_5 = 0$ we also find in a similar manner inequality (15).

In what follows, in $(H4)$ we assume that $\mathfrak{g}_\infty > c_4$ (in a similar manner we study the case $\mathfrak{f}_\infty > c_3$). Then there exists a positive constant $C_1 > 0$ such that

$$\mathfrak{g}(t,w_1,w_2,w_3,w_4) \ge c_4\varphi_{\rho_2}(t_1w_1+t_2w_2+t_3w_3+t_4w_4) - C_1, \tag{16}$$

for all $t\in[\sigma_1,\sigma_2]$ and $w_i \ge 0$, $i=1,\dots,4$. From the definition of $I_{0+}^{\nu_1}$, for any $(x,y)\in\mathfrak{Q}_0$ and $\tau\in[0,1]$, we find

$$\begin{aligned}&I_{0+}^{\nu_1}x(\tau) = \frac{1}{\Gamma(\nu_1)}\int_0^\tau(\tau-\zeta)^{\nu_1-1}x(\zeta)\,d\zeta \ge \frac{1}{\Gamma(\nu_1)}\int_0^\tau(\tau-\zeta)^{\nu_1-1}\zeta^{\gamma_1-1}\|x\|\,d\zeta\\ &\overset{\zeta=\tau z}{=} \frac{\|x\|}{\Gamma(\nu_1)}\int_0^1(\tau-\tau z)^{\nu_1-1}\tau^{\gamma_1-1}z^{\gamma_1-1}\tau\,dz = \frac{\|x\|}{\Gamma(\nu_1)}\tau^{\nu_1+\gamma_1-1}\int_0^1 z^{\gamma_1-1}(1-z)^{\nu_1-1}\,dz\\ &= \frac{\|x\|}{\Gamma(\nu_1)}\tau^{\nu_1+\gamma_1-1}B(\gamma_1,\nu_1) = \frac{\|x\|\tau^{\nu_1+\gamma_1-1}\Gamma(\gamma_1)}{\Gamma(\gamma_1+\nu_1)},\end{aligned} \tag{17}$$

and similarly

$$I_{0+}^{\nu_2}y(\tau) \ge \frac{\|y\|\tau^{\nu_2+\gamma_2-1}\Gamma(\gamma_2)}{\Gamma(\gamma_2+\nu_2)},$$

where $B(z_1,z_2)$ is the first Euler function defined by $B(z_1,z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,dt$, $z_1,z_2>0$. Then by using (16) and (17), for any $(x,y)\in\mathfrak{Q}_0$ and $t\in[\sigma_1,\sigma_2]$ we obtain

$$\begin{aligned}&\mathfrak{E}_2(x,y)(t) \ge \int_{\sigma_1}^{\sigma_2}\mathfrak{G}_6(t,\tau)\varphi_{\varrho_2}\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_2(\tau,\zeta)\mathfrak{g}(\zeta,x(\zeta),y(\zeta),I_{0+}^{\nu_1}x(\zeta),I_{0+}^{\nu_2}y(\zeta))\,d\zeta\right)d\tau\\ &\ge \sigma_1^{\gamma_2-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_6(\tau)\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_2(\tau,\zeta)\left[c_4\left(t_1x(\zeta)+t_2y(\zeta)+t_3I_{0+}^{\nu_1}x(\zeta)+t_4I_{0+}^{\nu_2}y(\zeta)\right)^{\rho_2-1}\right.\right.\\ &\quad\left.\left.-C_1\right]d\zeta\right)^{\varrho_2-1}d\tau\\ &\ge \sigma_1^{\gamma_2-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_6(\tau)\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_2(\tau,\zeta)\left[c_4\left(t_1\sigma_1^{\gamma_1-1}\|x\|+t_2\sigma_1^{\gamma_2-1}\|y\|\right.\right.\right.\\ &\quad\left.\left.\left.+t_3\frac{\sigma_1^{\nu_1+\gamma_1-1}\Gamma(\gamma_1)}{\Gamma(\gamma_1+\nu_1)}\|x\|+t_4\frac{\sigma_1^{\nu_2+\gamma_2-1}\Gamma(\gamma_2)}{\Gamma(\gamma_2+\nu_2)}\|y\|\right)^{\rho_2-1}-C_1\right]d\zeta\right)^{\varrho_2-1}d\tau\\ &\ge \sigma_1^{\gamma_2-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_6(\tau)\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_2(\tau,\zeta)\left[c_4\left(\min\left\{t_1\sigma_1^{\gamma_1-1},t_2\sigma_1^{\gamma_2-1},t_3\frac{\sigma_1^{\nu_1+\gamma_1-1}\Gamma(\gamma_1)}{\Gamma(\gamma_1+\nu_1)},\right.\right.\right.\right.\\ &\quad\left.\left.\left.\left.t_4\frac{\sigma_1^{\nu_2+\gamma_2-1}\Gamma(\gamma_2)}{\Gamma(\gamma_2+\nu_2)}\right\}2\|(x,y)\|_{\mathfrak{Y}}\right)^{\rho_2-1}-C_1\right]d\zeta\right)^{\varrho_2-1}d\tau\\ &= \sigma_1^{\gamma_2-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_6(\tau)\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_2(\tau,\zeta)\left[c_4(2d_4\|(x,y)\|_{\mathfrak{Y}})^{\rho_2-1}-C_1\right]d\zeta\right)^{\varrho_2-1}d\tau\\ &= \Xi_8\sigma_1^{\gamma_2-1}\left[c_4(2d_4\|(x,y)\|_{\mathfrak{Y}})^{\rho_2-1}-C_1\right]^{\varrho_2-1}\\ &= \left(\Xi_8^{\rho_2-1}\sigma_1^{(\gamma_2-1)(\rho_2-1)}c_4 2^{\rho_2-1}d_4^{\rho_2-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_2-1} - \Xi_8^{\rho_2-1}\sigma_1^{(\gamma_2-1)(\rho_2-1)}C_1\right)^{\varrho_2-1}\\ &= \left(\eta_2\|(x,y)\|_{\mathfrak{Y}}^{\rho_2-1}-C_2\right)^{\varrho_2-1},\ \ C_2 = \Xi_8^{\rho_2-1}\sigma_1^{(\gamma_1-1)(\rho_2-1)}C_1.\end{aligned}$$

So we find

$$\|\mathfrak{E}(x,y)\|_{\mathfrak{Y}} \geq \|\mathfrak{E}_2(x,y)\| \geq \mathfrak{E}_2(x,y)(\sigma_1) \geq \left(\eta_2\|(x,y)\|_{\mathfrak{Y}}^{\rho_2-1} - C_2\right)^{\varrho_2-1}, \ \forall\,(x,y) \in \mathfrak{Q}_0.$$

We choose $r_2 \geq \max\left\{1, C_2^{\varrho_2-1}/(\eta_2-1)^{\varrho_2-1}\right\}$ and we deduce

$$\|\mathfrak{E}(x,y)\|_{\mathfrak{Y}} \geq \|(x,y)\|_{\mathfrak{Y}}, \ \forall\,(x,y) \in \partial B_{r_2} \cap \mathfrak{Q}_0. \tag{18}$$

