



Article Controllability and Hyers–Ulam Stability of Fractional Systems with Pure Delay

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Abstract: Linear and nonlinear fractional-delay systems are studied. As an application, we derive the controllability and Hyers–Ulam stability results using the representation of solutions of these systems with the help of their delayed Mittag–Leffler matrix functions. We provide some sufficient and necessary conditions for the controllability of linear fractional-delay systems by introducing a fractional delay Gramian matrix. Furthermore, we establish some sufficient conditions of controllability and Hyers–Ulam stability of nonlinear fractional-delay systems by applying Krasnoselskii's fixed-point theorem. Our results improve, extend, and complement some existing ones. Finally, numerical examples of linear and nonlinear fractional-delay systems are presented to demonstrate the theoretical results.

Keywords: controllability; fractional-delay system; delayed Mittag–Leffler matrix function; Caputo fractional derivative; Hyers–Ulam stability; Krasnoselskii's fixed-point theorem

MSC: 34K37; 93B05; 93C23; 93D99

1. Introduction

The fractional delay differential equations and their applications have gained significant attention owing to their successful modeling in several fields of science and engineering, such as disease, control theory, signal analysis, diffusion processes, biology, forced oscillations, population dynamics, viscoelastic systems, computer engineering, and finance; see, for instance, [1–8]. Recently, the representation of solutions of time-delay systems has been considered. In particular, the pioneering study [9,10] produced several innovative findings on the representations of solutions of time-delay systems, which were used in the control problems and stability analysis; see, for instance, [11–21] and the references therein.

On the one hand, the controllability of systems is one of the most fundamental and significant concepts in modern control theory, which consists of determining the control parameters that steer the solutions of a control system from its initial state to its final state using a set of admissible controls, where initial and final states may vary over an entire space. In recent decades, there has been considerable interest in the controllability analysis of fractional-delay systems of order $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$, and several methods for studying the controllability results have been developed, for example, the robust and universal methods [22]; the Laplace transform technique, the Mittag–Leffler function and fixed-point argument [23]; Martelli's fixed-point theorem, multivalued functions, and cosine and sine families [24]; the Mittag–Leffler matrix functions and the Schauder fixed-point theorem [20,25,26]; the Mittag–Leffler matrix function, the Gramian matrix, and the iterative technique [27]; the solution operator theory, fractional calculations, and fixed point techniques [28]; and the delayed fractional Gram matrix and the explicit solution formula [29].



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Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). On the other hand, the Hyers–Ulam stability of fractional delay systems has been studied recently by many authors; see, for example, [19,30,31] and the references therein.

However, to the best of our knowledge, no research has been conducted on the controllability of linear fractional-delay systems of the form

and the controllability and Hyers–Ulam stability of the corresponding nonlinear fractionaldelay systems of the form

where ${}^{C}D_{0^+}^{\alpha}$ is called the Caputo fractional derivative of order $\alpha \in (1,2]$ with the lower index zero, h > 0 is a delay, $x_1 > (n-1)h$, $y(x) \in \mathbb{R}^n$, $\psi \in C([-h,0],\mathbb{R}^n)$, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are any matrices, $f \in C(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ is a given function, and $u(x) \in \mathbb{R}^m$ shows control vector.

Elshenhab and Wang [11] have presented a novel formulation of solutions to the linear fractional-delay systems

of the following form:

$$y(x) = \mathcal{H}_{h,\alpha} (A(x-h)^{\alpha}) \psi(0) + \mathcal{M}_{h,\alpha} (A(x-h)^{\alpha}) \psi'(0)$$

- $A \int_{-h}^{0} S_{h,\alpha} (A(x-2h-\downarrow)^{\alpha}) \psi(\downarrow) d\downarrow$
+ $\int_{0}^{x} S_{h,\alpha} (A(x-h-\downarrow)^{\alpha}) f(\downarrow) d\downarrow,$ (4)

where $\mathcal{H}_{h,\alpha}(Ax^{\alpha})$, $\mathcal{M}_{h,\alpha}(Ax^{\alpha})$, and $\mathcal{S}_{h,\alpha}(Ax^{\alpha})$ are known as the delayed Mittag–Lefflertype matrix functions formulated by

$$\mathcal{H}_{h,\alpha}(Ax^{\alpha}) := \begin{cases} \Theta, & -\infty < x < -h, \\ \mathbb{I}, & -h \le x < 0, \\ \mathbb{I} - A \frac{x^{\alpha}}{\Gamma(1+\alpha)}, & 0 \le x < h, \\ \vdots & \vdots & & \vdots \\ \mathbb{I} - A \frac{x^{\alpha}}{\Gamma(1+\alpha)} + A^2 \frac{(x-h)^{2\alpha}}{\Gamma(1+2\alpha)} \\ + \dots + (-1)^r A^r \frac{(x-(r-1)h)^{r\alpha}}{\Gamma(1+r\alpha)}, & (r-1)h \le x < rh, \end{cases}$$
(5)
$$\mathcal{M}_{h,\alpha}(Ax^{\alpha}) := \begin{cases} \Theta, & -\infty < x < -h, \\ \mathbb{I}(x+h), & -h \le x < 0, \\ \mathbb{I}(x+h) - A \frac{x^{\alpha+1}}{\Gamma(2+\alpha)}, & 0 \le x < h, \\ \vdots & \vdots \\ \mathbb{I}(x+h) - A \frac{x^{\alpha+1}}{\Gamma(2+\alpha)} + A^2 \frac{(x-h)^{2\alpha+1}}{\Gamma(2+2\alpha)} \\ + \dots + (-1)^r A^r \frac{(x-(r-1)h)^{r\alpha+1}}{\Gamma(2+r\alpha)}, & (r-1)h \le x < rh, \end{cases}$$
(6)

and

$$S_{h,\alpha}(Ax^{\alpha}) := \begin{cases} \Theta, & -\infty < x < -h, \\ \mathbb{I}\frac{(x+h)^{\alpha-1}}{\Gamma(\alpha)}, & -h \le x < 0, \\ \mathbb{I}\frac{(x+h)^{\alpha-1}}{\Gamma(\alpha)} - A\frac{x^{2\alpha-1}}{\Gamma(2\alpha)}, & 0 \le x < h, \\ \vdots & \vdots \\ \mathbb{I}\frac{(x+h)^{\alpha-1}}{\Gamma(\alpha)} - A\frac{x^{2\alpha-1}}{\Gamma(2\alpha)} + A^2\frac{(x-h)^{3\alpha-1}}{\Gamma(3\alpha)} \\ + \dots + (-1)^r A^r \frac{(x-(r-1)h)^{\alpha(r+1)-1}}{\Gamma(\alpha(r+1))}, & (r-1)h \le x < rh, \end{cases}$$
(7)

respectively, where the notation Θ and \mathbb{I} are the $n \times n$ null and identity matrix, respectively, Γ is a gamma function, and r = 0, 1, 2, ...

Applying Formula (4), the solution of (2) can be represented as

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$$y(x) = \mathcal{H}_{h,\alpha} (A(x-h)^{\alpha}) \psi(0) + \mathcal{M}_{h,\alpha} (A(x-h)^{\alpha}) \psi'(0)$$

- $A \int_{-h}^{0} S_{h,\alpha} (A(x-2h-\downarrow)^{\alpha}) \psi(\downarrow) d\downarrow$
+ $\int_{0}^{x} S_{h,\alpha} (A(x-h-\downarrow)^{\alpha}) f(\downarrow, y(\downarrow)) d\downarrow$
+ $\int_{0}^{x} S_{h,\alpha} (A(x-h-\downarrow)^{\alpha}) Bu(\downarrow) d\downarrow,$ (8)

Motivated by [11,16], the explicit solutions Formula (8) of (3) combined with the delayed Mittag–Leffler matrix functions are employed as an application to derive controllability results on $\Omega = [0, x_1]$.

The rest of this paper is structured as follows: in Section 2, we present some preliminaries, some basic notation and definitions, and some useful lemmas. In Section 3, we derive sufficient and necessary conditions for the controllability of (1) by introducing a fractional delay Gramian matrix. In Section 4, we establish sufficient conditions of the controllability of (2) by applying Krasnoselskii's fixed-point theorem. In Section 5, we discuss the Hyers–Ulam stability of (2) on the finite-time interval $[0, x_1]$. Finally, we provide numerical examples of linear and nonlinear fractional-delay systems to demonstrate the theoretical results.

2. Preliminaries

Throughout the paper, we refer to $C(\Omega, \mathbb{R}^n)$ as the Banach space of a vector-valued continuous function from $\Omega \to \mathbb{R}^n$ endowed with the norm $\|y\|_{C(\Omega)} = \max_{x \in \Omega} \|y(x)\|$ for a norm $\|\cdot\|$ on \mathbb{R}^n , and the matrix norm as $\|A\| = \max_{\|y\|=1} \|Ay\|$, where $A : \mathbb{R}^n \to \mathbb{R}^n$. We define a space $C^1(\Omega, \mathbb{R}^n) = \{y \in C(\Omega, \mathbb{R}^n) : y' \in C(\Omega, \mathbb{R}^n)\}$. Let *X*, *Y* be two Banach spaces and $L_b(X, Y)$ be the space of bounded linear operators from *X* to *Y*. Now, $L^p(\Omega, Y)$ indicates the Banach space of functions $f : \Omega \to Y$ that are Bochner integrable normed by $\|f\|_{L^p(\Omega,Y)}$ for some $1 . Furthermore, we let <math>\|\psi\|_C = \max_{s \in [-h,0]} \|\psi(s)\|$ and $\|\psi'\|_C = \max_{s \in [-h,0]} \|\psi'(s)\|$.

We mention some basic concepts and lemmas utilized throughout this paper.

Definition 1. ([5]). The Mittag–Leffler function with two parameters is given by

$$\mathbb{E}_{\sigma,\tau}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\sigma r + \tau)}, \quad \sigma, \ \tau > 0, \ x \in \mathbb{C}.$$

In the case of $\tau = 1$, then

$$\mathbb{E}_{\sigma,1}(x) = \mathbb{E}_{\sigma}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\sigma r+1)}, \quad \sigma > 0.$$

Definition 2. ([5]). A function $y : [-h, \infty) \to \mathbb{R}^n$ has the Caputo fractional derivative of order $\alpha \in (1, 2]$ with a lower index 0 given by

$$\binom{C}{0}D_{0^+}^{\alpha}y(x) = rac{1}{\Gamma(2-\alpha)}\int_0^x rac{y''(\downarrow)}{(x-\downarrow)^{\alpha-1}}d\downarrow, \quad x>0.$$

Definition 3. ([32]). The systems (1) or (2) are controllable on $\Omega = [0, x_1]$ if there is a control function $u \in L^2(\Omega, \mathbb{R}^m)$ such that (1) or (2) has a solution $y : [-h, x_1] \to \mathbb{R}^n$ with $y(0) = y_0$, $y'(0) = y'_0$ satisfies $y(x_1) = y_1$ for all $y_0, y'_0, y_1 \in \mathbb{R}^n$.

Definition 4. ([33]). The system (2) is Hyers–Ulam stable on $[0, x_1]$ if there is, for a given constant $\varepsilon > 0$, a function $\varphi \in C(\Omega, \mathbb{R}^n)$ satisfying the inequality

$$\left\| \left({}^{C}D_{0^{+}}^{\alpha}\varphi \right)(x) + A\varphi(x-h) - f(x,\varphi(x)) - Bu(x) \right\| \le \varepsilon, \quad x \in [0,x_{1}], \tag{9}$$

there exists a solution $y \in C(\Omega, \mathbb{R}^n)$ of (2) and a constant M > 0 such that

$$\|\varphi(x) - y(x)\| \le M\varepsilon$$
, for all $x \in [0, x_1]$.

Remark 1. ([33]). A function $\varphi \in C(\Omega, \mathbb{R}^n)$ is a solution of the inequality (9) if and only if there is a function $\pi \in C(\Omega, \mathbb{R}^n)$ such that

(i) $\|\pi(x)\| \leq \varepsilon, x \in \Omega.$

(ii) $(^{C}D_{0^{+}}^{\alpha}\varphi)(x) = -A\varphi(x-h) + f(x,\varphi(x)) + Bu(x) + \pi(x), x \in \Omega.$

Lemma 1. ([17]). The following inequalities hold:

$$\begin{aligned} \left\| \mathcal{H}_{h,\alpha}(Ax^{\alpha}) \right\| &\leq \mathbb{E}_{\alpha}(\|A\|x^{\alpha}), \\ \left\| \mathcal{M}_{h,\alpha}(Ax^{\alpha}) \right\| &\leq (x+h)\mathbb{E}_{\alpha,2}(\|A\|(x+h)^{\alpha}), \end{aligned}$$

and

$$\left\|\mathcal{S}_{h,\alpha}(Ax^{\alpha})\right\| \leq (x+h)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|A\|(x+h)^{\alpha}\right).$$

for any $x \in [(r-1)h, rh], r = 1, 2, ...$

Lemma 2. Let $\alpha > 0$ and $\varphi \in C(\Omega, \mathbb{R}^n)$ be a solution of the inequality (9). Then there exists, for a given constant $\varepsilon > 0$, a solution φ^* satisfying the inequality

$$\|\varphi(x)-\varphi^*(x)\|\leq \frac{x^{lpha}\varepsilon}{lpha}\mathbb{E}_{lpha,lpha}(\|A\|x^{lpha}).$$

where

$$\begin{split} \varphi^*(x) &= \mathcal{H}_{h,\alpha} \big(A(x-h)^{\alpha} \big) \psi(0) + \mathcal{M}_{h,\alpha} \big(A(x-h)^{\alpha} \big) \psi'(0) \\ &- A \int_{-h}^0 \mathcal{S}_{h,\alpha} \big(A(x-2h-\downarrow)^{\alpha} \big) \psi(\downarrow) d\downarrow \\ &+ \int_0^x \mathcal{S}_{h,\alpha} \big(A(x-h-\downarrow)^{\alpha} \big) f(\downarrow,\varphi(\downarrow)) d\downarrow \\ &+ \int_0^x \mathcal{S}_{h,\alpha} \big(A(x-h-\downarrow)^{\alpha} \big) Bu_{\varphi}(\downarrow) d\downarrow. \end{split}$$

Proof. From Remark 1, the solution of the equation

$$\left({}^{\mathsf{C}}D_{0^{+}}^{\alpha}\varphi\right)(x) = -A\varphi(x-h) + f(x,\varphi(x)) + Bu(x) + \pi(x), \quad x \in \Omega,$$

can be written as

$$\begin{split} \varphi(x) &= \mathcal{H}_{h,\alpha} \big(A(x-h)^{\alpha} \big) \psi(0) + \mathcal{M}_{h,\alpha} \big(A(x-h)^{\alpha} \big) \psi'(0) \\ &- A \int_{-h}^{0} \mathcal{S}_{h,\alpha} \big(A(x-2h-\downarrow)^{\alpha} \big) \psi(\downarrow) \mathrm{d} \downarrow \\ &+ \int_{0}^{x} \mathcal{S}_{h,\alpha} \big(A(x-h-\downarrow)^{\alpha} \big) f(\downarrow,\varphi(\downarrow)) \mathrm{d} \downarrow \\ &+ \int_{0}^{x} \mathcal{S}_{h,\alpha} \big(A(x-h-\downarrow)^{\alpha} \big) B u_{\varphi}(\downarrow) \mathrm{d} \downarrow \\ &+ \int_{0}^{x} \mathcal{S}_{h,\alpha} \big(A(x-h-\downarrow)^{\alpha} \big) \pi(\downarrow) \mathrm{d} \downarrow. \end{split}$$