Now based on Lemma 4, the relations (15), (18) and the Guo–Krasnosel'skii fixed point theorem we conclude that the operator $\mathfrak{E}$ has a fixed point $(x,y) \in (\overline{B}_{r_2} \setminus B_{r_1}) \cap \mathfrak{Q}_0$ with $r_1 \leq \|(x,y)\|_{\mathfrak{Y}} \leq r_2$ and $x(s) \geq s^{\gamma_1-1}\|x\|$, $y(s) \geq s^{\gamma_2-1}\|y\|$ for all $s \in [0,1]$. So $\|x\| > 0$ or $\|y\| > 0$, that is $x(s) > 0$ for all $s \in (0,1]$ or $y(s) > 0$ for all $s \in (0,1]$. Therefore, $(x(t), y(t)),\ t \in [0,1]$ is a positive solution of problem (1), (2). □

Proof of Theorem 2. From assumption $(H5)$ we deduce that there exist $C_3 > 0$, $C_4 > 0$ such that

$$\begin{aligned} &\psi_1(t,w_1,w_2,w_3,w_4) \leq e_1\varphi_{\rho_1}(u_1w_1+u_2w_2+u_3w_3+u_4w_4)+C_3,\\ &\psi_2(t,w_1,w_2,w_3,w_4) \leq e_2\varphi_{\rho_2}(u_1w_1+u_2w_2+u_3w_3+u_4w_4)+C_4, \end{aligned} \tag{19}$$

for any $t \in [0,1]$ and $w_i \geq 0,\ i = 1,\dots,4$. By using $(H2)$ and (19), for any $(x,y) \in \mathfrak{Q}_0$ and $t \in [0,1]$ we obtain

$$\begin{aligned}
\mathfrak{E}_1(x,y)(t) &\leq \int_0^1 \mathfrak{J}_3(\tau)\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta)\mathfrak{f}(\zeta,x(\zeta),y(\zeta),I_{0+}^{\mu_1}x(\zeta),I_{0+}^{\mu_2}y(\zeta))\,d\zeta\right)d\tau\\
&\quad+\int_0^1 \mathfrak{J}_4(\tau)\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta)\mathfrak{g}(\zeta,x(\zeta),y(\zeta),I_{0+}^{\nu_1}x(\zeta),I_{0+}^{\nu_2}y(\zeta))\,d\zeta\right)d\tau\\
&\leq \Xi_3\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta)\xi_1(\zeta)\psi_1(\zeta,x(\zeta),y(\zeta),I_{0+}^{\mu_1}x(\zeta),I_{0+}^{\mu_2}y(\zeta))\,d\zeta\right)\\
&\quad+\Xi_4\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta)\xi_2(\zeta)\psi_2(\zeta,x(\zeta),y(\zeta),I_{0+}^{\nu_1}x(\zeta),I_{0+}^{\nu_2}y(\zeta))\,d\zeta\right)\\
&\leq \Xi_3\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta)\xi_1(\zeta)\left[e_1\varphi_{\rho_1}\left(u_1x(\zeta)+u_2y(\zeta)+u_3I_{0+}^{\mu_1}x(\zeta)+u_4I_{0+}^{\mu_2}y(\zeta)\right)+C_3\right]d\zeta\right)\\
&\quad+\Xi_4\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta)\xi_2(\zeta)\left[e_2\varphi_{\rho_2}(v_1x(\zeta)+v_2y(\zeta)+v_3I_{0+}^{\nu_1}x(\zeta)+v_4I_{0+}^{\nu_2}y(\zeta))+C_4\right]d\zeta\right)\\
&\leq \Xi_3\varphi_{\varrho_1}\left(\int_0^1 \mathfrak{J}_1(\zeta)\xi_1(\zeta)\left[e_1\left(u_1\|x\|+u_2\|y\|+\frac{u_3\|x\|}{\Gamma(\mu_1+1)}+\frac{u_4\|y\|}{\Gamma(\mu_2+1)}\right)^{\rho_1-1}+C_3\right]d\zeta\right)\\
&\quad+\Xi_4\varphi_{\varrho_2}\left(\int_0^1 \mathfrak{J}_2(\zeta)\xi_2(\zeta)\left[e_2\left(v_1\|x\|+v_2\|y\|+\frac{v_3\|x\|}{\Gamma(\nu_1+1)}+\frac{v_4\|y\|}{\Gamma(\nu_2+1)}\right)^{\rho_2-1}+C_4\right]d\zeta\right)\\
&\leq \Xi_3\varphi_{\varrho_1}\left[e_1\left(\max\left\{u_1,u_2,\tfrac{u_3}{\Gamma(\mu_1+1)},\tfrac{u_4}{\Gamma(\mu_2+1)}\right\}2\|(x,y)\|_{\mathfrak{Y}}\right)^{\rho_1-1}+C_3\right]\\
&\quad\times\left(\int_0^1 \mathfrak{J}_1(\zeta)\xi_1(\zeta)\,d\zeta\right)^{\varrho_1-1}\\
&\quad+\Xi_4\varphi_{\varrho_2}\left[e_2\left(\max\left\{v_1,v_2,\tfrac{v_3}{\Gamma(\nu_1+1)},\tfrac{v_4}{\Gamma(\nu_2+1)}\right\}2\|(x,y)\|_{\mathfrak{Y}}\right)^{\rho_2-1}+C_4\right]\\
&\quad\times\left(\int_0^1 \mathfrak{J}_2(\zeta)\xi_2(\zeta)\,d\zeta\right)^{\varrho_2-1}\\
&= \Xi_1^{\varrho_1-1}\Xi_3\left(e_1k_1^{\rho_1-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_1-1}+C_3\right)^{\varrho_1-1}+\Xi_2^{\varrho_2-1}\Xi_4\left(e_2k_2^{\rho_2-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_2-1}+C_4\right)^{\varrho_2-1}.
\end{aligned}$$

In a similar way we find

$$\begin{aligned}
\mathfrak{E}_2(x,y)(t) &\leq \Xi_1^{\varrho_1-1}\Xi_5\left(e_1k_1^{\rho_1-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_1-1}+C_3\right)^{\varrho_1-1}\\
&\quad+\Xi_2^{\varrho_2-1}\Xi_6\left(e_2k_2^{\rho_2-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_2-1}+C_4\right)^{\varrho_2-1}.
\end{aligned}$$