From Lemma 1, we obtain

$$\begin{split} \|\varphi(x) - \varphi^*(x)\| &\leq \int_0^x \|\mathcal{S}_{h,\alpha} \big(A(x-h-\downarrow)^{\alpha}\big)\big\| \|\pi(\downarrow)\| \mathsf{d}\downarrow\\ &\leq \varepsilon \int_0^x (x-\downarrow)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \big(\|A\|(x-\downarrow)^{\alpha}\big) \mathsf{d}\downarrow\\ &\leq \frac{x^{\alpha} \varepsilon}{\alpha} \mathbb{E}_{\alpha,\alpha} (\|A\|x^{\alpha}), \end{split}$$

for all $x \in \Omega$. This ends the proof. \Box

Lemma 3. (*Krasnoselskii's fixed-point theorem,* [34]). Let *C* be a closed, convex, and non-empty subset of a Banach space X. Suppose that the operators A and B are maps from C into X such that $Ax + By \in C$ for every pair $x, y \in C$. If A is compact and continuous, B is a contraction mapping. Then, there exists $z \in C$ such that z = Az + Bz.

3. Controllability of Linear Fractional Delay System

In this section, we establish some sufficient and necessary conditions of controllability of (1) by introducing a fractional delay Gramian matrix defined by

$$W_{h,\alpha}[0,x_1] = \int_0^{x_1} S_{h,\alpha} \left(A(x_1 - h - \downarrow)^{\alpha} \right) B B^T S_{h,\alpha} \left(A^T (x_1 - h - \downarrow)^{\alpha} \right) d\downarrow.$$
(10)

It follows from the definition of the matrix $W_{h,\alpha}[0, x_1]$ that it is always positive semidefinite for $x_1 \ge 0$.

Theorem 1. The linear system (1) is controllable if and only if $W_{h,\alpha}[0, x_1]$ is positive definite.

Proof. Sufficiency. Let $W_{h,\alpha}[0, x_1]$ be positive definite; then, it will be non-singular and its inverse will be well-defined. As a result, we can derive the associated control input u(x), for any finite terminal conditions $y_1, y'_1 \in \mathbb{R}^n$, as

$$u(x) = B^{T} \mathcal{S}_{h,\alpha} \left(A^{T} (x_{1} - h - x)^{\alpha} \right) W_{h,\alpha}^{-1}[0, x_{1}] \beta,$$
(11)

where

$$\beta = y_1 - \mathcal{H}_{h,\alpha} \left(A(x_1 - h)^{\alpha} \right) \psi(0) - \mathcal{M}_{h,\alpha} \left(A(x_1 - h)^{\alpha} \right) \psi'(0) + A \int_{-h}^0 \mathcal{S}_{h,\alpha} \left(A(x_1 - 2h - \downarrow)^{\alpha} \right) \psi(\downarrow) \mathrm{d} \downarrow.$$
(12)

From (8), the solution $y(x_1)$ of (1) can be formulated as

$$y(x_{1}) = \mathcal{H}_{h,\alpha} \left(A(x_{1} - h)^{\alpha} \right) \psi(0) + \mathcal{M}_{h,\alpha} \left(A(x_{1} - h)^{\alpha} \right) \psi'(0)$$
$$- A \int_{-h}^{0} \mathcal{S}_{h,\alpha} \left(A(x_{1} - 2h - \downarrow)^{\alpha} \right) \psi(\downarrow) d\downarrow$$
$$+ \int_{0}^{x_{1}} \mathcal{S}_{h,\alpha} \left(A(x_{1} - h - \downarrow)^{\alpha} \right) Bu(\downarrow) d\downarrow.$$
(13)

Substituting (11) into (13), we obtain

$$y(x_{1}) = \mathcal{H}_{h,\alpha} \left(A(x_{1}-h)^{\alpha} \right) \psi(0) + \mathcal{M}_{h,\alpha} \left(A(x_{1}-h)^{\alpha} \right) \psi'(0) - A \int_{-h}^{0} S_{h,\alpha} \left(A(x_{1}-2h-\downarrow)^{\alpha} \right) \psi(\downarrow) d\downarrow$$

$$+ \int_{0}^{x_{1}} S_{h,\alpha} \left(A(x_{1}-h-\downarrow)^{\alpha} \right) B B^{T} S_{h,\alpha} \left(A^{T}(x_{1}-h-\downarrow)^{\alpha} \right) d\downarrow W_{h,\alpha}^{-1}[0,x_{1}] \beta.$$
(14)

From (10), (12), and (14), we obtain

$$y(x_1) = \mathcal{H}_{h,\alpha} \left(A(x_1 - h)^{\alpha} \right) \psi(0) + \mathcal{M}_{h,\alpha} \left(A(x_1 - h)^{\alpha} \right) \psi'(0)$$
$$- A \int_{-h}^{0} S_{h,\alpha} \left(A(x_1 - 2h - \downarrow)^{\alpha} \right) \psi(\downarrow) d\downarrow + \beta$$
$$= y_1.$$

We can see from (3) and (4) that the boundary conditions hold. Thus, (1) is controllable. **Necessity.** Assume that (1) is controllable. For the sake of a contradiction, suppose that $W_{h,\alpha}[0, x_1]$ is not positive definite, and there exists at least a nonzero vector $z \in \mathbb{R}^n$ such that $z^T W_{h,\alpha}[0, x_1]z = 0$, which implies that

$$0 = z^{T} W_{h,\alpha}[0, x_{1}]z$$

$$= \int_{0}^{x_{1}} z^{T} S_{h,\alpha} (A(x_{1} - h - \downarrow)^{\alpha}) BB^{T} S_{h,\alpha} (A^{T}(x_{1} - h - \downarrow)^{\alpha}) z d\downarrow$$

$$= \int_{0}^{x_{1}} \left[z^{T} S_{h,\alpha} (A(x_{1} - h - \downarrow)^{\alpha}) B \right] \left[z^{T} S_{h,\alpha} (A(x_{1} - h - \downarrow)^{\alpha}) B \right]^{T} d\downarrow$$

$$= \int_{0}^{x_{1}} \left[z^{T} S_{h,\alpha} (A(x_{1} - h - \downarrow)^{\alpha}) B \right] \left[z^{T} S_{h,\alpha} (A(x_{1} - h - \downarrow)^{\alpha}) B \right]^{T} d\downarrow$$

$$= \int_{0}^{x_{1}} \left\| z^{T} S_{h,\alpha} (A(x_{1} - h - \downarrow)^{\alpha}) B \right\| d\downarrow.$$

Hence

$$z^{T}\mathcal{S}_{h,\alpha}(A(x_{1}-h-\downarrow)^{\alpha})B = (0,\ldots,0) := \mathbf{0}^{T}, \text{ for all } \downarrow \in \Omega,$$
(15)

where **0** denotes the *n* dimensional zero vector. Consider the initial points $y_0 = y'_0 = \mathbf{0}$ and the final point $y_1 = z$ at $x = x_1$. Since (1) is controllable, from Definition 3, there exists a control function $u_1(x)$ that steers the response from **0** to $y_1 = z$ at $x = x_1$. Then,

$$y_{1} = z = -A \int_{-h}^{0} S_{h,\alpha} (A(x_{1} - 2h - \downarrow)^{\alpha}) \psi(\downarrow) d\downarrow$$
$$+ \int_{0}^{x_{1}} S_{h,\alpha} (A(x_{1} - h - \downarrow)^{\alpha}) Bu_{1}(\downarrow) d\downarrow.$$
(16)

Multiplying (16) by z^T and using (15), we obtain $z^T z = 0$. This is a contradiction to $z \neq \mathbf{0}$. Thus, $W_{h,\alpha}[0, x_1]$ is positive definite. This ends the proof. \Box

Remark 2. We note in the case of $\alpha = 2$ in (1) that Theorem 1 coincides with the conclusion of Corollary 1 in [16].

Remark 3. Under condition A, a nonsingular $n \times n$ matrix, we note in the case of $\alpha = 2$, $A = A^2$ in (1) that Theorem 1 coincides with the conclusion of Theorem 3.1 in [21] and Corollary 2 in [16].

4. Controllability of Nonlinear Fractional Delay System

In this section, we estabilish sufficient conditions of controllability of (2) using Krasnoselskii's fixed point theorem.

We impose the following assumptions:

(G1) The function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and there exists a constant $L_f \in L^q(\Omega, \mathbb{R}^+)$ and q > 1 such that

$$||f(x,y_1) - f(x,y_2)|| \le L_f(x)||y_1 - y_2||$$
, for all $x \in \Omega, y_1, y_2 \in \mathbb{R}^n$,

let $\sup_{x\in\Omega} f(x,0) = M_f < \infty$.

(G2) The linear operator $Y : L^2(\Omega, \mathbb{R}^m) \to \mathbb{R}^n$ defined by

$$Y = \int_0^{x_1} S_{h,\alpha} (A(x_1 - h - \downarrow)^{\alpha}) Bu(\downarrow) d\downarrow.$$

Suppose that Y^{-1} exists and takes values in $L^2(\Omega, \mathbb{R}^m) / \ker Y$, and there exists a constant $M_1 > 0$ such that $||Y^{-1}|| \le M_1$.

To establish our result, we now employ Krasnoselskii's fixed point theorem.

Theorem 2. Let (G1) and (G2) hold. Then, the nonlinear system (2) is controllable if

$$M_{2}\left[1 + \frac{M_{1}x_{1}^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x_{1}^{\alpha})\|B\|\right] < 1,$$
(17)

where

$$M_{2} = \frac{x_{1}^{\alpha - \frac{1}{q}}}{(\alpha p - p + 1)^{\frac{1}{p}}} \mathbb{E}_{\alpha, \alpha}(\|A\|x_{1}^{\alpha}) \|L_{f}\|_{L^{q}(\Omega, \mathbb{R}^{+})} \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \ p, q > 1$$

Proof. Before we start to prove this theorem, we shall use the following assumptions and estimates: We consider the set

$$B_{\epsilon} = \left\{ y \in C([-h, x_1], \mathbb{R}^n) : \|y\|_{C[-h, x_1]} = \sup_{x \in [-h, x_1]} \|y(x)\| \le \epsilon \right\}.$$

Let $x \in [0, x_1]$. From (**G1**) and Hölder inequality, we obtain

$$\int_{0}^{x} (x-\downarrow)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\|A\| (x-\downarrow)^{\alpha} \right) L_{f}(\downarrow) d\downarrow
\leq \left(\int_{0}^{x} \left((x-\downarrow)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \left(\|A\| (x-\downarrow)^{\alpha} \right) \right)^{p} d\downarrow \right)^{\frac{1}{p}} \left(\int_{0}^{x} L_{f}^{q}(\downarrow) d\downarrow \right)^{\frac{1}{q}}
\leq \mathbb{E}_{\alpha,\alpha} (\|A\| x^{\alpha}) \left(\int_{0}^{x} (x-\downarrow)^{(\alpha-1)p} d\downarrow \right)^{\frac{1}{p}} \left(\int_{0}^{x} L_{f}^{q}(\downarrow) d\downarrow \right)^{\frac{1}{q}}
= \frac{x^{\alpha-\frac{1}{q}}}{(\alpha p-p+1)^{\frac{1}{p}}} \mathbb{E}_{\alpha,\alpha} (\|A\| x^{\alpha}) \left\| L_{f} \right\|_{L^{q}(\Omega,\mathbb{R}^{+})}.$$
(18)

Furthermore, consider the following control function u_y :

$$u_{y}(x) = Y^{-1} \Big[y_{1} - \mathcal{H}_{h,\alpha} \big(A(x_{1} - h)^{\alpha} \big) \psi(0) - \mathcal{M}_{h,\alpha} \big(A(x_{1} - h)^{\alpha} \big) \psi'(0) \\ + A \int_{-h}^{0} S_{h,\alpha} \big(A(x_{1} - 2h - \downarrow)^{\alpha} \big) \psi(\downarrow) d\downarrow \\ - \int_{0}^{x_{1}} S_{h,\alpha} \big(A(x_{1} - h - \downarrow)^{\alpha} \big) f(\downarrow, y(\downarrow)) d\downarrow \Big](x),$$
(19)

for $x \in \Omega$. From (18), (19), (G1), (G2), and Lemma 1, we obtain

$$\begin{split} \|u_{y}(x)\| &\leq \left\|Y^{-1}\right\| \left(\|y_{1}\| + \left\|\mathcal{H}_{h,\alpha}\left(A(x_{1}-h)^{\alpha}\right)\right\| \|\psi(0)\| \\ &+ \left\|\mathcal{M}_{h,\alpha}\left(A(x_{1}-h)^{\alpha}\right)\right\| \|\psi'(0)\| \\ &+ \left\|\mathcal{M}\right\| \int_{-h}^{0} \left\|\mathcal{S}_{h,\alpha}\left(A(x_{1}-2h-\downarrow)^{\alpha}\right)\right\| \|\psi(\downarrow)\| d\downarrow \\ &+ \int_{0}^{x_{1}} \left\|\mathcal{S}_{h,\alpha}\left(A(x_{1}-h-\downarrow)^{\alpha}\right)\right\| \|f(\downarrow,y(\downarrow))\| d\downarrow) \\ &\leq M_{1} \|y_{1}\| + M_{1} \mathbb{E}_{\alpha}\left(\left\|\mathcal{A}\|\left(x_{1}-h\right)^{\alpha}\right)\right\|\psi\|_{C} \\ &+ M_{1} x_{1} \mathbb{E}_{\alpha,2}\left(\left\|\mathcal{A}\|x_{1}^{\alpha}\right)\right\|\psi'\|_{C} \\ &+ M_{1} \|\mathcal{A}\|\|\psi\|_{C} \int_{-h}^{0} (x_{1}-h-\downarrow)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\left(\left\|\mathcal{A}\|\left(x_{1}-h-\downarrow\right)^{\alpha}\right)d\downarrow \\ &+ M_{1} \int_{0}^{x_{1}} (x_{1}-\downarrow)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\left(\left\|\mathcal{A}\|\left(x_{1}-\downarrow\right)^{\alpha}\right)L_{f}(\downarrow)\|y(\downarrow)\| d\downarrow \\ &+ M_{1} \int_{0}^{x_{1}} (x_{1}-\downarrow)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\left(\left\|\mathcal{A}\|\left(x_{1}-\downarrow\right)^{\alpha}\right)\right\|f(\downarrow,0)\| d\downarrow \\ &\leq M_{1} \|y_{1}\| + M_{1} \mathbb{E}_{\alpha}\left(\left\|\mathcal{A}\|\left(x_{1}-h\right)^{\alpha}\right)\|\psi\|_{C} \\ &+ M_{1} x_{1} \mathbb{E}_{\alpha,2}\left(\left\|\mathcal{A}\|x_{1}^{\alpha}\right)\right\|\psi'\|_{C} + \frac{M_{1} \|\mathcal{A}\|\|\psi\|_{C} x_{1}^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha}\left(\left\|\mathcal{A}\|x_{1}^{\alpha}\right) \\ &+ \frac{M_{1} x^{\alpha-\frac{1}{q}}}{(\alpha p-p+1)^{\frac{1}{p}}} \mathbb{E}_{\alpha,\alpha}\left(\left\|\mathcal{A}\|x^{\alpha}\right)\right\|L_{f} \|_{L^{q}(\Omega,\mathbb{R}^{+})} \|y\|_{C(\Omega)} \\ &+ \frac{M_{1} M_{f} x_{1}^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha}\left(\left\|\mathcal{A}\|x_{1}^{\alpha}\right) \\ &\leq M_{1} \|y_{1}\| + M_{1} M_{2} \epsilon + M_{1} \theta(x_{1}), \end{split}$$

where

$$\theta(x) = \mathbb{E}_{\alpha} \left(\|A\| (x-h)^{\alpha} \right) \|\psi\|_{C} + x \mathbb{E}_{\alpha,2} \left(\|A\| x^{\alpha} \right) \|\psi'\|_{C} + \frac{x^{\alpha} \left(\|A\| \|\psi\|_{C} + M_{f} \right)}{\alpha} \mathbb{E}_{\alpha,\alpha} \left(\|A\| x^{\alpha} \right).$$