Hence we conclude

$$\begin{aligned}\|\mathfrak{E}_1(x,y)\| &\le \Xi_1^{\varrho_1-1}\Xi_3\Big(e_1k_1^{\rho_1-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_1-1}+C_3\Big)^{\varrho_1-1}\\ &+\Xi_2^{\varrho_2-1}\Xi_4\Big(e_2k_2^{\rho_2-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_2-1}+C_4\Big)^{\varrho_2-1},\\ \|\mathfrak{E}_2(x,y)\| &\le \Xi_1^{\varrho_1-1}\Xi_5\Big(e_1k_1^{\rho_1-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_1-1}+C_3\Big)^{\varrho_1-1}\\ &+\Xi_2^{\varrho_2-1}\Xi_6\Big(e_2k_2^{\rho_2-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_2-1}+C_4\Big)^{\varrho_2-1},\end{aligned}$$

and then

$$\begin{aligned}\|\mathfrak{E}(x,y)\|_{\mathfrak{Y}} &\le \Xi_1^{\varrho_1-1}(\Xi_3+\Xi_5)\Big(e_1k_1^{\rho_1-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_1-1}+C_3\Big)^{\varrho_1-1}\\ &+\Xi_2^{\varrho_2-1}(\Xi_4+\Xi_6)\Big(e_2k_2^{\rho_2-1}\|(x,y)\|_{\mathfrak{Y}}^{\rho_2-1}+C_4\Big)^{\varrho_2-1},\end{aligned} \tag{20}$$

for all $(x,y)\in\mathfrak{Q}_0$. We choose

$$r_3 \ge \max\left\{1, \frac{\Xi_1^{\varrho_1-1}(\Xi_3+\Xi_5)\Lambda_1C_3^{\varrho_1-1}+\Xi_2^{\varrho_2-1}(\Xi_4+\Xi_6)\Lambda_2C_4^{\varrho_2-1}}{1-\Big[\Xi_1^{\varrho_1-1}(\Xi_3+\Xi_5)\Lambda_1e_1^{\varrho_1-1}k_1+\Xi_2^{\varrho_2-1}(\Xi_4+\Xi_6)\Lambda_2e_2^{\varrho_2-1}k_2\Big]}\right\}.$$

Then by (20) and the inequalities $(a+b)^{\varrho_i-1}\le\Lambda_i(a^{\varrho_i-1}+b^{\varrho_i-1})$, for $a,b\ge0$, $i=1,2$ we deduce

$$\|\mathfrak{E}(x,y)\|_{\mathfrak{Y}}\le\|(x,y)\|_{\mathfrak{Y}},\ \ \forall\,(x,y)\in\partial B_{r_3}\cap\mathfrak{Q}_0. \tag{21}$$

Now, in $(H6)$ we assume that $\mathfrak{f}_0>e_3$ (the case $\mathfrak{g}_0>e_4$ is treated in a similar way). So there exists $\widetilde{r}_4\in(0,1]$ such that

$$\mathfrak{f}(t,w_1,w_2,w_3,w_4)\ge e_4\varphi_{\rho_1}((p_1w_1+p_2w_2+p_3w_3+p_4w_4)^{\varsigma_1}), \tag{22}$$

for all $t\in[\sigma_1,\sigma_2]$, $w_i\ge0$, $i=1,\dots,4$, $\sum_{i=1}^4 p_iw_i\le\widetilde{r}_4$. We define $r_4\le\min\{\widetilde{r}_4/\widetilde{k}_3,\widetilde{r}_4\}$, where $\widetilde{k}_3=2\max\Big\{p_1,p_2,\frac{p_3}{\Gamma(\mu_1+1)},\frac{p_4}{\Gamma(\mu_2+1)}\Big\}$. Hence for any $(x,y)\in\overline{B}_{r_4}\cap\mathfrak{Q}$ and $t\in[0,1]$ we find

$$\begin{aligned}&p_1x(\tau)+p_2y(\tau)+p_3I_{0+}^{\mu_1}x(\tau)+p_4I_{0+}^{\mu_2}y(\tau)\\ &\le 2\max\Big\{p_1,p_2,\frac{p_3}{\Gamma(\mu_1+1)},\frac{p_4}{\Gamma(\mu_2+1)}\Big\}\|(x,y)\|_{\mathfrak{Y}}=\widetilde{k}_3r_4\le\widetilde{r}_4.\end{aligned}$$

Therefore, by using (22) and the inequalities $I_{0+}^{\mu_1}x(\tau)\ge\|x\|\frac{\tau^{\mu_1+\gamma_1-1}\Gamma(\gamma_1)}{\Gamma(\gamma_1+\mu_1)}$ and $I_{0+}^{\mu_2}y(\tau)\ge\|y\|\frac{\tau^{\mu_2+\gamma_2-1}\Gamma(\gamma_2)}{\Gamma(\gamma_2+\mu_2)}$, for all $\tau\in[0,1]$ and $(x,y)\in\mathfrak{Q}_0$, we obtain for any $(x,y)\in\overline{B}_{r_4}\cap\mathfrak{Q}_0$ and $t\in[\sigma_1,\sigma_2]$

$$\begin{aligned}\mathfrak{E}_1(x,y)(t) &\ge \int_{\sigma_1}^{\sigma_2}\mathfrak{G}_3(t,\tau)\varphi_{\varrho_1}\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_1(\tau,\zeta)\mathfrak{f}(\zeta,x(\zeta),y(\zeta),I_{0+}^{\mu_1}x(\zeta),I_{0+}^{\mu_2}y(\zeta))\,d\zeta\right)d\tau\\ &\ge \sigma_1^{\gamma_1-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_3(\tau)\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_1(\tau,\zeta)e_3\Big(p_1x(\zeta)+p_2y(\zeta)+p_3I_{0+}^{\mu_1}x(\zeta)\right.\\ &\quad\left.+p_4I_{0+}^{\mu_2}y(\zeta)\Big)^{\varsigma_1(\rho_1-1)}d\zeta\right)^{\varrho_1-1}d\tau\\ &\ge \sigma_1^{\gamma_1-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_3(\tau)\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_1(\tau,\zeta)e_3\Big(p_1\sigma_1^{\gamma_1-1}\|x\|+p_2\sigma_1^{\gamma_2-1}\|y\|\right.\\ &\quad\left.+p_3\frac{\sigma_1^{\mu_1+\gamma_1-1}\Gamma(\gamma_1)}{\Gamma(\gamma_1+\mu_1)}\|x\|+p_4\frac{\sigma_1^{\mu_2+\gamma_2-1}\Gamma(\gamma_2)}{\Gamma(\gamma_2+\mu_2)}\|y\|\Big)^{\varsigma_1(\rho_1-1)}d\zeta\right)^{\varrho_1-1}d\tau\end{aligned}$$

$$\geq \sigma_1^{\gamma_1-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_3(\tau)\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_1(\tau,\zeta)e_3\left(\min\left\{p_1\sigma_1^{\gamma_1-1},p_2\sigma_1^{\gamma_2-1},p_3\frac{\sigma_1^{\mu_1+\gamma_1-1}\Gamma(\gamma_1)}{\Gamma(\gamma_1+\mu_1)},\right.\right.\right.$$
$$\left.\left.\left.p_4\frac{\sigma_1^{\mu_2+\gamma_2-1}\Gamma(\gamma_2)}{\Gamma(\gamma_2+\mu_2)}\right\}2\|(x,y)\|_{\mathfrak{V}}\right)^{\varsigma_1(\rho_1-1)}d\zeta\right)^{\varrho_1-1}d\tau$$
$$=\sigma_1^{\gamma_1-1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_3(\tau)\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_1(\tau,\zeta)e_3(2k_3\|(x,y)\|_{\mathfrak{V}})^{\varsigma_1(\rho_1-1)}d\zeta\right)^{\varrho_1-1}d\tau$$
$$=\sigma_1^{\gamma_1-1}e_3^{\varrho_1-1}2^{\varsigma_1}k_3^{\varsigma_1}\|(x,y)\|_{\mathfrak{V}}^{\varsigma_1}\int_{\sigma_1}^{\sigma_2}\mathfrak{J}_3(\tau)\left(\int_{\sigma_1}^{\tau}\mathfrak{G}_1(\tau,\zeta)\,d\zeta\right)^{\varrho_1-1}d\tau$$
$$=\sigma_1^{\gamma_1-1}e_3^{\varrho_1-1}2^{\varsigma_1}k_3^{\varsigma_1}\Xi_7\|(x,y)\|_{\mathfrak{V}}^{\varsigma_1}$$
$$\geq\sigma_1^{\gamma_1-1}e_3^{\varrho_1-1}2^{\varsigma_1}k_3^{\varsigma_1}\Xi_7\|(x,y)\|_{\mathfrak{V}}=\|(x,y)\|_{\mathfrak{V}}.$$