Furthermore,

$$\begin{aligned} &\|u_{y}(x) - u_{z}(x)\| \\ &\leq M_{1} \int_{0}^{x_{1}} \|\mathcal{S}_{h,\alpha} \left(A(x_{1} - h - \downarrow)^{\alpha}\right)\| \|f(\downarrow, y(\downarrow)) - f(\downarrow, z(\downarrow))\| d\downarrow \\ &\leq M_{1} \int_{0}^{x_{1}} \|\mathcal{S}_{h,\alpha} \left(A(x_{1} - h - \downarrow)^{\alpha}\right)\| L_{f}(\downarrow)\| y(\downarrow) - z(\downarrow)\| d\downarrow \\ &\leq M_{1} M_{2} \|y - z\|_{C(\Omega)}. \end{aligned}$$

$$(21)$$

We also define the operators \mathcal{L}_1 , \mathcal{L}_2 on B_{ϵ} as follows:

$$(\mathcal{L}_{1}y)(x) = \mathcal{H}_{h,\alpha} (A(x-h)^{\alpha}) \psi(0) + \mathcal{M}_{h,\alpha} (A(x-h)^{\alpha}) \psi'(0)$$

- $A \int_{-h}^{0} S_{h,\alpha} (A(x-2h-\downarrow)^{\alpha}) \psi(\downarrow) d\downarrow$
+ $\int_{0}^{x} S_{h,\alpha} (A(x-h-\downarrow)^{\alpha}) Bu_{y}(\downarrow) d\downarrow,$ (22)

$$(\mathcal{L}_2 y)(x) = \int_0^x \mathcal{S}_{h,\alpha} \big(A(x - h - \downarrow)^{\alpha} \big) f(\downarrow, y(\downarrow)) \mathrm{d} \downarrow.$$
(23)

Now, we see that B_{ϵ} is a closed, bounded, and convex set of $C([-h, x_1], \mathbb{R}^n)$. Therefore, our proof is divided into three main steps.

Step 1. We prove $\mathcal{L}_1 y + \mathcal{L}_2 z \in B_{\epsilon}$ for all $y, z \in B_{\epsilon}$. For each $x \in \Omega$ and $y, z \in B_{\epsilon}$, using (20), we obtain

$$\begin{split} \|\mathcal{L}_{1}y + \mathcal{L}_{2}z\|_{C[-h,x_{1}]} &= \sup_{x \in [-h,x_{1}]} \|(\mathcal{L}_{1}y + \mathcal{L}_{2}z)(x)\| \\ &\leq \sup_{x \in [-h,x_{1}]} \{ \|\mathcal{H}_{h,\alpha}(A(x-h)^{\alpha})\| \|\psi(0)\| + \|\mathcal{M}_{h,\alpha}(A(x-h)^{\alpha})\| \|\psi'(0)\| \\ &+ \|A\| \int_{-h}^{0} \|S_{h,\alpha}(A(x-2h-\downarrow)^{\alpha})\| \|\psi(\downarrow)\| d\downarrow \\ &+ \int_{0}^{x} \|S_{h,\alpha}(A(x-h-\downarrow)^{\alpha})\| \|B\| \|u_{y}(\downarrow)\| d\downarrow \\ &+ \int_{0}^{x} \|S_{h,\alpha}(A(x-h-\downarrow)^{\alpha})\| \|\|f(\downarrow,z(\downarrow))\| d\downarrow \} \\ &\leq \mathbb{E}_{\alpha}(\|A\|(x-h)^{\alpha})\|\psi\|_{C} + x\mathbb{E}_{\alpha,2}(\|A\|x^{\alpha})\|\psi'\|_{C} \\ &+ \frac{x^{\alpha}\|A\| \|\psi\|_{C}}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha}) + \frac{M_{f}x^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha}) \\ &+ \frac{x^{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha})\|B\| (M_{1}\|y_{1}\| + M_{1}M_{2}\epsilon + M_{1}\theta(x_{1})) d\downarrow \\ &+ \frac{x^{\alpha-\frac{1}{q}}}{(\alpha p - p + 1)^{\frac{1}{p}}} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha})\|L_{f}\|_{L^{q}(\Omega,\mathbb{R}^{+})} \|z\|_{C}(\Omega) \\ &\leq \theta(x_{1}) + M_{2}\epsilon + \frac{M_{1}x^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha}_{1})\|B\| \|y_{1}\| \\ &+ \frac{M_{1}M_{2}\epsilon x^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha}_{1})\|B\| + \frac{M_{1}\theta(x_{1})x^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha}_{1})\|B\| \\ &\leq \theta(x_{1}) \left[1 + \frac{M_{1}x^{\alpha}_{1}}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha}_{1})\|B\| \right] + \frac{M_{1}x^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha}_{1})\|B\| \|y_{1}\| \\ &+ M_{2} \left[1 + \frac{M_{1}x^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha}_{1})\|B\| \right]\epsilon. \end{split}$$

Thus, for some ϵ sufficiency large, and from (17), we have $\mathcal{L}_1 y + \mathcal{L}_2 z \in B_{\epsilon}$. **Step 2.** We prove $\mathcal{L}_1 : B_{\epsilon} \to C([-h, x_1], \mathbb{R}^n)$ is a contraction. For each $x \in \Omega$ and $y, z \in B_{\epsilon}$, using (21), we obtain

$$\begin{aligned} |(\mathcal{L}_{1}y)(x) - (\mathcal{L}_{1}z)(x)| &\leq \int_{0}^{x} \left\| \mathcal{S}_{h,\alpha} \left(A(x-h-\downarrow)^{\alpha} \right) \right\| \|B\| \left\| u_{y}(\downarrow) - u_{z}(\downarrow) \right\| d\downarrow \\ &\leq \|B\| M_{1}M_{2} \|y-z\|_{C(\Omega)} \int_{0}^{x} \left\| \mathcal{S}_{h,\alpha} \left(A(x-h-\downarrow)^{\alpha} \right) \right\| d\downarrow \\ &\leq \frac{x_{1}^{\alpha} \|B\| M_{1}M_{2}}{\alpha} \mathbb{E}_{\alpha,\alpha} (\|A\| x_{1}^{\alpha}) \|y-z\|_{C(\Omega)} \\ &\leq \mu \|y-z\|_{C(\Omega)'} \end{aligned}$$

where $\mu := \frac{x_1^{\alpha} \|B\| M_1 M_2}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\| x_1^{\alpha})$. From (17), note $\mu < 1$; we conclude that \mathcal{L}_1 is a contraction mapping.

Step 3. We prove $\mathcal{L}_2 : B_{\epsilon} \to C([-h, x_1], \mathbb{R}^n)$ is a continuous compact operator.

Firstly, we show that \mathcal{L}_2 is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ as $n \to \infty$ in B_{ϵ} . Thus, for each $x \in \Omega$, using (23) and Lebesgue's dominated convergence theorem, we obtain

$$\begin{split} \|(\mathcal{L}_{2}y_{n})(x) - (\mathcal{L}_{2}y)(x)\| \\ &\leq \int_{0}^{x} \|\mathcal{S}_{h,\alpha} \big(A(x-h-\downarrow)^{\alpha}\big)\| \|f(\downarrow, y_{n}(\downarrow)) - f(\downarrow, y(\downarrow))\| d\downarrow \\ &\leq \int_{0}^{x} (x-\downarrow)^{\alpha-1} \mathbb{E}_{\alpha,\alpha} \big(\|A\|(x-\downarrow)^{\alpha}\big) L_{f}(\downarrow)\| y_{n}(\downarrow) - y(\downarrow)\| d\downarrow \to 0, \text{ as } n \to \infty. \end{split}$$

Hence $\mathcal{L}_2 : B_{\epsilon} \to C([-h, x_1], \mathbb{R}^n)$ is a continuous.

Next, we prove that \mathcal{L}_2 is uniformly bounded on B_{ϵ} . For each $x \in \Omega$, $y \in B_{\epsilon}$, we have

$$\begin{split} \|\mathcal{L}_{2}y\| &= \sup_{x \in \Omega} \|(\mathcal{L}_{2}y)(x)\| \\ &\leq \sup_{x \in \Omega} \left\{ \int_{0}^{x} \|\mathcal{S}_{h,\alpha} \left(A(x-h-\downarrow)^{\alpha}\right)\| \|f(\downarrow, y(\downarrow))\| d\downarrow \right\} \\ &\leq \frac{x^{\alpha-\frac{1}{q}}}{(\alpha p-p+1)^{\frac{1}{p}}} \mathbb{E}_{\alpha,\alpha} (\|A\|x^{\alpha}) \|L_{f}\|_{L^{q}(\Omega,\mathbb{R}^{+})} \|y\|_{C(\Omega)} \\ &+ \frac{M_{f}x^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha} (\|A\|x^{\alpha}) \\ &\leq M_{2}\epsilon + \frac{M_{f}x_{1}^{\alpha}}{\alpha} \mathbb{E}_{\alpha,\alpha} (\|A\|x_{1}^{\alpha}), \end{split}$$

which implies that \mathcal{L}_2 is uniformly bounded on B_{ϵ} .

It remains to be shown that \mathcal{L}_2 is equicontinuous. For each $x_2, x_3 \in \Omega, 0 < x_2 < x_3 \leq x_1$ and $y \in B_{\epsilon}$, using (23), we obtain

$$\begin{aligned} (\mathcal{L}_{2}y)(x_{3}) &- (\mathcal{L}_{2}y)(x_{2}) \\ &\leq \int_{0}^{x_{3}} \mathcal{S}_{h,\alpha} \big(A(x_{3} - h - \downarrow)^{\alpha} \big) f(\downarrow, y(\downarrow)) \mathrm{d} \downarrow \\ &- \int_{0}^{x_{2}} \mathcal{S}_{h,\alpha} \big(A(x_{2} - h - \downarrow)^{\alpha} \big) f(\downarrow, y(\downarrow)) \mathrm{d} \downarrow \\ &= \Psi_{1} + \Psi_{2}, \end{aligned}$$

where

$$\Psi_1 = \int_{x_2}^{x_3} \mathcal{S}_{h,\alpha} \big(A(x_3 - h - \downarrow)^{\alpha} \big) f(\downarrow, y(\downarrow)) d\downarrow,$$

and

$$\Psi_2 = \int_0^{x_2} \left[S_{h,\alpha} \left(A(x_3 - h - \downarrow)^{\alpha} \right) - S_{h,\alpha} \left(A(x_2 - h - \downarrow)^{\alpha} \right) \right] f(\downarrow, y(\downarrow)) d\downarrow.$$

Thus

$$\|(\mathcal{L}_2 y)(x_3) - (\mathcal{L}_2 y)(x_2)\| \le \|\Psi_1\| + \|\Psi_2\|.$$
(24)

Now, we can check $\|\Psi_i\| \to 0$ as $x_2 \to x_3$, i = 1, 2. For Ψ_1 , we obtain

$$\begin{split} \Psi_{1} &\leq \int_{x_{2}}^{x_{3}} (x_{3} - \downarrow)^{\alpha - 1} \mathbb{E}_{\alpha, \alpha} \big(\|A\| (x_{3} - \downarrow)^{\alpha} \big) L_{f}(\downarrow) \|y(\downarrow)\| d\downarrow \\ &+ \int_{x_{2}}^{x_{3}} (x_{3} - \downarrow)^{\alpha - 1} \mathbb{E}_{\alpha, \alpha} \big(\|A\| (x_{3} - \downarrow)^{\alpha} \big) \|f(\downarrow, 0)\| d\downarrow \\ &\leq \frac{(x_{3} - x_{2})^{\alpha - \frac{1}{q}}}{(\alpha p - p + 1)^{\frac{1}{p}}} \mathbb{E}_{\alpha, \alpha} \big(\|A\| x_{3}^{\alpha} \big) \Big\| L_{f} \Big\|_{L^{q}(\Omega, \mathbb{R}^{+})} \|y\|_{C(\Omega)} \\ &+ \frac{M_{f}(x_{3} - x_{2})^{\alpha}}{\alpha} \mathbb{E}_{\alpha, \alpha} \big(\|A\| x_{3}^{\alpha} \big) \to 0, \text{ as } x_{2} \to x_{3}. \end{split}$$

For Ψ_2 , we obtain

$$\begin{split} |\Psi_2\| &\leq \epsilon \int_0^{x_2} \left\| \mathcal{S}_{h,\alpha} \big(A(x_3 - h - \downarrow)^{\alpha} \big) - \mathcal{S}_{h,\alpha} \big(A(x_2 - h - \downarrow)^{\alpha} \big) \right\| L_f(\downarrow) \mathrm{d} \downarrow \\ &+ M_f \int_0^{x_2} \left\| \mathcal{S}_{h,\alpha} \big(A(x_3 - h - \downarrow)^{\alpha} \big) - \mathcal{S}_{h,\alpha} \big(A(x_2 - h - \downarrow)^{\alpha} \big) \right\| \mathrm{d} \downarrow. \end{split}$$

From (7), we know that the delayed Mittag–Leffler type matrix function $S_{h,\alpha}(Ax^{\alpha})$ is uniformly continuous for $x \in \Omega$. Thus,

$$\left\|\mathcal{S}_{h,\alpha}\left(A(x_3-h-\downarrow)^{\alpha}\right)-\mathcal{S}_{h,\alpha}\left(A(x_2-h-\downarrow)^{\alpha}\right)\right\|\to 0, \text{ as } x_2\to x_3.$$

Therefore, we have $\|\Psi_i\| \to 0$ as $x_2 \to x_3$, i = 1, 2, which implies that, using (24),

$$\|(\mathcal{L}_2 y)(x_3) - (\mathcal{L}_2 y)(x_2)\| \to 0$$
, as $x_2 \to x_3$,

for all $y \in B_{\epsilon}$. Thus, the Arzelà–Ascoli theorem tells us that \mathcal{L}_2 is compact on B_{ϵ} .

Therefore, according to Krasnoselskii's fixed-point theorem (Lemma 3), $\mathcal{L}_1 + \mathcal{L}_2$ has a fixed point y on B_{ϵ} . In addition, y is also a solution of (2) and $(\mathcal{L}_1y + \mathcal{L}_2y)(x_1) = y_1$. This means that u_y steers the system (2) from y_0 to y_1 in finite time x_1 , which implies that (2) is controllable on Ω . This completes the proof. \Box

Remark 4. We note in the case of $\alpha = 2$ in (2) that Theorem 2 coincides with the conclusion of Corollary 3 in [16].