Then we deduce

$$\|\mathfrak{E}(x,y)\|_{\mathfrak{V}}\geq\|\mathfrak{E}_1(x,y)\|\geq\mathfrak{E}_1(x,y)(\sigma_1)\geq\|(x,y)\|_{\mathfrak{V}},\ \ \forall\,(x,y)\in\partial B_{r_4}\cap\mathfrak{Q}_0. \tag{23}$$

By Lemma 4, (21), (23) and the Guo–Krasnosel'skii fixed point theorem, we conclude that $\mathfrak{E}$ has a fixed point $(x,y)\in(\overline{B}_{r_3}\setminus B_{r_4})\cap\mathfrak{Q}_0$, so $r_4\leq\|(x,y)\|_{\mathfrak{V}}\leq r_3$, and $x(s)\geq s^{\gamma_1-1}\|x\|$, $y(s)\geq s^{\gamma_2-1}\|y\|$ for all $s\in[0,1]$, which is a positive solution of problem (1), (2). □

Proof of Theorem 3. Because assumptions $(H1)$, $(H2)$ and $(H4)$ hold, then by Theorem 1 we deduce that there exists $r_2>1$ such that

$$\|\mathfrak{E}(x,y)\|_{\mathfrak{V}}\geq\|(x,y)\|_{\mathfrak{V}},\ \ \forall\,(x,y)\in\partial B_{r_2}\cap\mathfrak{Q}_0. \tag{24}$$

Next because assumptions $(H1)$, $(H2)$ and $(H6)$ hold, then by Theorem 2 we conclude that there exists $r_4<1$ such that

$$\|\mathfrak{E}(x,y)\|_{\mathfrak{V}}\geq\|(x,y)\|_{\mathfrak{V}},\ \ \forall\,(x,y)\in\partial B_{r_4}\cap\mathfrak{Q}_0. \tag{25}$$

Now, consider the set $B_1=\{(x,y)\in\mathfrak{V},\ \|(x,y)\|_{\mathfrak{V}}<1\}$. By assumption $(H7)$ for any $(x,y)\in\partial B_1\cap\mathfrak{Q}_0$ and $t\in[0,1]$ we find

$$\mathfrak{E}_1(x,y)(t)\leq\int_0^1\mathfrak{J}_3(\tau)\varphi_{\varrho_1}\left(\int_0^1\mathfrak{J}_1(\zeta)\xi_1(\zeta)\psi_1\left(\zeta,x(\zeta),y(\zeta),I_{0+}^{\mu_1}x(\zeta),I_{0+}^{\mu_2}y(\zeta)\right)d\zeta\right)d\tau$$
$$+\int_0^1\mathfrak{J}_4(\tau)\varphi_{\varrho_2}\left(\int_0^1\mathfrak{J}_2(\zeta)\xi_2(\zeta)\psi_2(\zeta,x(\zeta),y(\zeta),I_{0+}^{\nu_1}x(\zeta),I_{0+}^{\nu_2}y(\zeta))d\zeta\right)d\tau$$
$$\leq A_0^{\varrho_1-1}\int_0^1\mathfrak{J}_3(\tau)\varphi_{\varrho_1}\left(\int_0^1\mathfrak{J}_1(\zeta)\xi_1(\zeta)\,d\zeta\right)d\tau+A_0^{\varrho_2-1}\int_0^1\mathfrak{J}_4(\tau)\varphi_{\varrho_2}\left(\int_0^1\mathfrak{J}_2(\zeta)\xi_2(\zeta)\,d\zeta\right)d\tau$$
$$=A_0^{\varrho_1-1}\left(\int_0^1\mathfrak{J}_3(\tau)\,d\tau\right)\left(\int_0^1\mathfrak{J}_1(\zeta)\xi_1(\zeta)\,d\zeta\right)^{\varrho_1-1}$$
$$+A_0^{\varrho_2-1}\left(\int_0^1\mathfrak{J}_4(\tau)\,d\tau\right)\left(\int_0^1\mathfrak{J}_2(\zeta)\xi_2(\zeta)\,d\zeta\right)^{\varrho_2-1}$$
$$=A_0^{\varrho_1-1}\Xi_3\Xi_1^{\varrho_1-1}+A_0^{\varrho_2-1}\Xi_4\Xi_2^{\varrho_2-1}<\tfrac{1}{4}+\tfrac{1}{4}=\tfrac{1}{2},$$
$$\mathfrak{E}_2(x,y)(t)\leq\int_0^1\mathfrak{J}_5(\tau)\varphi_{\varrho_1}\left(\int_0^1\mathfrak{J}_1(\zeta)\xi_1(\zeta)\psi_1\left(\zeta,x(\zeta),y(\zeta),I_{0+}^{\mu_1}x(\zeta),I_{0+}^{\mu_2}y(\zeta)\right)d\zeta\right)d\tau$$
$$+\int_0^1\mathfrak{J}_6(\tau)\varphi_{\varrho_2}\left(\int_0^1\mathfrak{J}_2(\zeta)\xi_2(\zeta)\psi_2(\zeta,x(\zeta),y(\zeta),I_{0+}^{\nu_1}x(\zeta),I_{0+}^{\nu_2}y(\zeta))d\zeta\right)d\tau$$
$$\leq A_0^{\varrho_1-1}\int_0^1\mathfrak{J}_5(\tau)\varphi_{\varrho_1}\left(\int_0^1\mathfrak{J}_1(\zeta)\xi_1(\zeta)\,d\zeta\right)d\tau+A_0^{\varrho_2-1}\int_0^1\mathfrak{J}_6(\tau)\varphi_{\varrho_2}\left(\int_0^1\mathfrak{J}_2(\zeta)\xi_2(\zeta)\,d\zeta\right)d\tau$$
$$=A_0^{\varrho_1-1}\left(\int_0^1\mathfrak{J}_5(\tau)\,d\tau\right)\left(\int_0^1\mathfrak{J}_1(\zeta)\xi_1(\zeta)\,d\zeta\right)^{\varrho_1-1}$$
$$+A_0^{\varrho_2-1}\left(\int_0^1\mathfrak{J}_6(\tau)\,d\tau\right)\left(\int_0^1\mathfrak{J}_2(\zeta)\xi_2(\zeta)\,d\zeta\right)^{\varrho_2-1}$$
$$=A_0^{\varrho_1-1}\Xi_5\Xi_1^{\varrho_1-1}+A_0^{\varrho_2-1}\Xi_6\Xi_2^{\varrho_2-1}<\tfrac{1}{4}+\tfrac{1}{4}=\tfrac{1}{2}.$$

Therefore we deduce $\|\mathfrak{E}_1(x,y)\| < \frac{1}{2}$, $\|\mathfrak{E}_2(x,y)\| < \frac{1}{2}$ for all $(x,y) \in \partial B_1 \cap \mathfrak{Q}_0$. So we obtain

$$\|\mathfrak{E}(x,y)\|_{\mathfrak{Y}} = \|\mathfrak{E}_1(x,y)\| + \|\mathfrak{E}_2(x,y)\| < 1 = \|(x,y)\|_{\mathfrak{Y}}, \ \forall (x,y) \in \partial B_1 \cap \mathfrak{Q}_0. \tag{26}$$

Then, by (24) and (26) we conclude that there exists a positive solution $(x_1, y_1) \in \mathfrak{Q}_0$ with $1 < \|(x_1,y_1)\|_{\mathfrak{Y}} \le r_2$ for problem (1), (2). By (25) and (26) we deduce that there exists another positive solution $(x_2,y_2) \in \mathfrak{Q}_0$ with $r_4 \le \|(x_2,y_2)\|_{\mathfrak{Y}} < 1$ for problem (1), (2). Hence problem (1), (2) has at least two positive solutions $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$, $t \in [0,1]$. □

5. Examples

Let $\delta_1 = \frac{7}{4}$, $\delta_2 = \frac{5}{3}$, $p = 3$, $q = 4$, $\gamma_1 = \frac{5}{2}$, $\gamma_2 = \frac{17}{5}$, $n = 1$, $m = 2$, $\mu_1 = \frac{23}{6}$, $\mu_2 = \frac{19}{7}$, $\nu_1 = \frac{47}{9}$, $\nu_2 = \frac{22}{3}$, $\alpha_0 = \frac{4}{3}$, $\alpha_1 = \frac{2}{3}$, $\beta_0 = \frac{9}{4}$, $\beta_1 = \frac{3}{4}$, $\beta_2 = \frac{5}{6}$, $\rho_1 = \frac{27}{8}$, $\rho_2 = \frac{38}{9}$, $\varrho_1 = \frac{27}{19}$, $\varrho_2 = \frac{38}{29}$, $\mathfrak{M}_0(\tau) = \frac{5\tau}{7}$, $\tau \in [0,1]$, $\mathfrak{N}_0(\tau) = \left\{\frac{1}{2},\ \tau \in \left[0, \frac{1}{3}\right);\ \frac{11}{10},\ \tau \in \left[\frac{1}{3}, 1\right]\right\}$, $\mathfrak{M}_1(\tau) = \left\{\frac{3}{4},\ \tau \in \left[0, \frac{1}{2}\right);\ \frac{93}{28},\ \tau \in \left[\frac{1}{2}, 1\right]\right\}$, $\mathfrak{N}_1(\tau) = \left\{\frac{1}{3},\ \tau \in \left[0, \frac{4}{5}\right);\ \frac{29}{24},\ \tau \in \left[\frac{4}{5}, 1\right]\right\}$, $\mathfrak{N}_2(\tau) = \frac{3\tau}{2}$, $\tau \in [0,1]$.

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{7/4}\left(\varphi_{27/8}\left(D_{0+}^{5/2}x(t)\right)\right) = \mathfrak{f}\left(t, x(t), y(t), I_{0+}^{23/6}x(t), I_{0+}^{19/7}y(t)\right), \ t \in (0,1), \\ D_{0+}^{5/3}\left(\varphi_{38/9}\left(D_{0+}^{17/5}y(t)\right)\right) = \mathfrak{g}\left(t, x(t), y(t), I_{0+}^{47/9}x(t), I_{0+}^{22/3}y(t)\right), \ t \in (0,1), \end{cases} \tag{27}$$

with the boundary conditions

$$\begin{cases} x(0) = x'(0) = 0, \ D_{0+}^{5/2}x(0) = 0, \ \varphi_{27/8}\left(D_{0+}^{5/2}x(1)\right) = \frac{5}{7}\int_0^1 \varphi_{27/8}\left(D_{0+}^{5/2}x(\tau)\right)d\tau, \\ D_{0+}^{4/3}x(1) = \frac{18}{7}D_{0+}^{2/3}y\left(\frac{1}{2}\right), \\ y(0) = y'(0) = y''(0) = 0, \ D_{0+}^{17/5}y(0) = 0, \ D_{0+}^{17/5}y(1) = \left(\frac{3}{5}\right)^{9/29}D_{0+}^{17/5}y\left(\frac{1}{3}\right), \\ D_{0+}^{9/4}y(1) = \frac{7}{8}D_{0+}^{3/4}x\left(\frac{4}{5}\right) + \frac{3}{2}\int_0^1 D_{0+}^{5/6}x(\tau)\,d\tau. \end{cases} \tag{28}$$

We obtain here $\mathfrak{a}_1 \approx 0.59183673 > 0$, $\mathfrak{a}_2 \approx 0.71155008 > 0$, $\mathfrak{b}_1 \approx 1.45311179$, $\mathfrak{b}_2 \approx 2.39587178$, $\mathfrak{b} \approx 1.09690108 > 0$. Then assumption $(H1)$ is satisfied. We also find