Remark 5. Under condition A, there is a nonsingular $n \times n$ matrix; we note in the case of $\alpha = 2$ and $A = A^2$ in (2) that Theorem 2 coincides with the conclusion of Theorem 4.1 in [21] and Corollary 4 in [16].

5. Hyers–Ulam Stability of Nonlinear Fractional Delay System

In this section, we discuss the Hyers–Ulam stability of (2) on the finite time interval $[0, x_1]$.

Theorem 3. Let (G1), (G2), and (17) be satisfied. Then, the system (2) is Hyers–Ulam stable.

Proof. With the help of Theorem 2, let $\omega \in C(\Omega, \mathbb{R}^n)$ be a solution of the inequality (9) and *y* be the unique solution of (2), that is,

$$\begin{split} y(x) &= \mathcal{H}_{h,\alpha} \big(A(x-h)^{\alpha} \big) \psi(0) + \mathcal{M}_{h,\alpha} \big(A(x-h)^{\alpha} \big) \psi'(0) \\ &- A \int_{-h}^{0} \mathcal{S}_{h,\alpha} \big(A(x-2h-\downarrow)^{\alpha} \big) \psi(\downarrow) \mathrm{d} \downarrow \\ &+ \int_{0}^{x} \mathcal{S}_{h,\alpha} \big(A(x-h-\downarrow)^{\alpha} \big) f(\downarrow, y(\downarrow)) \mathrm{d} \downarrow \\ &+ \int_{0}^{x} \mathcal{S}_{h,\alpha} \big(A(x-h-\downarrow)^{\alpha} \big) Bu_{y}(\downarrow) \mathrm{d} \downarrow. \end{split}$$

From Lemma 2, and by a similar way in the proof of Theorem 2 and by virtue of (21), we obtain

$$\begin{split} \| \varpi(x) - y(x) \| &\leq \| \varpi(x) - \varpi^*(x) \| + \| \varpi^*(x) - y(x) \| \\ &\leq \frac{x^{\alpha} \varepsilon}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x^{\alpha}) \\ &+ \int_0^x \| \mathcal{S}_{h,\alpha} (A(x - h - \downarrow)^{\alpha}) \| \|B\| \| u_{\varpi}(\downarrow) - u_y(\downarrow) \| d\downarrow \\ &+ \int_0^x \| \mathcal{S}_{h,\alpha} (A(x - h - \downarrow)^{\alpha}) \| \|f(\downarrow, \varpi(\downarrow)) - f(\downarrow, y(\downarrow)) \| d\downarrow \\ &\leq \frac{x_1^{\alpha} \varepsilon}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x_1^{\alpha}) \\ &+ \frac{x_1^{\alpha} \|B\| M_1 M_2}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x_1^{\alpha}) \| \varpi - y \|_{C(\Omega)} \\ &+ M_2 \| \varpi - y \|_{C(\Omega)} \\ &= \frac{x_1^{\alpha} \varepsilon}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x_1^{\alpha}) \\ &+ M_2 \Big(1 + \frac{x_1^{\alpha} \|B\| M_1}{\alpha} \mathbb{E}_{\alpha,\alpha}(\|A\|x_1^{\alpha}) \Big) \| \varpi - y \|_{C(\Omega)}. \end{split}$$

So,

$$\|\omega - y\|_{C(\Omega)} \leq \frac{x_1^{\alpha}\varepsilon}{\alpha(1-\rho)} \mathbb{E}_{\alpha,\alpha}(\|A\|x_1^{\alpha}),$$

where

$$\rho := M_2 \bigg(1 + \frac{x_1^{\alpha} \|B\| M_1}{\alpha} \mathbb{E}_{\alpha, \alpha}(\|A\| x_1^{\alpha}) \bigg).$$

Thus,

$$\|arphi(x)-y(x)\|\leq Narepsilon, \ \ N=rac{x_1^lpha}{lpha(1-
ho)}\mathbb{E}_{lpha,lpha}(\|A\|x_1^lpha).$$

This completes the proof. \Box

Remark 6. Let $\alpha = 2$ in (2). Then, Theorem 3 coincides with the conclusion of Theorem 3 in [16].

Remark 7. We note that Theorems 1–3 improve, extend, and complement some existing results in [16,19,21,35].

6. Examples

In this section, we present applications of the results derived.

Example 1. Consider the following linear delay fractional controlled system:

where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \psi(x) = \begin{pmatrix} 2x \\ x \end{pmatrix}, \psi'(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Wenote that $B \in \mathbb{R}^{2 \times 1}$ and $u(x) \in \mathbb{R}$ show the control vector. Constructing the corresponding fractional delay Gramian matrix of (25) via (10), we obtain

$$W_{0.5,1.5}[0,1] = \int_0^1 S_{0.5,1.5} \Big(A(0.5-\downarrow)^{1.5} \Big) B B^T S_{0.5,1.5} \Big(A^T (0.5-\downarrow)^{1.5} \Big) d\downarrow$$

=: $O_1 + O_2$,

where

$$O_{1} = \int_{0}^{0.5} \mathcal{S}_{0.5,1.5} \Big(A(0.5 - \downarrow)^{1.5} \Big) B B^{T} \mathcal{S}_{0.5,1.5} \Big(A^{T} (0.5 - \downarrow)^{1.5} \Big) d\downarrow,$$

for $(0.5 - \downarrow) \in (0, 0.5)$,

$$O_2 = \int_{0.5}^1 S_{0.5,1.5} \Big(A(0.5 - \downarrow)^{1.5} \Big) B B^T S_{0.5,1.5} \Big(A^T (0.5 - \downarrow)^{1.5} \Big) d\downarrow,$$

for $(0.5 - \downarrow) \in (-0.5, 0)$, and

$$\mathcal{H}_{0.5,1.5}\Big(Ax^{1.5}\Big) := \begin{cases} \Theta, & -\infty < x < -0.5, \\ \mathbb{I}, & -0.5 \le x < 0, \\ \mathbb{I} - A\frac{x^{1.5}}{\Gamma(2.5)} & 0 \le x < 0.5, \\ \mathbb{I} - A\frac{x^{1.5}}{\Gamma(2.5)} + A^2\frac{(x-0.5)^3}{\Gamma(4)}, & 0.5 \le x < 1, \end{cases}$$

$$\mathcal{M}_{0.5,1.5} \Big(A x^{1.5} \Big) := \begin{cases} \Theta, & -\infty < x < -0.5, \\ \mathbb{I}(x+0.5), & -0.5 \le x < 0, \\ \mathbb{I}(x+0.5) - A \frac{x^{2.5}}{\Gamma(3.5)}, & 0 \le x < 0.5, \\ \mathbb{I}(x+0.5) - A \frac{x^{2.5}}{\Gamma(3.5)} + A^2 \frac{(x-0.5)^4}{\Gamma(5)}, & 0.5 \le x < 1, \end{cases}$$

and

$$\mathcal{S}_{0.5,1.5}\left(Ax^{1.5}\right) := \begin{cases} \Theta, & -\infty < x < -0.5, \\ \mathbb{I}\frac{(x+0.5)^{0.5}}{\Gamma(1.5)}, & -0.5 \le x < 0, \\ \mathbb{I}\frac{(x+0.5)^{0.5}}{\Gamma(1.5)} - A\frac{x^2}{\Gamma(3)}, & 0 \le x < 0.5, \\ \mathbb{I}\frac{(x+0.5)^{0.5}}{\Gamma(1.5)} - A\frac{x^2}{\Gamma(3)} + A^2\frac{(x-0.5)^{3.5}}{\Gamma(4.5)}, & 0.5 \le x < 1. \end{cases}$$