$$\begin{aligned}
\mathfrak{g}_1(t,\tau) &= \frac{1}{\Gamma(7/4)}\begin{cases} t^{3/4}(1-\tau)^{3/4} - (t-\tau)^{3/4}, \ 0 \le \tau \le t \le 1, \\ t^{3/4}(1-\tau)^{3/4}, \ 0 \le t \le \tau \le 1, \end{cases} \\
\mathfrak{g}_2(t,\tau) &= \frac{1}{\Gamma(5/3)}\begin{cases} t^{2/3}(1-\tau)^{2/3} - (t-\tau)^{2/3}, \ 0 \le \tau \le t \le 1, \\ t^{2/3}(1-\tau)^{2/3}, \ 0 \le t \le \tau \le 1, \end{cases} \\
\mathfrak{g}_3(t,\tau) &= \frac{1}{\Gamma(5/2)}\begin{cases} t^{3/2}(1-\tau)^{1/6} - (t-\tau)^{3/2}, \ 0 \le \tau \le t \le 1, \\ t^{3/2}(1-\tau)^{1/6}, \ 0 \le t \le \tau \le 1, \end{cases} \\
\mathfrak{g}_{31}(\vartheta,\tau) &= \frac{1}{\Gamma(7/4)}\begin{cases} \vartheta^{3/4}(1-\tau)^{1/6} - (\vartheta-\tau)^{3/4}, \ 0 \le \tau \le \vartheta \le 1, \\ \vartheta^{3/4}(1-\tau)^{1/6}, \ 0 \le \vartheta \le \tau \le 1, \end{cases} \\
\mathfrak{g}_{32}(t,\tau) &= \frac{1}{\Gamma(5/3)}\begin{cases} \vartheta^{2/3}(1-\tau)^{1/6} - (\vartheta-\tau)^{2/3}, \ 0 \le \tau \le \vartheta \le 1, \\ \vartheta^{2/3}(1-\tau)^{1/6}, \ 0 \le \vartheta \le \tau \le 1, \end{cases} \\
\mathfrak{g}_4(t,\tau) &= \frac{1}{\Gamma(17/5)}\begin{cases} t^{12/5}(1-\tau)^{3/20} - (t-\tau)^{12/5}, \ 0 \le \tau \le t \le 1, \\ t^{12/5}(1-\tau)^{3/20}, \ 0 \le t \le \tau \le 1, \end{cases} \\
\mathfrak{g}_{41}(\vartheta,\tau) &= \frac{1}{\Gamma(41/15)}\begin{cases} \vartheta^{26/15}(1-\tau)^{3/20} - (\vartheta-\tau)^{26/15}, \ 0 \le \tau \le \vartheta \le 1, \\ \vartheta^{26/15}(1-\tau)^{3/20}, \ 0 \le \vartheta \le \tau \le 1, \end{cases} \\
\mathfrak{G}_1(t,\tau) &= \mathfrak{g}_1(t,\tau) + \frac{5t^{3/4}}{7\mathfrak{a}_1}\int_0^1 \mathfrak{g}_1(\zeta,\tau)\,d\zeta, \\
\mathfrak{G}_2(t,\tau) &= \mathfrak{g}_2(t,\tau) + \frac{3t^{2/3}}{5\mathfrak{a}_2}\mathfrak{g}_2\left(\frac{1}{3},\tau\right),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{G}_3(t,\tau) &= \mathfrak{g}_3(t,\tau) + \frac{t^{3/2}\mathfrak{b}_1}{\mathfrak{b}}\left[\frac{7}{8}\mathfrak{g}_{31}\left(\frac{4}{5},\tau\right) + \frac{3}{2}\int_0^1 \mathfrak{g}_{32}(\vartheta,\tau)\,d\vartheta\right],\\
\mathfrak{G}_4(t,\tau) &= \frac{18t^{3/2}\Gamma(17/5)}{7\mathfrak{b}\Gamma(23/20)}\mathfrak{g}_{41}\left(\frac{1}{2},\tau\right),\\
\mathfrak{G}_5(t,\tau) &= \frac{t^{12/5}\Gamma(5/2)}{\mathfrak{b}\Gamma(7/6)}\left[\frac{7}{8}\mathfrak{g}_{31}\left(\frac{4}{5},\tau\right) + \frac{3}{2}\int_0^1 \mathfrak{g}_{32}(\vartheta,\tau)\,d\vartheta\right],\\
\mathfrak{G}_6(t,\tau) &= \mathfrak{g}_4(t,\tau) + \frac{18t^{12/5}\mathfrak{b}_2}{7\mathfrak{b}}\mathfrak{g}_{41}\left(\frac{1}{2},\tau\right),\\
\mathfrak{h}_1(\tau) &= \tfrac{1}{\Gamma(7/4)}(1-\tau)^{3/4},\ \ \mathfrak{h}_2(\tau) = \tfrac{1}{\Gamma(5/3)}(1-\tau)^{2/3},\\
\mathfrak{h}_3(\tau) &= \tfrac{1}{\Gamma(5/2)}(1-\tau)^{1/6}(1-(1-\tau)^{4/3}),\\
\mathfrak{h}_4(\tau) &= \tfrac{1}{\Gamma(17/5)}(1-\tau)^{3/20}(1-(1-\tau)^{9/4}),
\end{aligned}$$

for all $t,\tau,\vartheta \in [0,1]$. In addition we deduce

$$\begin{aligned}
\mathfrak{J}_1(\tau) &= \mathfrak{h}_1(\tau) + \tfrac{5}{7\mathfrak{a}_1\Gamma(11/4)}\left[(1-\tau)^{3/4} - (1-\tau)^{7/4}\right],\ \ \tau\in[0,1],\\
\mathfrak{J}_2(\tau) &= \begin{cases} \mathfrak{h}_2(\tau) + \tfrac{3}{5\mathfrak{a}_2\Gamma(5/3)}\left[\left(\tfrac{1}{3}\right)^{2/3}(1-\tau)^{2/3} - \left(\tfrac{1}{3}-\tau\right)^{2/3}\right],\ \ 0\le\tau\le\tfrac{1}{3},\\ \mathfrak{h}_2(\tau) + \tfrac{3}{5\mathfrak{a}_2\Gamma(5/3)}\left(\tfrac{1}{3}\right)^{2/3}(1-\tau)^{2/3},\ \ \tfrac{1}{3}<\tau\le 1,\end{cases}\\
\mathfrak{J}_3(\tau) &= \begin{cases} \mathfrak{h}_3(\tau) + \tfrac{\mathfrak{b}_1}{\mathfrak{b}}\Big\{\tfrac{7}{8\Gamma(7/4)}\left[\left(\tfrac{4}{5}\right)^{3/4}(1-\tau)^{1/6} - \left(\tfrac{4}{5}-\tau\right)^{3/4}\right]\\ \quad +\tfrac{3}{2\Gamma(8/3)}\left[(1-\tau)^{1/6} - (1-\tau)^{5/3}\right]\Big\},\ \ 0\le\tau\le\tfrac{4}{5},\\ \mathfrak{h}_3(\tau) + \tfrac{\mathfrak{b}_1}{\mathfrak{b}}\Big\{\tfrac{7}{8\Gamma(7/4)}\left(\tfrac{4}{5}\right)^{3/4}(1-\tau)^{1/6}\\ \quad +\tfrac{3}{2\Gamma(8/3)}\left[(1-\tau)^{1/6} - (1-\tau)^{5/3}\right]\Big\},\ \ \tfrac{4}{5}<\tau\le 1,\end{cases}\\
\mathfrak{J}_4(\tau) &= \begin{cases} \tfrac{18\Gamma(17/5)}{7\mathfrak{b}\Gamma(23/20)\Gamma(41/15)}\left[\left(\tfrac{1}{2}\right)^{26/15}(1-\tau)^{3/20} - \left(\tfrac{1}{2}-\tau\right)^{26/15}\right],\ \ 0\le\tau\le\tfrac{1}{2},\\ \tfrac{18\Gamma(17/5)}{7\mathfrak{b}\Gamma(23/20)\Gamma(41/15)}\left(\tfrac{1}{2}\right)^{26/15}(1-\tau)^{3/20},\ \ \tfrac{1}{2}<\tau\le 1,\end{cases}\\
\mathfrak{J}_5(\tau) &= \begin{cases} \tfrac{\Gamma(5/2)}{\mathfrak{b}\Gamma(7/6)}\Big\{\tfrac{7}{8\Gamma(7/4)}\left[\left(\tfrac{4}{5}\right)^{3/4}(1-\tau)^{1/6} - \left(\tfrac{4}{5}-\tau\right)^{3/4}\right]\\ +\tfrac{3}{2\Gamma(8/3)}\left[(1-\tau)^{1/6} - (1-\tau)^{5/3}\right]\Big\},\ \ 0\le\tau\le\tfrac{4}{5},\\ \tfrac{\Gamma(5/2)}{\mathfrak{b}\Gamma(7/6)}\Big\{\tfrac{7}{8\Gamma(7/4)}\left(\tfrac{4}{5}\right)^{3/4}(1-\tau)^{1/6}\\ +\tfrac{3}{2\Gamma(8/3)}\left[(1-\tau)^{1/6} - (1-\tau)^{5/3}\right]\Big\},\ \ \tfrac{4}{5}<\tau\le 1,\end{cases}\\
\mathfrak{J}_6(\tau) &= \begin{cases} \mathfrak{h}_4(\tau) + \tfrac{18\mathfrak{b}_2}{7\mathfrak{b}\Gamma(41/15)}\left[\left(\tfrac{1}{2}\right)^{26/15}(1-\tau)^{3/20} - \left(\tfrac{1}{2}-\tau\right)^{26/15}\right],\ \ 0\le\tau\le\tfrac{1}{2},\\ \mathfrak{h}_4(\tau) + \tfrac{18\mathfrak{b}_2}{7\mathfrak{b}\Gamma(41/15)}\left(\tfrac{1}{2}\right)^{26/15}(1-\tau)^{3/20},\ \ \tfrac{1}{2}<\tau\le 1.\end{cases}
\end{aligned}$$

Example 1. *We introduce the functions*

$$\begin{aligned}
\mathfrak{f}(t,w_1,w_2,w_3,w_4) &= \frac{(3w_1+2w_2+w_3+5w_4)^{19a/8}}{t^{z_1}(1-t)^{z_2}},\\
\mathfrak{g}(t,w_1,w_2,w_3,w_4) &= \frac{(w_1+7w_2+4w_3+2w_4)^{29b/9}}{t^{z_3}(1-t)^{z_4}},
\end{aligned} \tag{29}$$

for $t\in(0,1)$, $w_i\ge 0$, $i=1,\ldots,4$, *where* $a>1$, $b>1$, $z_1\in(0,1)$, $z_2\in\left(0,\frac{7}{4}\right)$, $z_3\in(0,1)$, $z_4\in\left(0,\frac{5}{3}\right)$. *Here* $\xi_1(t)=\frac{1}{t^{z_1}(1-t)^{z_2}}$, $\xi_2(t)=\frac{1}{t^{z_3}(1-t)^{z_4}}$ *for* $t\in(0,1)$, $\psi_1(t,w_1,w_2,w_3,w_4)=(3w_1+2w_2+w_3+5w_4)^{19a/8}$ *and* $\psi_2(t,w_1,w_2,w_3,w_4)=(w_1+7w_2+4w_3+2w_4)^{29b/9}$ *for* $t\in[0,1]$, $w_i\ge 0$, $i=1,\ldots,4$. *We also obtain* $M_1=B(1-z_1,7/4-z_2)\in(0,\infty)$, $M_2=B(1-z_3,5/3-z_4)\in(0,\infty)$. *Then assumption* $(H2)$ *is satisfied. In addition, in* $(H3)$, *for* $l_1=3$, $l_2=2$, $l_3=1$, $l_4=5$, $\theta_1=1$, $m_1=1$, $m_2=7$, $m_3=4$, $m_4=2$, $\theta_2=1$, *we find* $\psi_{10}=0$ *and* $\psi_{20}=0$. *In* $(H4)$, *for* $[\sigma_1,\sigma_2]\subset(0,1)$, $s_1=3$, $s_2=2$, $s_3=1$, $s_4=5$, *we obtain*

$\mathfrak{f}_\infty = \infty$. *Then by Theorem 1 we deduce that problem (27), (28) with the nonlinearities (29) has at least one solution* $(x_1(t), y_1(t))$, $t \in [0,1]$.

Example 2. *We define the functions*

$$\begin{aligned}
&\mathfrak{f}(t,w_1,w_2,w_3,w_4) = \frac{p_0(t+3)}{(t^2+8)\sqrt[4]{t^3}}\left[\left(\frac{1}{2}w_1+w_2+\frac{1}{4}w_3+\frac{1}{5}w_4\right)^{v_1}\right.\\
&\quad\left.+\left(\tfrac{1}{2}w_1+w_2+\tfrac{1}{4}w_3+\tfrac{1}{5}w_4\right)^{v_2}\right],\ t\in(0,1],\ w_i\ge 0,\ i=1,\dots,4,\\
&\mathfrak{g}(t,w_1,w_2,w_3,w_4) = \frac{q_0(2+\sin t)}{(t+6)^4\sqrt[7]{(1-t)^5}}\left(w_1^{v_3}+e^{w_2}+\ln(w_3+w_4+1)\right),\\
&\quad t\in[0,1),\ w_i\ge 0,\ i=1,\dots,4,
\end{aligned} \tag{30}$$

where $p_0 > 0$, $q_0 > 0$, $v_1 > 19/8$, $v_2 \in (0, 19/8)$, $v_3 > 0$. *Here we have* $\zeta_1(t) = \frac{1}{\sqrt[4]{t^3}}$, $t \in (0,1]$, $\psi_1(t,w_1,w_2,w_3,w_4) = \frac{p_0(t+3)}{(t^2+8)}\left[\left(\frac{1}{2}w_1+w_2+\frac{1}{4}w_3+\frac{1}{5}w_4\right)^{v_1}+\left(\frac{1}{2}w_1+w_2+\frac{1}{4}w_3+\frac{1}{5}w_4\right)^{v_2}\right]$, $t\in[0,1]$, $w_i \ge 0$, $i=1,\dots,4$, $\zeta_2(t) = \frac{1}{\sqrt[7]{(1-t)^5}}$, $t \in [0,1)$, $\psi_2(t,w_1,w_2,w_3,w_4) = \frac{q_0(2+\sin t)}{(t+6)^4}(w_1^{v_3}+e^{w_2}+\ln(w_3+w_4+1))$, $t\in[0,1]$, $w_i\ge 0$, $i=1,\dots,4$. *We obtain* $M_1 = B(1/4, 7/4) \in (0,\infty)$, $M_2 = \frac{21}{20} \in (0,\infty)$. *Then assumption* $(H2)$ *is satisfied. For* $[\sigma_1,\sigma_2] \subset (0,1)$, $s_1 = \frac{1}{2}$, $s_2 = 1$, $s_3 = \frac{1}{4}$, $s_4 = \frac{1}{5}$, *we find* $\mathfrak{f}_\infty = \infty$ *(in* $(H4)$*), and for* $p_1 = \frac{1}{2}$, $p_2 = 1$, $p_3 = \frac{1}{4}$, $p_4 = \frac{1}{5}$, $\varsigma_1 \in \left(\frac{8v_2}{19}, 1\right]$, *we have* $\mathfrak{f}_0 = \infty$ *(in* $(H6)$*). So assumptions* $(H4)$ *and* $(H6)$ *are satisfied. Then after some computations we deduce* $\Xi_1 \approx 3.93816256$, $\Xi_2 \approx 1.53523525$, $\Xi_3 \approx 1.40740842$, $\Xi_4 \approx 0.97489748$, $\Xi_5 \approx 1.04873754$, $\Xi_6 \approx 0.92404828$, $\varpi = 1$, *and* $A_0 = \max\left\{\frac{4p_0}{9}\left(\left(\frac{39}{20}\right)^{v_1}+\left(\frac{39}{20}\right)^{v_2}\right), q_0 m_0(1+e+\ln 3)\right\}$, *where* $m_0 = \max_{t\in[0,1]}\frac{2+\sin t}{(t+6)^4} \approx 2.00035047$. *If*