Next, we can calculate that

$$O_1 = \begin{pmatrix} \frac{50961}{40000} & \frac{503601}{100000} \\ \frac{503601}{1000000} & \frac{-57031}{500000} \end{pmatrix}, \quad O_2 = \begin{pmatrix} \frac{3183}{20000} & \frac{3183}{10000} \\ \frac{3183}{10000} & \frac{3183}{5000} \end{pmatrix}.$$

Then,we obtain

$$W_{0.5,1.5}[0,1] = O_1 + O_2 = \left(\begin{array}{cc} \frac{114621}{400000} & \frac{821901}{1000000} \\ \frac{821901}{1000000} & \frac{261269}{500000} \end{array}\right),$$

and

$$W_{0.5,1.5}^{-1}[0,1] = \begin{pmatrix} \frac{-130634500000}{131446670889} & \frac{68491750000}{43815555963} \\ \frac{68491750000}{43815556963} & \frac{-23879375000}{43815556963} \end{pmatrix}.$$

Therefore, we see that $W_{0.5,1.5}[0,1]$ is positive definite. Furthermore, for any finite terminal conditions $y_1, y'_1 \in \mathbb{R}^2$ such that $y(x_1) = y_1 = (y_{11}, y_{12})^T$, $y'(x_1) = y'_1 = (y'_{11}, y'_{12})^T$, as a result we can establish the corresponding control as follows:

$$u(x) = B^{T} S_{0.5,1.5} \left(A^{T} (0.5 - x)^{1.5} \right) W_{0.5,1.5}^{-1}[0,1] \beta,$$

where

$$\beta = y_1 - \mathcal{M}_{0.5,1.5} \left(A(0.5)^{1.5} \right) \psi'(0) + A \int_{-0.5}^0 \mathcal{S}_{0.5,1.5} \left(A(-\downarrow)^{1.5} \right) \psi(\downarrow) d\downarrow$$
$$= \left(\begin{array}{c} y_{11} - \frac{1127}{500} \\ y_{12} - \frac{5337}{5000} \end{array} \right).$$

Hence, the system (25) *is controllable on* [0, 1] *by Theorem* 1.

Example 2. Consider the following nonlinear delay fractional controlled system:

$$\binom{{}^{C}D_{0^{+}}^{1.8}y}{y(x)} + Ay(x-0.6) = f(x,y(x)) + Bu(x), \text{ for } x \in \Omega_1 := [0,1.2], y(x) \equiv \psi(x), y'(x) \equiv \psi'(x) \text{ for } -0.6 \le x \le 0,$$
 (26)

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, B = \mathbb{I}^{2 \times 2}, \ \psi(x) = \begin{pmatrix} 3x+1 \\ x^2 \end{pmatrix}, \ \psi'(x) = \begin{pmatrix} 3 \\ 2x \end{pmatrix},$$
$$f(x, y(x)) = \begin{pmatrix} 0.5(x - 0.6)\cos[y_1(x)] \\ 0.5(x - 0.6)\cos[y_2(x)] \end{pmatrix}.$$

Now, we set $u(x) = \tilde{y}$ *, where* $\tilde{y} \in \mathbb{R}^2$ *. From the definition of* Y *in* (**G2**)*, we obtain*

$$\begin{split} \mathbf{Y} &= \int_{0}^{1.2} \mathcal{S}_{0.6,1.8} \Big(A(0.6 - \downarrow)^{1.8} \Big) B \mathrm{d} \downarrow \widetilde{y} \\ &= \int_{0}^{0.6} \mathcal{S}_{0.6,1.8} \Big(A(0.6 - \downarrow)^{1.8} \Big) \mathrm{d} \downarrow \widetilde{y} + \int_{0.6}^{1.2} \mathcal{S}_{0.6,1.8} \Big(A(0.6 - \downarrow)^{1.8} \Big) \mathrm{d} \downarrow \widetilde{y} \\ &= \Big(\begin{array}{c} \frac{578469}{100000} & 0 \\ 0 & \frac{141647}{250000} \end{array} \Big) \widetilde{y} + \Big(\begin{array}{c} \frac{23783}{100000} & 0 \\ 0 & \frac{23783}{100000} \end{array} \Big) \widetilde{y} \\ &= \Big(\begin{array}{c} \frac{816299}{1000000} & 0 \\ 0 & \frac{402209}{500000} \end{array} \Big) \widetilde{y}. \end{split}$$

Define the inverse $Y^{-1}:\mathbb{R}^2\to L^2(\Omega_1,\mathbb{R}^2)$ by

$$(Y^{-1}\widetilde{y})(x) := \begin{pmatrix} \frac{100000}{816299} & 0\\ 0 & \frac{50000}{402209} \end{pmatrix} \widetilde{y}.$$

Then, we obtain

$$\left\| \left(Y^{-1} \widetilde{y} \right)(x) \right\| \le \left\| \left(\begin{array}{cc} \frac{1000000}{816299} & 0\\ 0 & \frac{500000}{402209} \end{array} \right) \right\| \| \widetilde{y} \| = 1.243 \| \widetilde{y} \|.$$

and thus we obtain $\|Y^{-1}\| \leq 1.243 =: M_1$. Hence, the assumption (G2) is satisfied by Y.

Next, keep in mind that $|\cos \lambda - \cos \delta| \le |\lambda - \delta|$ *, for all* $\lambda, \delta \in \mathbb{R}$ *, we obtain*

$$\begin{split} \|f(x,y) - f(x,z)\| \\ &= |0.5(x-0.6)|\sqrt{(\cos[y_1(x)] - \cos[z_1(x)])^2 + (\cos[y_2(x)] - \cos[z_2(x)])^2} \\ &\leq |0.5(x-0.6)|\sqrt{(y_1(x) - z_1(x))^2 + (y_2(x) - z_2(x))^2} \\ &= |0.5(x-0.6)| \|y - z\|, \end{split}$$

for all $x \in \Omega_1$, and $y(x), z(x) \in \mathbb{R}^2$. We set $L_f(x) = |0.5(x - 0.6)|$ such that $L_f \in L^q(\Omega_1, \mathbb{R}^+)$ in (G1). By choosing p = q = 2, we have

$$\left\|L_f\right\|_{L^2(\Omega_1,\mathbb{R}^+)} = \left(\int_0^{1.2} [0.5(\downarrow - 0.6)]^2 d\downarrow\right)^{\frac{1}{2}} = 0.18974.$$

Then, we obtain

$$M_{2} = \frac{(1.2)^{1.3}}{(2.6)^{\frac{1}{2}}} \mathbb{E}_{1.8,1.8} \left(2(1.2)^{1.8} \right) \left\| L_{f} \right\|_{L^{q}(\Omega, \mathbb{R}^{+})} = 0.301$$

Finally, we calculate that

$$M_2 \left[1 + rac{M_1 (1.2)^{1.8}}{1.8} \mathbb{E}_{1.8, 1.8} \left(2(1.2)^{1.8}
ight) \|B\|
ight] = 0.8815 < 1.6$$

which implies that all the conditions of Theorems 2 and 3 are satisfied. Therefore, the system (26) is controllable and Hyers–Ulam stable.

7. Conclusions

In this work, we established some sufficient and necessary conditions for the controllability of linear fractional-delay systems by using a fractional delay Gramian matrix and the representation of solutions of these systems with the help of their delayed Mittag– Leffler matrix functions. Furthermore, we established some sufficient conditions for the controllability and Hyers–Ulam stability of nonlinear fractional-delay systems by applying Krasnoselskii's fixed-point theorem and the representation of the solutions of these systems. Finally, the effectiveness of the obtained results was illustrated by numerical examples.

Our future work includes extending and complementing the results of this paper to derive the controllability and Hyers–Ulam stability results of fractional stochastic delay systems with compact analytic semigroups or using the delayed Mittag–Leffler matrix functions with various behaviors such as impulses and delays in multi-states.

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