$$\begin{aligned}
p_0 &< \frac{9}{\left(\frac{39}{20}\right)^{v_1}+\left(\frac{39}{20}\right)^{v_2}}\min\left\{\frac{1}{4^{27/8}\Xi_3^{19/8}\Xi_1}, \frac{1}{4^{38/9}\Xi_4^{29/9}\Xi_2}, \frac{1}{4^{27/8}\Xi_5^{19/8}\Xi_1}, \frac{1}{4^{38/9}\Xi_6^{29/9}\Xi_2}\right\},\\
q_0 &< \frac{1}{m_0(1+e+\ln 3)}\min\left\{\frac{1}{4^{19/8}\Xi_3^{19/8}\Xi_1}, \frac{1}{4^{29/9}\Xi_4^{29/9}\Xi_2}, \frac{1}{4^{19/8}\Xi_5^{19/8}\Xi_1}, \frac{1}{4^{29/9}\Xi_6^{29/9}\Xi_2}\right\},
\end{aligned}$$

then the inequalities $A_0^{8/19}\Xi_3\Xi_1^{8/19} < \frac{1}{4}$, $A_0^{9/29}\Xi_4\Xi_2^{9/29} < \frac{1}{4}$, $A_0^{8/19}\Xi_5\Xi_1^{8/19} < \frac{1}{4}$, $A_0^{9/29}\Xi_6\Xi_2^{9/29} < \frac{1}{4}$ *are satisfied, (that is, assumption* $(H7)$ *is satisfied). For example, if* $v_1 = 2$, $v_2 = 3$ *and* $p_0 \le 0.0008$, $q_0 \le 0.0004$, *then the above inequalities are verified. By Theorem 3, we conclude that problem (27), (28) with the nonlinearities (30) has at least two positive solutions* $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$, $t \in [0,1]$.

6. Conclusions

In this paper we investigated the system of coupled fractional differential equations (1) with ρ-Laplacian operators and Riemann–Liouville fractional derivatives of varied orders, supplemented with general nonlocal boundary conditions (2) containing Riemann–Stieltjes integrals and fractional derivatives of differing orders. The nonlinearities from the system are dependent on various fractional integrals and they are nonnegative and singular in the points $t = 0$ and $t = 1$. The last boundary conditions for the unknown functions x and y are coupled in the point 1, in contrast to the boundary conditions from paper [6] in which they are uncoupled in the point 1. We presented diverse assumptions on the functions $\mathfrak{f}$ and $\mathfrak{g}$ so that problem (1), (2) has one positive solution (in Theorems 1 and 2), and two positive solutions (in Theorem 3). We also gave the corresponding Green functions and their properties used in the proof of the main results. We transformed our problem into a system of integral equations and we associated an operator $\mathfrak{C}$ for which we looked for the fixed points by applying the Guo–Krasnosel'skii fixed point theorem of cone expansion and compression of norm type. We presented finally two examples for illustrating our main theorems. For some future research directions we have in mind the study of some systems of fractional differential equations with other nonlocal coupled or uncoupled boundary conditions.

Author Contributions: Conceptualization, R.L.; Formal analysis, A.T. and R.L.; Methodology, A.T. and R.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Wang, G.; Ren, X.; Zhang, L.; Ahmad, B. Explicit iteration and unique positive solution for a Caputo-Hadamard fractional turbulent flow model. *IEEE Access* **2019**, *7*, 109833–109839. [CrossRef]
2. Guo, D.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cones*; Academic Press: New York, NY, USA, 1988.
3. Wang, H.; Jiang, J. Existence and multiplicity of positive solutions for a system of nonlinear fractional multi-point boundary value problems with p-Laplacian operator. *J. Appl. Anal. Comput.* **2021**, *11*, 351–366.
4. Tudorache, A.; Luca, R. Positive solutions for a system of Riemann–Liouville fractional boundary value problems with p-Laplacian operators. *Adv. Differ. Equ.* **2020**, *292*, 1–30. 10.1186/s13662-020-02750-6. [CrossRef]
5. Alsaedi, A.; Luca, R.; Ahmad, B. Existence of positive solutions for a system of singular fractional boundary value problems with p-Laplacian operators. *Mathematics* **2020**, *8*, 1890. [CrossRef]
6. Tudorache, A.; Luca, R. Positive solutions of a singular fractional boundary value problem with r-Laplacian operators. *Fractal Fract.* **2022**, *6*, 18. [CrossRef]
7. Prasad, K.R.; Leela, I.D.; Khuddush, M. Existence and uniqueness of positive solutions for system of (p,q,r)-Laplacian fractional order boundary value problems. *Adv. Theory Nonlinear Anal. Appl.* **2021**, *5*, 138–157.
8. Tan, J.; Li, M. Solutions of fractional differential equations with p-Laplacian operator in Banach spaces. *Bound. Value Prob.* **2018**, *15*, 1–13. [CrossRef]
9. Tang, X.; Wang, X.; Wang, Z.; Ouyang, P. The existence of solutions for mixed fractional resonant boundary value problem with $p(t)$-Laplacian operator. *J. Appl. Math. Comput.* **2019**, *61*, 559–572. [CrossRef]
10. Tian, Y.; Sun, S.; Bai, Z. Positive Solutions of Fractional Differential Equations with p-Laplacian. *J. Funct. Spaces* **2017**, *2017*, 3187492. [CrossRef]
11. Wang, H.; Liu, S.; Li, H. Positive solutions to p-Laplacian fractional differential equations with infinite-point boundary value conditions. *Adv. Differ. Equ.* **2018**, *425*, 1–15. [CrossRef]
12. Wang, Y.; Liu, S.; Han, Z. Eigenvalue problems for fractional differential equationswith mixed derivatives and generalized p-Laplacian. *Nonlinear Anal. Model. Control* **2018**, *23*, 830–850. [CrossRef]
13. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer: Cham, Switzerland, 2017.
14. Ahmad, B.; Henderson, J.; Luca, R. *Boundary Value Problems for Fractional Differential Equations and Systems*; Trends in Abstract and Applied Analysis; World Scientific: Hackensack, NJ, USA, 2021; Volume 9.
15. Ahmad, B.; Ntouyas, S.K. *Nonlocal Nonlinear Fractional-Order Boundary Value Problems*; World Scientific: Hackensack, NJ, USA, 2021.
16. Henderson, J.; Luca, R. *Boundary Value Problems for Systems of Differential, Difference and Fractional Equations. Positive Solutions*; Elsevier: Amsterdam, The Netherlands, 2016.
17. Zhou, Y.; Wang, J.R.; Zhang, L. *Basic Theory of Fractional Differential Equations*, 2nd ed.; World Scientific: Singapore, 2016.